Chapter 9
Overlapping Generations Models

This chapter describes the pure exchange overlapping generations model of Paul Samuelson (1958). We begin with an abstract presentation that treats the overlapping generations model as a special case of the chapter 8 general equilibrium model with complete markets and all trades occurring at time 0. A peculiar type of heterogeneity across agents distinguishes the model. Each individual cares about consumption only at two adjacent dates, and the set of individuals who care about consumption at a particular date includes some who care about consumption one period earlier and others who care about consumption one period later. We shall study how this special preference and demographic pattern affects some of the outcomes of the chapter 8 model.

While it helps to reveal the fundamental structure, allowing complete markets with time 0 trading in an overlapping generations model strains credulity. The formalism envisions that equilibrium price and quantity sequences are set at time 0, before the participants who are to execute the trades have been born. For that reason, most applied work with the overlapping generations model adopts a sequential-trading arrangement, like the sequential trade in Arrow securities described in chapter 8. The sequential-trading arrangement has all trades executed by agents living in the here and now. Nevertheless, equilibrium quantities and intertemporal prices are equivalent between these two trading arrangements. Therefore, analytical results found in one setting transfer to the other.

Later in the chapter, we use versions of the model with sequential trading to tell how the overlapping generations model provides a framework for thinking about equilibria with government debt and/or valued fiat currency, intergenerational transfers, and fiscal policy.

9.1. Endowments and preferences

Time is discrete, starts at $t = 1$, and lasts forever, so $t = 1, 2, \ldots$. There is an infinity of agents named $i = 0, 1, \ldots$. We can also regard i as agent i's period of birth. There is a single good at each date. The good is not storable. There is no uncertainty. Each agent has a strictly concave, twice continuously differentiable, one-period utility function $u(c)$, which is strictly increasing in consumption c of the one good. Agent i consumes a vector $c^i = \{c_t^i\}_{t=1}^{\infty}$ and has the special utility function

$$U^i(c^i) = u(c_i^i) + u(c_{i+1}^i), \quad i \geq 1, \tag{9.1.1a}$$

$$U^0(c^0) = u(c_1^0). \tag{9.1.1b}$$

Notice that agent i only wants goods dated i and $i+1$. The interpretation of equations (9.1.1) is that agent i lives during periods i and $i+1$ and wants to consume only when he is alive.

Each household has an endowment sequence y^i satisfying $y_i^i \geq 0, y_{i+1}^i \geq 0, y_t^i = 0 \; \forall t \neq i$ or $i+1$. Thus, households are endowed with goods only when they are alive.

9.2. Time 0 trading

We use the definition of competitive equilibrium from chapter 8. Thus, we temporarily suspend disbelief and proceed in the style of Debreu (1959) with time 0 trading. Specifically, we imagine that there is a "clearinghouse" at time 0 that posts prices and, at those prices, aggregates demands and supplies for goods in different periods. An equilibrium price vector makes markets for all periods $t \geq 2$ clear, but there may be excess supply in period 1; that is, the clearinghouse might end up with goods left over in period 1. Any such excess supply of goods in period 1 can be given to the initial old generation without any effects on the equilibrium price vector, since those old agents optimally consume all their wealth in period 1 and do not want to buy goods in future periods. The reason for our special treatment of period 1 will become clear as we proceed.

Thus, at date 0, there are complete markets in time t consumption goods with date 0 price q_t^0. A household's budget constraint is

$$\sum_{t=1}^{\infty} q_t^0 c_t^i \leq \sum_{t=1}^{\infty} q_t^0 y_t^i. \tag{9.2.1}$$

Letting μ^i be a Lagrange multiplier attached to consumer i's budget constraint, the consumer's first-order conditions are

$$\mu^i q_i^0 = u'(c_i^i), \tag{9.2.2a}$$

$$\mu^i q_{i+1}^0 = u'(c_{i+1}^i), \tag{9.2.2b}$$

$$c_t^i = 0 \text{ if } t \notin \{i, i+1\}. \tag{9.2.2c}$$

Evidently an allocation is feasible if for all $t \geq 1$,

$$c_t^t + c_t^{t-1} \leq y_t^t + y_t^{t-1}. \tag{9.2.3}$$

DEFINITION: An allocation is *stationary* if $c_{i+1}^i = c_o, c_i^i = c_y \ \forall i \geq 1$.

Here the subscript o denotes old and y denotes young. Note that we do not require that $c_1^0 = c_o$. We call an equilibrium with a stationary allocation a *stationary equilibrium.*

9.2.1. Example equilibria

Let $\epsilon \in (0, .5)$. The endowments are

$$\begin{aligned} y_i^i &= 1 - \epsilon, \ \forall i \geq 1, \\ y_{i+1}^i &= \epsilon, \ \forall i \geq 0, \\ y_t^i &= 0 \text{ otherwise.} \end{aligned} \tag{9.2.4}$$

This economy has many equilibria. We describe two stationary equilibria now, and later we shall describe some nonstationary equilibria. We can use a guess-and-verify method to confirm the following two equilibria.

