Mathematics III
Problem Set 1: Differential Equations
Suggested Solutions

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November 28, 2017

Deadline is *Mon 27 November at 17:00.* Submission via email: jose.elias.gallegos@iies.su.se or in class. By that same time, I will upload solutions to my webpage, `www.joseeliasgallegos.com`

Exercise 1: First-Order Differential Equations

Solve these linear equations in the form $y = y_n + y_p$ with $y_n = y(0)e^{at}$.
(a) $\dot{y} - 4y = -5$
Answer:

(a)

Let me first solve a FDE for the general case, and then we can move to this particular case. Write

$$\dot{y} + ay = b$$

This is the simplest case. Multiply this equation by e^{at} (integrating factor)

$$\dot{y}e^{at} + aye^{at} = be^{at}$$

Notice that the LHS happens to be $\frac{d(ye^{at})}{dt}$. Thus,

$$\frac{d(ye^{at})}{dt} = be^{at}$$

Multiplying by dt,

$$d(ye^{at}) = be^{at}dt$$

Integrating,

$$\begin{aligned}
\int d(ye^{at}) &= \int be^{at}dt \\
ye^{at} + C &= b\int e^{at}dt \\
ye^{at} + C &= \frac{b}{a}e^{at} + D \\
ye^{at} &= \frac{b}{a}e^{at} + E
\end{aligned}$$

where $E = D - C$. Divide by e^{at},

$$y = \frac{b}{a} + Ee^{-at}$$

Okay, so the general case is now done. You can refer to it when you solve the rest of the exercise. But lets go to the particular case

$$\dot{y} - 4y = -5$$

This is the simplest case. Multiply this equation by e^{-4t} (integrating factor)

$$\dot{y}e^{-4t} - 4ye^{-4t} = -5e^{-4t}$$

Notice that the LHS happens to be $\frac{d(ye^{-4t})}{dt}$. Thus,

$$\frac{d(ye^{-4t})}{dt} = -5e^{-4t}$$

Multiplying by dt,

$$d(ye^{-4t}) = -5e^{-4t}dt$$

Integrating,

$$\begin{aligned}
\int d(ye^{-4t}) &= \int -5e^{-4t}dt \\
ye^{-4t} + C &= -5\int e^{-4t}dt \\
ye^{-4t} + C &= \frac{-5}{-4}e^{-4t} + D \\
ye^{-4t} &= \frac{5}{4}e^{-4t} + E
\end{aligned}$$

where $E = D - C$. Divide by e^{-4t},

$$y = \frac{5}{4} + Ee^{4t}$$

Finally, notice that $y \underset{t\to\infty}{\to} \infty$.

(b) $\dot{y} + 4y = 6$

Answer:

(b) This is the simplest case. Multiply this equation by e^{4t} (integrating factor)

$$\dot{y}e^{4t} + 4ye^{4t} = 6e^{4t}$$

Notice that the LHS happens to be $\frac{d(ye^{4t})}{dt}$. Thus,

$$\frac{d(ye^{4t})}{dt} = 6e^{4t}$$

Multiplying by dt,

$$d(ye^{4t}) = 6e^{4t}dt$$

Integrating,

$$\begin{aligned}\int d(ye^{4t}) &= \int 6e^{4t}dt \\ ye^{4t} + C &= 6\int e^{4t}dt \\ ye^{4t} + C &= \frac{6}{4}e^{4t} + D \\ ye^{4t} &= \frac{3}{2}e^{4t} + E\end{aligned}$$

where $E = D - C$. Divide by e^{4t},

$$y = \frac{3}{2} + Ee^{-4t}$$

Finally, notice that $y \underset{t\to\infty}{\to} \frac{3}{2}$.

(c) $\dot{y} - y = 5e^{3t}$, with $y(0) = 2$.

Answer:

(c) This is the simplest case. Multiply this equation by e^{-t} (integrating factor)

$$\dot{y}e^{-t} - ye^{-t} = 5e^{3t}e^{-t}$$

Notice that the LHS happens to be $\frac{d(ye^{-t})}{dt}$. Thus,

$$\frac{d(ye^{-t})}{dt} = 5e^{2t}$$

Multiplying by dt,

$$d(ye^{-t}) = 5e^{2t}dt$$

Integrating,

$$\begin{aligned}\int d(ye^{-t}) &= \int 5e^{2t}dt \\ ye^{-t} + C &= 5\int e^{2t}dt \\ ye^{-t} + C &= \frac{5}{2}e^{2t} + D \\ ye^{-t} &= \frac{5}{2}e^{2t} + E\end{aligned}$$

where $E = D - C$. Divide by e^{-t},

$$y = \frac{5}{2}e^{3t} + Ee^{t}$$

Notice that in this exercise we are given extra information: $y(0) = 2$. Thanks to this we can obtain E,

$$\begin{aligned}y(0) &= 2 \\ \frac{5}{2}e^{0} + Ee^{0} &= 2 \\ \frac{5}{2} + E &= 2 \implies E = -\frac{1}{2}\end{aligned}$$

