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Version A

Microeconomics II

SDPE, Stockholm School of Economics

Fifth Assignment

“On Contract Theory"

Section 1. Hidden Information

A car seller is considering opening its first stores in city x and has to determine their location as well as the price of the cars at each store. There is a population of potential costumers (or buyers) of total mass equal to 1 who live in the city centre. Each buyer may potentially buy at most one car. Each store can be located at a distance $t \in [0,1]$ from the city centre and more than one store can be open at a given location. The seller knows that there are two types of buyers: type h and type l. Type h's buyers are a fraction $q \in (0,1)$ of the population. If a type h's buyer buys a car at a price p from the store located at distance t, his/her net payoff is

$$u_h(t,p) = 3 - 2t_h - p_h$$

whereas the net payoff for a type l's buyer who buys a car at a price p from the store located at distance t is

$$u_l(t,p) = 2 - t_l - p_l$$

The seller's cost of operating a store located at t is $C(t) = \frac{t^2}{2}$, and the seller's profit is equal to the expected revenue minus the total cost of operating the stores. For example, if the seller opens two stores at t_l and t_h, and sets the prices at p_l and p_h, respectively, so that all type l's buyers buy from t_l and type h's buyers from t_h, its expected profit is equal to

$$qp_h + (1-q)p_l - C(t_h) - C(t_l)$$

Assume that buyers buy a car as long as doing so brings a non-negative net payoff.

1. Assume that the seller observes each buyer's type and that he/she can charge different prices to different buyers. Find the optimal solution to the seller's problem (i.e., find the optimal pair (t_i, p_i) for $i \in \{l, h\}$) and his/her profit at the solution. *(Hint: Please, remember that two stores can be opened at the same location).*

   Hereafter, let us assume that the seller cannot observe the buyers' types.

2. Find the optimal location, prices and profits when the seller is restricted to only one store. *(Hint: You might find useful your previous result).*

3. Suppose that the seller implements the policy obtained in Q1. Which store, if any, would each type of customer choose to buy from?

4. Formulate the seller's problem of choosing the optimal (t_l, p_l) and (t_h, p_h) so that type l's buyer buys from store t_l at price p_l and type h's buyer buys from store t_h at price p_h. *(Hint: Don't forget constraints on the location decisions).*

5. Show, as formally as you can, that at the solution of the seller's problem in Q4 the following are true:

   (a) The participation constraint of type l's buyer is binding.

   (b) The monotonicity conditions $t_l \geq t_h$ and $p_h \geq p_l$ hold.

   (c) The incentive compatibility constraint of type h's buyer holds with equality.

   (d) The incentive compatibility constraint of type h's buyer and the individual rationality constraint for the type l's buyer imply the individual rationality constraint of the type h's buyer.

   (e) The incentive compatibility constraint of type l's buyer can be ignored.

6. Using your findings in Q4 and Q5, write down the seller's relaxed problem and find a complete solution to this problem.

Answer Key for Exam A

Section 1. Hidden Information

A car seller is considering opening its first stores in city x and has to determine their location as well as the price of the cars at each store. There is a population of potential costumers (or buyers) of total mass equal to 1 who live in the city centre. Each buyer may potentially buy at most one car. Each store can be located at a distance $t \in [0,1]$ from the city centre and more than one store can be open at a given location. The seller knows that there are two types of buyers: type h and type l. Type h's buyers are a fraction $q \in (0,1)$ of the population. If a type h's buyer buys a car at a price p from the store located at distance t, his/her net payoff is

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The seller's cost of operating a store located at t is $C(t) = \frac{t^2}{2}$, and the seller's profit is equal to the expected revenue minus the total cost of operating the stores. For example, if the seller opens two stores at t_l and t_h, and sets the prices at p_l and p_h, respectively, so that all type l's buyers buy from t_l and type h's buyers from t_h, its expected profit is equal to

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Assume that buyers buy a car as long as doing so brings a non-negative net payoff.

1. Assume that the seller observes each buyer's type and that he/she can charge different prices to different buyers. Find the optimal solution to the seller's problem (i.e., find the optimal pair (t_i, p_i) for $i \in \{l, h\}$) and his/her profit at the solution. *(Hint: Please, remember that two stores can be opened at the same location).*

   Answer: Assume seller can observe types

$$\begin{aligned} \max_{p_l,p_h,t_l,t_h} \quad & qp_h + (1-q)p_l - \frac{t_h^2}{2} - \frac{t_l^2}{2} \\ s.t. \quad & 3 - 2t_h - p_h \geq 0 && (IR_h) \\ & 2 - t_h - p_l \geq 0 && (IR_l) \end{aligned}$$

Both (??)-(??) will be binding in equilibrium. Hence, the program becomes

$$\max_{t_l, t_h} \quad q(3-2t_h) + (1-q)(2-t_l) - \frac{t_h^2}{2} - \frac{t_l^2}{2} \tag{1}$$

One can verify that the optimal choice is $t_h^* = t_l^* = 0$. Hence, $p_h^* = 3$ and $p_l^* = 2$. Expected profit is

$$\mathbb{E}\pi = 3q + 2(1-q) = 2 + q \tag{2}$$

Hereafter, let us assume that the seller cannot observe the buyers' types.