1. Equilibrium H: a high-interest-rate equilibrium. Set $q_t^0 = 1 \ \forall t \geq 1$ and $c_i^i = c_{i+1}^i = .5$ for all $i \geq 1$ and $c_1^0 = \epsilon$. To verify that this is an equilibrium,

notice that each household's first-order conditions are satisfied and that the allocation is feasible. Extensive intergenerational trade occurs at time 0 at the equilibrium price vector q_t^0. Constraint $(9.2.3)$ holds with equality for all $t \geq 2$ but with strict inequality for $t = 1$. Some of the $t = 1$ consumption good is left unconsumed.

2. Equilibrium L: a low-interest-rate equilibrium. Set $q_1^0 = 1$, $\frac{q_{t+1}^0}{q_t^0} = \frac{u'(\epsilon)}{u'(1-\epsilon)} = \alpha > 1$. Set $c_t^i = y_t^i$ for all i, t. This equilibrium is autarkic, with prices being set to eradicate all trade.

9.2.2. Relation to welfare theorems

As we shall explain in more detail later, equilibrium H Pareto dominates equilibrium L. In equilibrium H every generation after the initial old one is better off and no generation is worse off than in equilibrium L. The equilibrium H allocation is strange because some of the time 1 good is not consumed, leaving room to set up a giveaway program to the initial old that makes them better off and costs subsequent generations nothing. We shall see how the institution of either perpetual government debt or of fiat money can accomplish this purpose.[1]

Equilibrium L is a competitive equilibrium that evidently fails to satisfy one of the assumptions needed to deliver the first fundamental theorem of welfare economics, which identifies conditions under which a competitive equilibrium allocation is Pareto optimal.[2] The condition of the theorem that is violated by equilibrium L is the assumption that the value of the aggregate endowment at the equilibrium prices is finite.[3]

[1] See Karl Shell (1971) for an investigation that characterizes why some competitive equilibria in overlapping generations models fail to be Pareto optimal. Shell cites earlier studies that had sought reasons why the welfare theorems seem to fail in the overlapping generations structure.

[2] See Mas-Colell, Whinston, and Green (1995) and Debreu (1954).

[3] Note that if the horizon of the economy were finite, then the counterpart of equilibrium H would not exist and the allocation of the counterpart of equilibrium L would be Pareto optimal.

9.2.3. Nonstationary equilibria

Our example economy has more equilibria. To construct more equilibria, we summarize preferences and consumption decisions in terms of an offer curve. We describe a graphical apparatus proposed by David Gale (1973) and used to good advantage by William Brock (1990).

DEFINITION: The household's *offer curve* is the locus of (c_i^i, c_{i+1}^i) that solves

$$\max_{\{c_i^i, c_{i+1}^i\}} U(c^i)$$

subject to

$$c_i^i + \alpha_i c_{i+1}^i \leq y_i^i + \alpha_i y_{i+1}^i.$$

Here $\alpha_i \equiv \frac{q_{i+1}^0}{q_i^0}$, the reciprocal of the one-period gross rate of return from period i to $i+1$, is treated as a parameter.

Evidently, the offer curve solves the following pair of equations:

$$c_i^i + \alpha_i c_{i+1}^i = y_i^i + \alpha_i y_{i+1}^i \tag{9.2.5a}$$

$$\frac{u'(c_{i+1}^i)}{u'(c_i^i)} = \alpha_i \tag{9.2.5b}$$

for $\alpha_i > 0$. We denote the offer curve by

$$\psi(c_i^i, c_{i+1}^i) = 0.$$

The graphical construction of the offer curve is illustrated in Figure 9.2.1. We trace it out by varying α_i in the household's problem and reading tangency points between the household's indifference curve and the budget line. The resulting locus depends on the endowment vector and lies above the indifference curve through the endowment vector. By construction, the following property is also true: at the intersection between the offer curve and a straight line through the endowment point, the straight line is tangent to an indifference curve.[4]

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[4] Given our assumptions on preferences and endowments, the conscientious reader will note that Figure 9.2.1 appears distorted because the offer curve really ought to intersect the feasibility line along the 45 degree line with $c_t^t = c_{t+1}^t$, i.e., at the allocation affiliated with equilibrium H above.

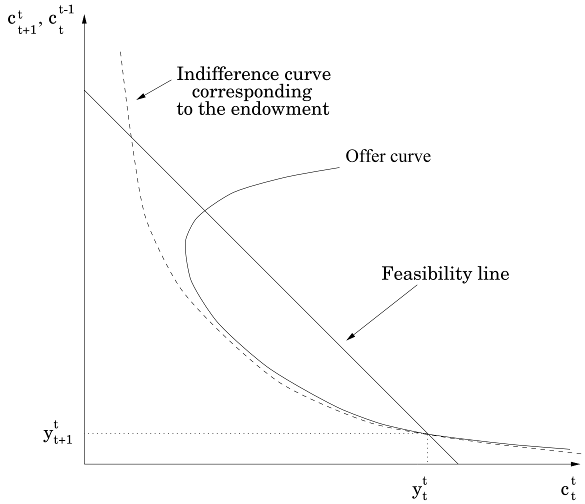


Figure 9.2.1: The offer curve and feasibility line.