Hence,

$$y = \frac{5}{2}e^{3t} - \frac{1}{2}e^{t}$$

Finally, notice that $y \underset{t\to\infty}{\to} \infty$.
(d) $\dot{y} + y = 8e^{-2t}$, with $y(0) = 2$.
Answer:

(d) This is the simplest case. Multiply this equation by e^{t} (integrating factor)

$$\dot{y}e^{t} + ye^{t} = 8e^{-2t}e^{t}$$

Notice that the LHS happens to be $\frac{d(ye^{t})}{dt}$. Thus,

$$\frac{d(ye^{t})}{dt} = 8e^{-t}$$

Multiplying by dt,

$$d(ye^{t}) = 8e^{-t}dt$$

Integrating,

$$\int d(ye^t) = \int 8e^{-t}dt$$
$$ye^t + C = 8\int e^{-t}dt$$
$$ye^t + C = \frac{8}{-1}e^{-t} + D$$
$$ye^t = -8e^{-t} + E$$

where $E = D - C$. Divide by e^t,

$$y = -8e^{-2t} + Ee^{-t}$$

Notice that in this exercise we are given extra information: $y(0) = 2$. Thanks to this we can obtain E,

$$y(0) = 2$$
$$-8e^0 + Ee^0 = 2$$
$$-8 + E = 2 \implies E = 10$$

[Solution manual says $E = 2$. Shout out if someone spots a typo]
Hence,

$$y = -8e^{-2t} + 6e^{-t}$$

Finally, notice that $y \underset{t\to\infty}{\to} 0$.

Exercise 2: Separable Differential Equations

Solve the following differential equations:
(a) $\dot{K} = An_0^{\alpha}a^bK^{b-c}e^{(\alpha\nu+\varepsilon)t}$, with $b - c \neq 1$, $\alpha\nu + \varepsilon \neq 0$.
Answer:

(a) Lets rewrite $An_0^{\alpha}a^b \equiv P$ and $\alpha\nu + \varepsilon \equiv Q$. The differential equation now looks

$$\dot{K}K^{c-b} = Pe^{Qt}$$

Proceeding,

$$\begin{aligned}
\frac{dK}{dt}K^{c-b} &= Pe^{Qt} \\
dK\ K^{c-b} &= Pe^{Qt}dt \\
\int K^{c-b}dK &= \int Pe^{Qt}dt \\
\frac{1}{c-b+1}K^{c-b+1} + C &= P\int e^{Qt}dt \\
\frac{1}{c-b+1}K^{c-b+1} + C &= \frac{P}{Q}e^{Qt} + D \\
\frac{1}{c-b+1}K^{c-b+1} &= \frac{P}{Q}e^{Qt} + E \implies K = \left[\left(\frac{P}{Q}e^{Qt} + E\right)(c-b+1)\right]^{\frac{1}{c-b+1}}
\end{aligned}$$

(b) $\dot{x} = \frac{(\beta-\alpha x)(x-a)}{x}$ with $\alpha, \beta, a > 0$, $\alpha a \neq \beta$. *Hint:* $\frac{x}{(\beta-\alpha x)(x-a)} = \frac{1}{\beta-\alpha a}\left(\frac{\beta}{\beta-\alpha x} + \frac{a}{x-a}\right)$.

Answer:

