Mathematics III
Problem Set 4: Dynamic Optimization II

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Deadline is *Tue 19 December at 9:00.* Submission via email: jose.elias.gallegos@iies.su.se or in class. By that same time, I will upload solutions to my webpage, `www.joseeliasgallegos.com`

Exercise 1: Value Function Iterations

Consider the dynamic optimization problem,

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \quad s.t. \quad k_{t+1} + c_t = A k_t^{\alpha},\ c_t > 0,$$

where $A > 0$, α, $\beta \in (0, 1)$ and $k_0 > 0$ is given.

(a) Let V_n be the sequence of value function iterations obtained by starting from $V_0 = 0$. Show that these functions all have the format:

$$V_n(k) = X_n + Y_n \log k \quad \text{for some constants } X_n,\, Y_n \in \mathbb{R}.$$

(b) Determine the $\lim_{n\to\infty} Y_n$ and $\lim_{n\to\infty} X_n$.

Use these results to find the policy functions for k' and c.

(c) Show that the policy function for c satisfies an extended Euler equation.

Is the solution unique?

Exercise 2: Guess and Verify (20 points)

Consider a problem with a binary shock $\theta \in \{\theta_1, \theta_2\} > 0$. The transition from the current shock i to next period's shock j happens with probability π_{ij}. The Bellman equation for the problem is:

$$V(k, i) = \max_{c,k'} \left\{ \log c + \beta \sum_j \pi_{ij} V(k', j) \right\} \quad s.t. \quad c + k' = \theta_i k^{\alpha}.$$

Find the value function using the guess-and-verify method with the guess:

$$V(k,\, i) = \gamma_{1,i} + \gamma_2 \log k + \gamma_3 \log \theta_i.$$

Note: *1)* The solution for $\gamma_{1,i}$ will depend on the transition probabilities π_{ij} and cannot be solved in closed form. *2)* You don't need to verify that the solution to the Bellman equation is unique.

Exercise 3: Contraction Mapping (20 points)

Show that the Bellman equation for *McCall's Job Search Model* covered in lectures defines a contraction using *Blackwell's Sufficient Conditions.* What are the conclusions from proving this?

Exercise 4: Savings Under Uncertainty (30 points)

Suppose a person seeks to maximize,

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\alpha_0 c_t - \frac{\alpha_1}{2} c_t^2 \right) \quad s.t. \quad a_{t+1} = (1+r)(a_t + y_t - c_t), \tag{1}$$

where $y_{t+1} = y + \epsilon_{t+1}$, $\epsilon_{t+1} \sim i.i.d.F\left(0,\, \sigma_\epsilon^2\right)$ and $\alpha_0\,, \alpha_1 > 0$.

(a) Formulate this problem as a dynamic programming problem.

(b) Show that there exits a value function $V(k)$ and that $V(k)$ is continuous and strictly concave.

(c) Assuming that $V(k)$ is differentiable, characterize the extended Euler equation that determines the optimal path of c and a.

(d) Is this Euler equation enough to determine the path of c and a? If not, what other condition do we need to impose? Write down this condition and explain it intuitively.