Following Gale (1973), we can use the offer curve and a straight line depicting feasibility in the (c_i^i, c_i^{i-1}) plane to construct a machine for computing equilibrium allocations and prices. In particular, we can use the following pair of difference equations to solve for an equilibrium allocation. For $i \geq 1$, the equations are[5]

$$\psi(c_i^i, c_{i+1}^i) = 0, \tag{9.2.6a}$$

$$c_i^i + c_i^{i-1} = y_i^i + y_i^{i-1}. \tag{9.2.6b}$$

We take c_1^1 as an initial condition. After the allocation has been computed, the equilibrium price system can be computed from

$$q_i^0 = u'(c_i^i)$$

for all $i \geq 1$.

[5] By imposing equation (9.2.6*b*) with equality, we are implicitly possibly including a give-away program to the initial old.

9.2.4. Computing equilibria

Example 1: Gale's equilibrium computation machine: A procedure for constructing an equilibrium is illustrated in Figure 9.2.2, which reproduces a version of a graph of David Gale (1973). Start with a proposed c_1^1, a time 1 allocation to the initial young. Then use the feasibility line to find the *maximal* feasible value for c_0^1, the time 1 allocation to the initial old. In the Arrow-Debreu equilibrium, the allocation to the initial old will be less than this maximal value, so that some of the time 1 good is thrown away. The reason for this is that the budget constraint of the initial old, $q_1^0(c_1^0 - y_1^0) \leq 0$, implies that $c_1^0 = y_1^0$.[6] The candidate time 1 allocation is thus feasible, but the time 1 young will choose c_1^1 only if the price α_1 is such that (c_2^1, c_1^1) lies on the offer curve. Therefore, we choose c_2^1 from the point on the offer curve that cuts a vertical line through c_1^1. Then we proceed to find c_2^2 from the intersection of a horizontal line through c_2^1 and the feasibility line. We continue recursively in this way, choosing c_i^i as the intersection of the feasibility line with a horizontal line through c_i^{i-1}, then choosing c_{i+1}^i as the intersection of a vertical line through c_i^i and the offer curve. We can construct a sequence of α_i's from the slope of a straight line through the endowment point and the sequence of (c_i^i, c_{i+1}^i) pairs that lie on the offer curve.

If the offer curve has the shape drawn in Figure 9.2.2, any c_1^1 between the upper and lower intersections of the offer curve and the feasibility line is an equilibrium setting of c_1^1. Each such c_1^1 is associated with a distinct allocation and α_i sequence, all but one of them converging to the *low*-interest-rate stationary equilibrium allocation and interest rate.

Example 2: Endowment at $+\infty$: Take the preference and endowment structure of the previous example and modify only one feature. Change the endowment of the initial old to be $y_1^0 = \epsilon > 0$ *and* "$\delta = 1 - \epsilon > 0$ units of consumption at $t = +\infty$," by which we mean that we take

$$\sum_t q_t^0 y_t^0 = q_1^0 \epsilon + \delta \lim_{t \to \infty} q_t^0 .$$

It is easy to verify that the only competitive equilibrium of the economy with this specification of endowments has $q_t^0 = 1 \ \forall t \geq 1$, and thus $\alpha_t = 1 \ \forall t \geq 1$. The

[6] Soon we shall discuss another market structure that avoids throwing away any of the initial endowment by augmenting the endowment of the initial old with a particular zero-dividend infinitely durable asset.

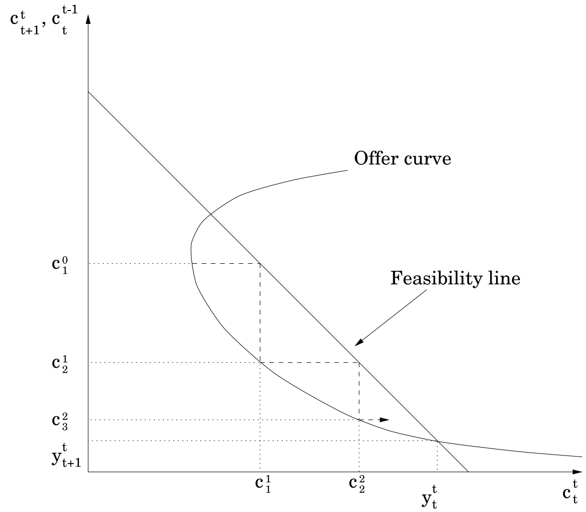


Figure 9.2.2: A nonstationary equilibrium allocation.

reason is that all the "low-interest-rate" equilibria that we computed in example 1 would assign an infinite value to the endowment of the initial old. Confronted with such prices, the initial old would demand unbounded consumption. That is not feasible. Therefore, such a price system cannot be an equilibrium.