(b) We have

$$\frac{dx}{dt} = \frac{(\beta-\alpha x)(x-a)}{x}$$

Separate,

$$\begin{aligned}
\frac{x}{(\beta-\alpha x)(x-a)}dx &= dt \\
\frac{1}{\beta-\alpha a}\left(\frac{\beta}{\beta-\alpha x} + \frac{a}{x-a}\right)dx &= dt \\
\left(\frac{\beta}{\beta-\alpha x} + \frac{a}{x-a}\right)dx &= (\beta-\alpha a)dt \\
\int\left(\frac{\beta}{\beta-\alpha x} + \frac{a}{x-a}\right)dx &= \int(\beta-\alpha a)dt \\
\beta\int\frac{1}{\beta-\alpha x}dx + a\int\frac{1}{x-a}dx &= (\beta-\alpha a)\int dt \\
\beta\left(-\frac{\ln|\beta-\alpha x|}{\alpha}\right) + C + a\ln|x-a| + D &= (\beta-\alpha a)(t+E) \\
\frac{\beta}{\alpha}\ln|\beta-\alpha x| - a\ln|x-a| &= -(\beta-\alpha a)t + F \\
\ln|\beta-\alpha x|^{\frac{\beta}{\alpha}} + \ln|x-a|^{-a} &= -(\beta-\alpha a)t + F \\
\ln\left(|\beta-\alpha x|^{\frac{\beta}{\alpha}}|x-a|^{-a}\right) &= -(\beta-\alpha a)t + F \\
|\beta-\alpha x|^{\frac{\beta}{\alpha}}|x-a|^{-a} &= e^{-(\beta-\alpha a)t+F} \\
|\beta-\alpha x|^{\frac{\beta}{\alpha}}|x-a|^{-a} &= e^{-(\beta-\alpha a)t}e^{F} \\
|\beta-\alpha x|^{\frac{\beta}{\alpha}}|x-a|^{-a} &= Ge^{(\alpha a-\beta)t}
\end{aligned}$$

where $F = -(\beta - \alpha a)E + C + D$ and $G = e^F$.

Exercise 3: The General Case

Given $x(T) = x_T$, show that the solution of $\dot{x} + a(t)x = b(t)$ is

$$x(t) = x_T e^{\int_t^T a(\xi)d\xi} - \int_t^T b(s)e^{\int_t^s a(\xi)d\xi}ds$$

Answer:

Multiply the differential equation by $e^{A(t)}$,

$$\dot{x}e^{A(t)} + a(t)xe^{A(t)} = b(t)e^{A(t)}$$

where, in the LHS, $A(t)$ is such that LHS= $\frac{dxe^{A(t)}}{dt}$,

$$\frac{dxe^{A(t)}}{dt} = \dot{x}e^{A(t)} + \dot{A}xe^{A(t)}$$

Therefore,

$$\dot{A} = a(t) \implies \frac{dA}{dt} = a(t) \implies dA = a(t)dt$$

Integrating,

$$\int dA = \int a(t)dt \implies A = \int a(t)dt$$

Hence, we can rewrite it as

$$\begin{aligned}
\frac{dxe^{A(t)}}{dt} &= b(t)e^{A(t)} \\
dxe^{A(t)} &= b(t)e^{A(t)}dt \\
\int dxe^{A(t)} &= \int b(t)e^{A(t)}dt \\
xe^{A(t)} + C &= \int b(t)e^{A(t)}dt \implies xe^{A(t)} = \int b(t)e^{A(t)}dt - C \\
x &= e^{-A(t)}\int b(t)e^{A(t)}dt - Ce^{-A(t)}
\end{aligned}$$

Let's now find C. Define $F(t) = \int b(t)e^{A(t)}dt$. We can write

$$A(t) - A(s) = \int_s^t a(\xi)d\xi$$

Rewritting x,

$$x = e^{-A(t)}F(t) - Ce^{-A(t)}$$

Now let $t = T$ and solve for C

$$Ce^{-A(T)} = e^{-A(T)}F(T) - x(T) \implies C = F(T) - e^{A(T)}x(T)$$

Introducing C into x

$$\begin{aligned} x(t) &= -\left[F(T) - e^{A(T)}x(T)\right] e^{-A(t)} + e^{-A(t)}F(t) = \\ &= x(T)e^{A(T)-A(t)} - [F(T) - F(t)]e^{-A(t)} \end{aligned}$$

By definition of $F(t)$ we have

$$F(T) - F(t) = \int_t^T b(s)e^{A(s)}ds$$

Hence,

$$\begin{aligned} x(t) &= x(T)e^{A(T)-A(t)} - \left[\int_t^T b(s)e^{A(s)}ds\right] e^{-A(t)} = \\ &= x(T)e^{A(T)-A(t)} - \int_t^T b(s)e^{A(s)-A(t)}ds \end{aligned}$$

Finally,

$$x(t) = x_T e^{\int_t^T a(\xi)d\xi} - \int_t^T b(s)e^{\int_t^s a(\xi)d\xi}ds$$

Exercise 4: Second-Order Linear Equations

Find a real solution to the following second-order equation

$$\ddot{y} + 2\dot{y} + 10y = 0$$

with $y(0) = 2$ and $\dot{y}(0) = 1$.