2. Find the optimal location, prices and profits when the seller is restricted to only one store. *(Hint: You might find useful your previous result).*

Answer: If there can only be one price, let us check the previous expected profits with each price

$$\mathbb{E}(\pi|p=2) = 2q + 2(1-q) = 2$$
$$\mathbb{E}(\pi|p=3) = 3q$$

Hence, the optimal choice is to set $p^* = 3$ if $q \geq \frac{2}{3}$ and $p^* = 2$ otherwise.

3. Suppose that the seller implements the policy obtained in Q1. Which store, if any, would each type of customer choose to buy from?

Answer:

$$u_h(t_h, p_h) = 3 - 2t_h - p_h = 0$$
$$u_h(t_l, p_l) = 3 - 2t_l - p_l = 1$$

Type h would like to be mistakenly thought as l!

$$u_l(t_l, p_l) = 2 - t_l - p_l = 0$$
$$u_l(t_h, p_h) = 2 - t_h - p_h = -1$$

Type l would like to be thought as l.

4. Formulate the seller's problem of choosing the optimal (t_l, p_l) and (t_h, p_h) so that type l's buyer buys from store t_l at price p_l and type h's buyer buys from store t_h at price p_h. *(Hint: Don't forget constraints on the location decisions).*

   Answer:

$$\begin{aligned} \max_{p_l,p_h,t_l,t_h} \quad & qp_h + (1-q)p_l - \frac{t_h^2}{2} - \frac{t_l^2}{2} & \\ s.t. \quad & 3 - 2t_h - p_h \geq 0 & (IR_h) \\ & 2 - t_l - p_l \geq 0 & (IR_l) \\ & 3 - 2t_h - p_h \geq 3 - 2t_l - p_l & (IC_h) \\ & 2 - t_l - p_l \geq 2 - t_h - p_h & (IC_l) \\ & 0 \leq t_l, t_h \leq 1 & \end{aligned}$$

5. Show, as formally as you can, that at the solution of the seller's problem in Q4 the following are true:

   (a) The participation constraint of type l's buyer is binding.

   (b) The monotonicity conditions $t_l \geq t_h$ and $p_h \geq p_l$ hold.

   (c) The incentive compatibility constraint of type h's buyer holds with equality.

   (d) The incentive compatibility constraint of type h's buyer and the individual rationality constraint for the type l's buyer imply the individual rationality constraint of the type h's buyer.

   (e) The incentive compatibility constraint of type l's buyer can be ignored.

   Answer: (a) IR_l binding:
   Suppose it is not. Then

$$3 - 2t_h - p_h \geq 3 - 2t_l - p_l \geq 2 - t_l - p_l > 0$$

   Then, increase simultaneously p_h and p_l by the same amount, the strict inequality would still hold and expected profits would increase. Hence, it *is* binding.

   (b) $t_l \geq t_h$ and $p_h \geq p_l$:
   By (**??**)-(**??**) we know

$$\begin{aligned} 2(t_l - t_h) &\geq p_h - p_l \\ p_h - p_l &\geq t_l - t_h \end{aligned}$$

which implies

$$t_l - t_h \le p_h - p_l \le 2(t_l - t_h) \implies t_l \ge t_h, p_h \ge p_l \tag{3}$$

(c) IC_h binding:
Suppose not. Then, increasing p_h, the inequality would still hold, and expected profits would increase. Hence, it *is* binding.

(d) IC_h and IR_l make redundant IR_h:
Taking IR_l with equality,

$$3 - 2t_h - p_h = 3 - 2t_l - p_l \ge 2 - t_l - p_l = 0 \tag{4}$$

(e) IC_l can be ignored (redundant):
Taking IC_h with equality,

$$p_h = p_l + 2(t_l - t_h) \tag{5}$$

Take IC_l,

$$\underbrace{2 - t_l - p_l}_{0} \ge 2 - t_h - p_h \tag{6}$$

Substitute p_h from IC_h on the RHS

$$\underbrace{2 - t_l - p_l}_{0} \ge \underbrace{2 - t_h - p_l - 2(t_l - t_h)}_{\underbrace{2 - t_l - p_l}_{0} + \underbrace{t_h - t_l}_{<0}} \tag{7}$$

Hence, IR_l, IC_h and $t_l \ge t_h$ implies IC_l.

6. Using your findings in Q4 and Q5, write down the seller's relaxed problem and find a complete solution to this problem.

Answer:

$$\begin{aligned}
\max_{p_l, p_h, t_l, t_h} \quad & q p_h + (1-q) p_l - \frac{t_h^2}{2} - \frac{t_l^2}{2} & \\
& 2 - t_l - p_l = 0 & (IR_l) \\
& 3 - 2t_h - p_h = 3 - 2t_l - p_l & (IC_h) \\
& p_h \ge p_l & (8) \\
& 0 \le t_h \le t_l \le 1 &
\end{aligned}$$

From IR_l, $p_l = 2 - t_l$. From IC_h, $p_h = p_l + 2(t_l - t_h) = 2 + t_l - 2t_h$. Hence, the program is

$$\begin{aligned} \max_{t_l, t_h} \quad & q(2 + t_l - 2t_h) + (1-q)(2 - t_l) - \frac{t_h^2}{2} - \frac{t_l^2}{2} \\ & p_h \geq p_l \\ & 0 \leq t_l, t_h \leq 1 \end{aligned} \tag{9}$$

And the optimum is $t_h^* = 0$ and $t_l^* = 2q - 1$ if $q > \frac{1}{2}$, $t_l^* = 0$ otherwise.