Example 3: A Lucas tree: Take the preference and endowment structure to be the same as example 1 and modify only one feature. Endow the initial old with a "Lucas tree," namely, a claim to a constant stream of $d > 0$ units of consumption for each $t \geq 1$.[7] Thus, the budget constraint of the initial old person now becomes

$$q^0_1 c^0_1 = d \sum_{t=1}^{\infty} q^0_t + q^0_1 y^0_1.$$

The offer curve of each young agent remains as before, but now the feasibility line is

$$c^i_i + c^{i-1}_i = y^i_i + y^{i-1}_i + d$$

[7] This is a version of an example of Brock (1990). The 'Lucas tree' refers to a colorful interpretation of a dividend stream as 'fruit' falling from a 'tree' in a pure exchange economy studied by Lucas (1978). See chapter 13.

for all $i \geq 1$. Note that young agents are endowed below the feasibility line. From Figure 9.2.3, it seems that there are two candidates for stationary equilibria, one with constant $\alpha < 1$, another with constant $\alpha > 1$. The one with $\alpha < 1$ is associated with the steeper budget line in Figure 9.2.3. However, the candidate stationary equilibrium with $\alpha > 1$ cannot be an equilibrium for a reason similar to that encountered in example 2. At the price system associated with an $\alpha > 1$, the wealth of the initial old would be unbounded, which would prompt them to consume an unbounded amount, which is not feasible. This argument rules out not only the stationary $\alpha > 1$ equilibrium but also all nonstationary candidate equilibria that converge to that constant α. Therefore, there is a unique equilibrium; it is stationary and has $\alpha < 1$.

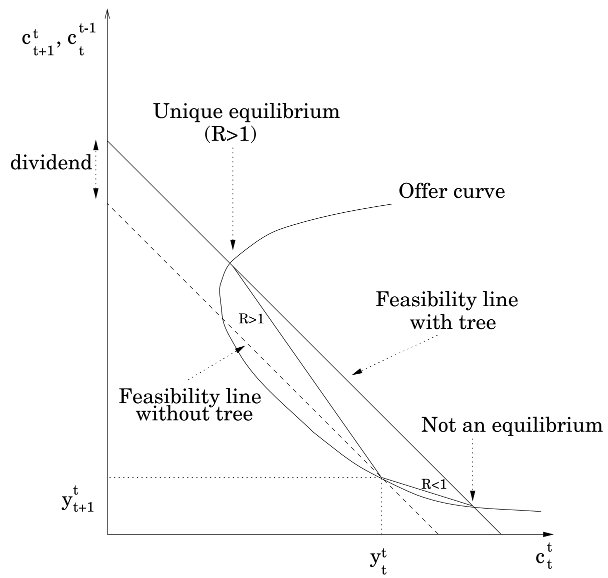


Figure 9.2.3: Unique equilibrium with a fixed-dividend asset.

If we interpret the gross rate of return on the tree as $\alpha^{-1} = \frac{p+d}{p}$, where $p = \sum_{t=1}^{\infty} q^0_t d$, we can compute that $p = \frac{d}{R-1}$ where $R = \alpha^{-1}$. Here p is the price of the Lucas tree.

In terms of the logarithmic preference example 5 below, the difference equation (9.2.9) becomes modified to

$$\alpha_i = \frac{1+2d}{\epsilon} - \frac{\epsilon^{-1}-1}{\alpha_{i-1}}. \tag{9.2.7}$$

Example 4: Government expenditures: Take the preferences and endowments to be as in example 1 again, but now alter the feasibility condition to be

$$c_i^i + c_i^{i-1} + g = y_i^i + y_i^{i-1}$$

for all $i \geq 1$ where $g > 0$ is a positive level of government purchases. The "clearinghouse" is now looking for an equilibrium price vector such that this feasibility constraint is satisfied. We assume that government purchases do not give utility. The offer curve and the feasibility line look as in Figure 9.2.4. Notice that the endowment point (y_i^i, y_{i+1}^i) lies *outside* the relevant feasibility line. Formally, this graph looks like example 3, but with a "negative dividend d." Now there are two stationary equilibria with $\alpha > 1$, and a continuum of equilibria converging to the higher α equilibrium (the one with the lower slope α^{-1} of the associated budget line). Equilibria with $\alpha > 1$ cannot be ruled out by the argument in example 3 because no one's endowment sequence receives infinite value when $\alpha > 1$.

Later, we shall interpret this example as one in which a government finances a constant deficit either by money creation or by borrowing at a negative real net interest rate. We shall discuss this and other examples in a setting with sequential trading.

Example 5: Log utility: Suppose that $u(c) = \ln c$ and that the endowment is described by equations (9.2.4). Then the offer curve is given by the recursive formulas $c_i^i = .5(1-\epsilon+\alpha_i\epsilon), c_{i+1}^i = \alpha_i^{-1} c_i^i$. Let α_i be the gross rate of return facing the young at i. Feasibility at i and the offer curves then imply

$$\frac{1}{2\alpha_{i-1}}(1-\epsilon+\alpha_{i-1}\epsilon) + .5(1-\epsilon+\alpha_i\epsilon) = 1. \tag{9.2.8}$$