Answer:

Notice that $a = 2$ and $b = 10$. Hence,

$$\frac{a^2}{4} - b = -9 < 0$$

then, we know that

$$y = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)]$$

where $\alpha = -\frac{a}{2} = -1$ and $\beta = \sqrt{b - \frac{a^2}{4}} = 3$. We still need to find A and B,

$$\dot{y} = \alpha e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)] + e^{\alpha t}\left\{A[-\sin(\beta t)]\beta + B\cos(\beta t)\beta\right\} =$$
$$= e^{\alpha t}\left[(A\alpha + B\beta)\cos(\beta t) + (\alpha B - \beta A)\sin(\beta t)\right]$$

We know that $y(0) = 2$. Hence,

$$y(0) = 2$$
$$e^0(A \cdot 1 + B \cdot 0) = 2 \implies A = 2$$

And, thanks to $\dot{y}(0) = 1$,

$$\dot{y}(0) = 1$$
$$e^0[(A\alpha + B\beta) \cdot 1 + (\alpha B - \beta A) \cdot 0] = 1 \implies B = \frac{1 - A\alpha}{\beta} = \frac{1 - 2\alpha}{\beta}$$

Exercise 5: Sinusoidal functions

The differential equation $\dot{x} + 2x = 2\cos(2t)$ has complete solution

$$x = Ce^{-2t} + \frac{\cos(2t) + \sin(2t)}{2}$$

(a) Plot this function for different C's. How is the behavior of the function affected by changes in C?

Answer:

(b) Show analytically that the complete solution is equivalent to

$$x = Ce^{-2t} + \frac{1}{\sqrt{2}}\cos\left(2t - \frac{\pi}{4}\right)$$

Answer:

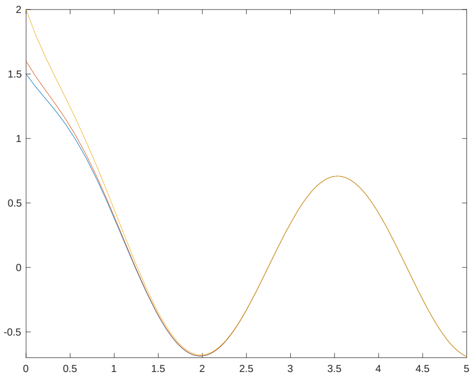
Using the sinusoidal relation

$$A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \phi)$$

where $\phi = \arctan\frac{B}{A}$ and $C = \sqrt{A^2 + B^2}$,

$$\frac{\cos(2t) + \sin(2t)}{2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\cos\left(2t - \arctan\frac{1/2}{1/2}\right) \equiv \frac{1}{\sqrt{2}}\cos\left(2t - \frac{\pi}{4}\right)$$

(notice that 1 is the 45% degree line, which in radians is $\frac{\pi}{4}$).



(c) Confirm this graphically by plotting both versions of the complete solution.

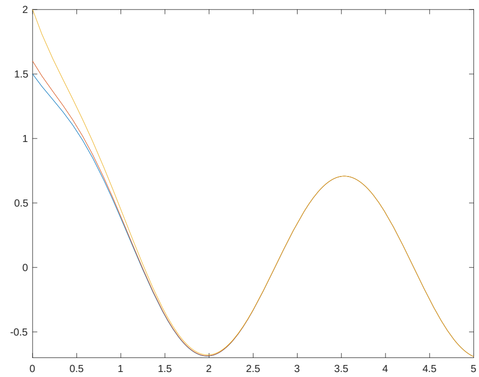
Answer:

```
clc
clear all


C1=1;
C2=1.1;
C3=1.5;

x1 = @(t) C1*exp(-2*t)+(cos(2*t)+sin(2*t))/2;
x2 = @(t) C2*exp(-2*t)+(cos(2*t)+sin(2*t))/2;
x3 = @(t) C3*exp(-2*t)+(cos(2*t)+sin(2*t))/2;

figure(1)
fplot(x1,[0,5])
hold on
fplot(x2,[0,5])
hold on
fplot(x3,[0,5])
```




```
hold off
print(1,'ex5a','-dpdf')

x1 = @(t) C1*exp(-2*t)+1/sqrt(2)*cos(2*t-pi/4);
x2 = @(t) C2*exp(-2*t)+1/sqrt(2)*cos(2*t-pi/4);
x3 = @(t) C3*exp(-2*t)+1/sqrt(2)*cos(2*t-pi/4);

figure(2)
fplot(x1,[0,5])
hold on
fplot(x2,[0,5])
hold on
fplot(x3,[0,5])
hold off
print(2,'ex5c','-dpdf')
```