Mathematics III

Problem Set 2: Differential and Difference Equations

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November 30, 2017

Deadline is *Mon 4 December at 10:00.* Submission via email: jose.elias.gallegos@iies.su.se or in class. By that same time, I will upload solutions to my webpage, www.joseeliasgallegos.com

Exercise 1: Direction Fields

(a) Draw the direction field (using isoclines or otherwise) and the curve of $\dot{y} = 1 + t - y$.
(b) Draw the direction field, phase portrait and stability line of $\dot{y} = y(5 - y)$.

Exercise 2: Difference Equations

A loan of amount $\$L$ is taken out on January 1 of year 0. Instalment payments for the principal and interest are paid annually, commencing on January 1 of year 1. Let the interest rate be $r < 2$, so that the interest amounts to rL for the first payment.
(a) The contract states that the principal share of the repayment will be half the size of the interest share. Show that the debt after January 1 of year n is $\left(1 - \frac{r}{2}\right)^n L$.
(b) Find r when it is known that exactly half the original loan is paid after 10 years.
(c) What will the remaining payments be each year if the contract is not changed?

Exercise 3: Linear First-Order Difference Equations

Consider the first-order difference equation

$$x_{t+1} = a_t x_t + b_t, \qquad t = 0, 1, 2, \dots$$

where $(a_t)_t$ and $(b_t)_t$ are sequences of real numbers. Show (using a proof by induction) that the general solution is given by

$$x_t = \left(\prod_{s=0}^{t-1} a_s\right) x_0 + \sum_{k=0}^{t-1} b_k \left(\prod_{s=k+1}^{t-1} a_s\right), \qquad t = 0, 1, 2, \dots$$

Recall: We define $\prod_{s=k}^{l} a_s = a_k a_{k+1} \cdots a_l$ for $l = k, k+1, k+2, ...$, and $\prod_{s=k}^{l} a_s = 1$ for $l < k$.

Exercise 4: Second-Order Linear Difference Equations

(a) Determine the general solution of the second-order equation

$$x_{t+2} + 4x_t = 0$$

Is the equation globally asymptotically stable?

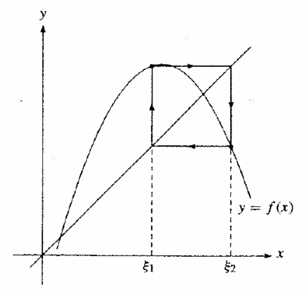
(b) Find one particular solution of the equation

$$x_{t+2} + 4x_t = 8 \cdot 2^t \cdot \cos\left(\frac{t\pi}{2}\right)$$

and construct the general solution of this equation using your result from part (a).

Exercise 5: Cycle of Period 2

The function f in the figure is given by $f(x) = -x^2 + 4x - \frac{4}{5}$.



Find the values of the cycle points ξ_1 and ξ_2, and use

$$|f'(\xi_1)f'(\xi_2)| < 1 \iff 4 < (b-1)^2 - 4ac < 6$$

to determine whether the cycle is stable. It is clear from the figure that the difference equation $x_{t+1} = f(x_t)$ has two equilibrium states. Find these equilibria, show that they are both unstable, and *verify* that if $f : I \to I$ is continuous and the difference equation $x_{t+1} = f(x_t)$ admits a cycle ξ_1, ξ_2 of period 2, it also has at least one equilibrium solution between ξ_1 and ξ_2. (*Hint: Consider the function $f(x) - x$ over the interval with endpoints ξ_1 and ξ_2.*)