This implies the difference equation

$$\alpha_i = \epsilon^{-1} - \frac{\epsilon^{-1}-1}{\alpha_{i-1}}. \tag{9.2.9}$$

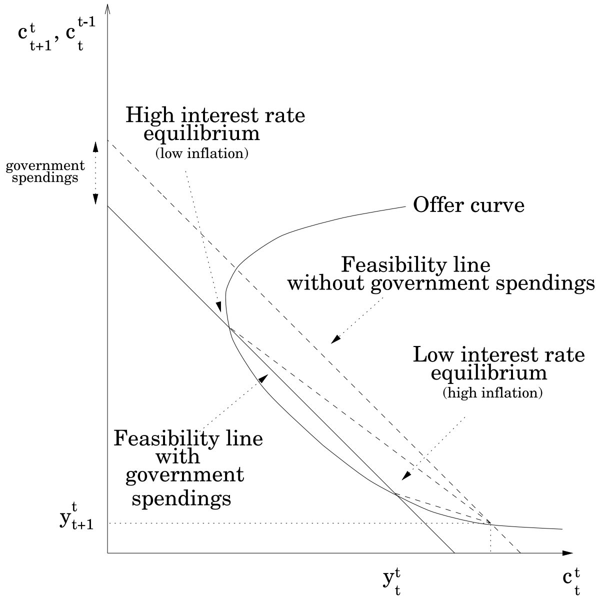


Figure 9.2.4: Equilibria with debt- or money-financed government deficit finance.

See Figure 9.2.2. An equilibrium α_i sequence must satisfy equation (9.2.8) and have $\alpha_i > 0$ for all i. Evidently, $\alpha_i = 1$ for all $i \geq 1$ is an equilibrium α sequence. So is any α_i sequence satisfying equation (9.2.8) and $\alpha_1 \geq 1$; $\alpha_1 < 1$ will not work because equation (9.2.8) implies that the tail of $\{\alpha_i\}$ is an unbounded negative sequence. The limiting value of α_i for any $\alpha_1 > 1$ is $\frac{1-\epsilon}{\epsilon} = u'(\epsilon)/u'(1-\epsilon)$, which is the interest factor associated with the stationary autarkic equilibrium. Notice that Figure 9.2.2 suggests that the stationary $\alpha_i = 1$ equilibrium is not stable, while the autarkic equilibrium is.

9.3. Sequential trading

We now alter the trading arrangement to bring them into line with standard presentations of the overlapping generations model. We abandon the time 0, complete markets trading arrangement and replace it with sequential trading in which a durable asset, either government debt or unbacked fiat money or claims on a Lucas tree, is passed from old to young. Some cross-generation transfers occur with voluntary exchanges, while others are engineered by government tax and transfer programs.

9.4. Money

In Samuelson's (1958) version of the model, trading occurs sequentially through a medium of exchange, an inconvertible (or "fiat") currency. In Samuelson's model, preferences and endowments are as described above, with one important additional component of the endowment. At date $t = 1$, old agents are endowed in the aggregate with $M > 0$ units of intrinsically worthless currency. No one has promised to redeem the currency for goods. The currency is not "backed" by any government promise to redeem it for goods. But as Samuelson showed, there exists a system of expectations that makes unbacked currency be valued. Currency will be valued today if people expect it to be valued tomorrow. Samuelson thus envisioned a situation in which currency is backed by expectations without promises.

For each date $t \geq 1$, young agents purchase m_t^i units of currency at a price of $1/p_t$ units of the time t consumption good. Here $p_t \geq 0$ is the time t price level. At each $t \geq 1$, each old agent exchanges his holdings of currency for the time t consumption good. The budget constraints of a young agent born in period $i \geq 1$ are

$$c_i^i + \frac{m_i^i}{p_i} \leq y_i^i, \tag{9.4.1}$$

$$c_{i+1}^i \leq \frac{m_i^i}{p_{i+1}} + y_{i+1}^i, \tag{9.4.2}$$

$$m_i^i \geq 0. \tag{9.4.3}$$

If $m_i^i \geq 0$, inequalities (9.4.1) and (9.4.2) imply

$$c_i^i + c_{i+1}^i \left(\frac{p_{i+1}}{p_i}\right) \leq y_i^i + y_{i+1}^i \left(\frac{p_{i+1}}{p_i}\right). \tag{9.4.4}$$

Provided that we set

$$\frac{p_{i+1}}{p_i} = \alpha_i = \frac{q_{i+1}^0}{q_i^0},$$

this budget set is identical with equation (9.2.1).

We use the following definitions:

DEFINITION: A nominal price sequence is a positive sequence $\{p_i\}_{i\geq 1}$.

DEFINITION: An equilibrium with valued fiat money is a feasible allocation and a nominal price sequence with $p_t < +\infty$ for all t such that given the price sequence, the allocation solves the household's problem for each $i \geq 1$.

The qualification that $p_t < +\infty$ for all t means that fiat money is valued. Sometimes we call an equilibrium with valued fiat money a 'monetary equilibrium'. If $\frac{1}{p_t} = +\infty$, we sometimes call it a 'nonmonetary equilibrium'.

9.4.1. Computing more equilibria with valued fiat currency

Summarize the household's optimal decisions with a saving function

$$y_i^i - c_i^i = s(\alpha_i; y_i^i, y_{i+1}^i). \tag{9.4.5}$$

Then the equilibrium conditions for the model are

$$\frac{M}{p_i} = s(\alpha_i; y_i^i, y_{i+1}^i) \tag{9.4.6a}$$

$$\alpha_i = \frac{p_{i+1}}{p_i}, \tag{9.4.6b}$$

where it is understood that $c_{i+1}^i = y_{i+1}^i + \frac{M}{p_{i+1}}$. Equation (9.4.6$a$) states that at time i the net of saving of generation i (the expression on the right side) equals the net dissaving of generation $i-1$ (the expression on the left side). To compute an equilibrium, we solve the difference equations (9.4.6) for $\{p_i\}_{i=1}^{\infty}$, then get the allocation from the household's budget constraints evaluated at equality at the equilibrium level of real balances. As an example, suppose that

$u(c) = \ln(c)$, and that $(y_i^i, y_{i+1}^i) = (w_1, w_2)$ with $w_1 > w_2$. The saving function is $s(\alpha_i) = .5(w_1 - \alpha_i w_2)$. Then equation $(9.4.6a)$ becomes

$$.5(w_1 - w_2 \frac{p_{t+1}}{p_t}) = \frac{M}{p_t}$$

or

$$p_t = 2M/w_1 + \left(\frac{w_2}{w_1}\right) p_{t+1}. \tag{9.4.7}$$

This is a difference equation whose solutions with a positive price level are

$$p_t = \frac{2M}{w_1(1 - \frac{w_2}{w_1})} + c \left(\frac{w_1}{w_2}\right)^t, \tag{9.4.8}$$

for any scalar $c > 0$.[8] The solution for $c = 0$ is the unique stationary solution. The solutions with $c > 0$ have uniformly higher price levels than the $c = 0$ solution, and have the value of currency approaching zero in the limit as $t \to +\infty$.

9.4.2. Equivalence of equilibria

We briefly look back at the equilibria with time 0 trading and note that the equilibrium allocations are the same under time 0 and sequential trading. Thus, the following proposition asserts that with an adjustment to the endowment and the consumption allocated to the initial old, a competitive equilibrium allocation with time 0 trading is an equilibrium allocation in the fiat money economy (with sequential trading).

PROPOSITION: Let $\overline{c}^i$ denote a competitive equilibrium allocation (with time 0 trading) and suppose that it satisfies $\overline{c}_1^1 < y_1^1$. Then there exists an equilibrium (with sequential trading) of the monetary economy with allocation that satisfies $c_i^i = \overline{c}_i^i, c_{i+1}^i = \overline{c}_{i+1}^i$ for $i \geq 1$.

PROOF: Take the competitive equilibrium allocation and price system and form $\alpha_i = q_{i+1}^0 / q_i^0$. Set $m_i^i / p_i = y_i^i - \overline{c}_i^i$. Set $m_i^i = M$ for all $i \geq 1$, and determine p_1 from $\frac{M}{p_1} = y_1^1 - \overline{c}_1^1$. This last equation determines a positive initial price level p_1 provided that $y_1^1 - \overline{c}_1^1 > 0$. Determine subsequent price levels from $p_{i+1} = \alpha_i p_i$.

[8] See the appendix to chapter 2.

Determine the allocation to the initial old from $c_1^0 = y_1^0 + \frac{M}{p_1} = y_1^0 + (y_1^1 - \overline{c}_1^1)$.
$\blacksquare$

In the monetary equilibrium, time t real balances equal the per capita *saving* of the young and the per capita *dissaving* of the old. To be a monetary equilibrium, both quantities must be positive for all $t \geq 1$.

A converse of the proposition is true.

PROPOSITION: Let $\overline{c}^i$ be an equilibrium allocation for the fiat money economy. Then there is a competitive equilibrium with time 0 trading with the same allocation, provided that the endowment of the initial old is augmented with an appropriate transfer from the clearinghouse.

To verify this proposition, we have to construct the required transfer from the clearinghouse to the initial old. Evidently, it is $y_1^1 - \overline{c}_1^1$. We invite the reader to complete the proof.

9.5. Deficit finance

For the rest of this chapter, we shall assume sequential trading. With sequential trading of fiat currency, this section reinterprets one of our earlier examples with time 0 trading, the example with government spending.

Consider the following overlapping generations model: The population is constant. At each date $t \geq 1$, N identical young agents are endowed with $(y_t^t, y_{t+1}^t) = (w_1, w_2)$, where $w_1 > w_2 > 0$. A government levies lump-sum taxes of τ_1 on each young agent and τ_2 on each old agent alive at each $t \geq 1$. There are N old people at time 1 each of whom is endowed with w_2 units of the consumption good and $M_0 > 0$ units of inconvertible, perfectly durable fiat currency. The initial old have utility function c_1^0. The young have utility function $u(c_t^t) + u(c_{t+1}^t)$. For each date $t \geq 1$ the government augments the currency supply according to

$$M_t - M_{t-1} = p_t(g - \tau_1 - \tau_2), \tag{9.5.1}$$

where g is a constant stream of government expenditures per capita and $0 < p_t \leq +\infty$ is the price level. If $p_t = +\infty$, we intend that equation (9.5.1) be interpreted as

$$g = \tau_1 + \tau_2. \tag{9.5.2}$$

For each $t \geq 1$, each young person's behavior is summarized by

$$s_t = f(R_t; \tau_1, \tau_2) = \arg\max_{s \geq 0} \left[u(w_1 - \tau_1 - s) + u(w_2 - \tau_2 + R_t s) \right]. \tag{9.5.3}$$

DEFINITION: An equilibrium with valued fiat currency is a pair of positive sequences $\{M_t, p_t\}$ such that (a) given the price level sequence, $M_t/p_t = f(R_t)$ (the dependence on τ_1, τ_2 being understood); (b) $R_t = p_t/p_{t+1}$; and (c) the government budget constraint (9.5.1) is satisfied for all $t \geq 1$.

The condition $f(R_t) = M_t/p_t$ can be written as $f(R_t) = M_{t-1}/p_t + (M_t - M_{t-1})/p_t$. The left side is the saving of the young. The first term on the right side is the dissaving of the old (the real value of currency that they exchange for time t consumption). The second term on the right is the dissaving of the government (its deficit), which is the real value of the additional currency that the government prints at t and uses to purchase time t goods from the young.

To compute an equilibrium, define $d = g - \tau_1 - \tau_2$ and write equation (9.5.1) as

$$\frac{M_t}{p_t} = \frac{M_{t-1}}{p_{t-1}} \frac{p_{t-1}}{p_t} + d$$

for $t \geq 2$ and

$$\frac{M_1}{p_1} = \frac{M_0}{p_1} + d$$

for $t = 1$. Substitute the equilibrium condition $M_t/p_t = f(R_t)$ into these equations to get

$$f(R_t) = f(R_{t-1}) R_{t-1} + d \tag{9.5.4a}$$

for $t \geq 2$ and

$$f(R_1) = \frac{M_0}{p_1} + d. \tag{9.5.4b}$$

Given p_1, which determines an initial R_1 by means of equation (9.5.4b), equations (9.5.4) form an autonomous difference equation in R_t. With appropriate transformations of variables, this system can be solved using Figure 9.2.4.

9.5.1. Steady states and the Laffer curve

Let's seek a stationary solution of equations (9.5.4), a quest rendered reasonable by the fact that $f(R_t)$ is time invariant (because the endowment and the tax patterns as well as the government deficit d are time-invariant). Guess that $R_t = R$ for $t \geq 1$. Then equations (9.5.4) become

$$f(R)(1 - R) = d, \tag{9.5.5a}$$

$$f(R) = \frac{M_0}{p_1} + d. \tag{9.5.5b}$$

For example, suppose that $u(c) = \ln(c)$. Then $f(R) = \frac{w_1 - \tau_1}{2} - \frac{w_2 - \tau_2}{2R}$. We have graphed $f(R)(1 - R)$ against d in Figure 9.5.1. Notice that if there is one solution for equation (9.5.5a), then there are at least two.

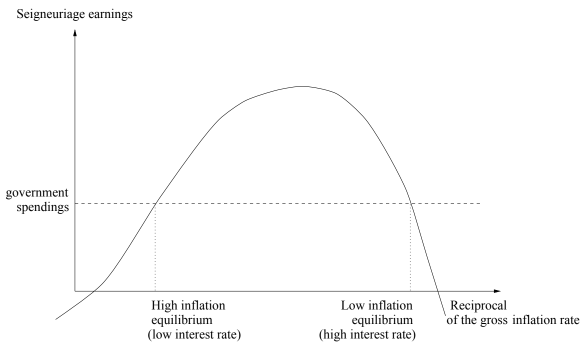


Figure 9.5.1: The Laffer curve in revenues from the inflation tax.

Here $(1-R)$ can be interpreted as a tax rate on real balances, and $f(R)(1-R)$ is a Laffer curve for the inflation tax rate. The high-return (low-tax) $R = \overline{R}$ is associated with the good Laffer curve stationary equilibrium, and the low-return (high-tax) $R = \underline{R}$ comes with the bad Laffer curve stationary equilibrium. Once R is determined, we can determine p_1 from equation (9.5.5b).

Figure 9.5.1 is isomorphic with Figure 9.2.4. The saving rate function $f(R)$ can be deduced from the offer curve. Thus, a version of Figure 9.2.4 can be used to solve the difference equation (9.5.4a) graphically. If we do so, we discover a continuum of nonstationary solutions of equation (9.5.4a), all but one of which have $R_t \to \underline{R}$ as $t \to \infty$. Thus, the bad Laffer curve equilibrium is stable.

The stability of the bad Laffer curve equilibrium arises under perfect foresight dynamics. Bruno and Fischer (1990) and Marcet and Sargent (1989) analyze how the system behaves under two different types of adaptive dynamics. They find that either under a crude form of adaptive expectations or under a least-squares learning scheme, R_t converges to $\overline{R}$. This finding is comforting because the comparative dynamics are more plausible at $\overline{R}$ (larger deficits bring higher inflation). Furthermore, Marimon and Sunder (1993) present experimental evidence pointing toward the selection made by the adaptive dynamics. Marcet and Nicolini (2003) build and calibrate an adaptive model of several Latin American hyperinflations that rests on this selection. Sargent, Williams, and Zha (2009) extend and estimate the model.

9.6. Equivalent setups

This section describes some alternative asset structures and trading arrangements that support the same equilibrium allocation. We take a model with a government deficit and show how it can be supported with sequential trading in government-indexed bonds, sequential trading in fiat currency, or time 0 trading of Arrow-Debreu dated securities.

9.6.1. *The economy*

Consider an overlapping generations economy with one agent born at each $t \geq 1$ and an initial old person at $t = 1$. Young agents born at date t have endowment pattern (y_t^t, y_{t+1}^t) and the utility function described earlier. The initial old person is endowed with $M_0 > 0$ units of unbacked currency and y_1^0 units of the consumption good. There is a stream of per-young-person government purchases $\{g_t\}$.

Definition: An equilibrium with money-financed government deficits is a sequence $\{M_t, p_t\}_{t=1}^\infty$ with $0 < p_t < +\infty$ and $M_t > 0$ that satisfies (a) given $\{p_t\}$,

$$M_t = \operatorname*{argmax}_{\tilde{M} \geq 0} \left[u(y_t^t - \tilde{M}/p_t) + u(y_{t+1}^t + \tilde{M}/p_{t+1}) \right]; \qquad (9.6.1a)$$

and (b)

$$M_t - M_{t-1} = p_t g_t. \tag{9.6.1b}$$

Now consider a version of the same economy in which there is no currency but rather indexed government bonds. The demographics and endowments are identical with the preceding economy, but now each initial old person is endowed with B_1 units of a maturing bond, denominated in units of time 1 consumption good. In period t, the government sells new one-period bonds to the young to finance its purchases g_t of time t goods and to pay off the one-period debt falling due at time t. Let $R_t > 0$ be the gross real one-period rate of return on government debt between t and $t+1$.

DEFINITION: An equilibrium with bond-financed government deficits is a sequence $\{B_{t+1}, R_t\}_{t=1}^{\infty}$ that satisfies (a) given $\{R_t\}$,

$$B_{t+1} = \arg\max_{\tilde{B}} [u(y_t^t - \tilde{B}/R_t) + u(y_{t+1}^t + \tilde{B})]; \tag{9.6.2a}$$

and (b)

$$B_{t+1}/R_t = B_t + g_t, \tag{9.6.2b}$$

with $B_1 \geq 0$ given.

These two types of equilibria are isomorphic in the following sense: Take an equilibrium of the economy with money-financed deficits and transform it into an equilibrium of the economy with bond-financed deficits as follows: set $B_t = M_{t-1}/p_t, R_t = p_t/p_{t+1}$. It can be verified directly that these settings of bonds and interest rates, together with the original consumption allocation, form an equilibrium of the economy with bond-financed deficits.

Each of these two types of equilibria is evidently also isomorphic to the following equilibrium formulated with time 0 markets:

DEFINITION: Let B_1^g represent claims to time 1 consumption owed by the government to the old at time 1. An equilibrium with time 0 trading is an initial level of government debt B_1^g, a price system $\{q_t^0\}_{t=1}^{\infty}$, and a sequence $\{s_t\}_{t=1}^{\infty}$ such that (a) given the price system,

$$s_t = \arg\max_{\tilde{s}} \left\{ u(y_t^t - \tilde{s}) + u\left[y_{t+1}^t + \left(\frac{q_t^0}{q_{t+1}^0} \right) \tilde{s} \right] \right\};$$

and (b)

$$q_1^0 B_1^g + \sum_{t=1}^{\infty} q_t^0 g_t = 0. \tag{9.6.3}$$

Condition b is the Arrow-Debreu version of the government budget constraint. Condition a is the optimality condition for the intertemporal consumption decision of the young of generation t.

The government budget constraint in condition b can be represented recursively as

$$q_{t+1}^0 B_{t+1}^g = q_t^0 B_t^g + q_t^0 g_t. \tag{9.6.4}$$

If we solve equation (9.6.4) forward and impose $\lim_{T\to\infty} q_{t+T}^0 B_{t+T}^g = 0$, we obtain the budget constraint (9.6.3) for $t = 1$. Condition (9.6.3) makes it evident that when $\sum_{t=1}^{\infty} q_t^0 g_t > 0$, $B_1^g < 0$, so that the government has negative net worth. This negative net worth corresponds to the unbacked claims that the market nevertheless values in the sequential-trading version of the model.

9.6.2. Growth

It is easy to extend these models to the case in which there is growth in the population. Let there be $N_t = nN_{t-1}$ identical young people at time t, with $n > 0$. For example, consider the economy with money-financed deficits. The total money supply is $N_t M_t$, and the government budget constraint is

$$N_t M_t - N_{t-1} M_{t-1} = N_t p_t g,$$

where g is per-young-person government purchases. Dividing both sides of the budget constraint by N_t and rearranging gives

$$\frac{M_t}{p_{t+1}} \frac{p_{t+1}}{p_t} = n^{-1} \frac{M_{t-1}}{p_t} + g. \tag{9.6.5}$$

This equation replaces equation $(9.6.1b)$ in the definition of an equilibrium with money-financed deficits. (Note that in a steady state, $R = n$ is the high-interest-rate equilibrium.) Similarly, in the economy with bond-financed deficits, the government budget constraint would become

$$\frac{B_{t+1}}{R_t} = n^{-1} B_t + g_t.$$

It is also easy to modify things to permit the government to tax young and old people at t. In that case, with government bond finance the government budget constraint becomes

$$\frac{B_{t+1}}{R_t} = n^{-1}B_t + g_t - \tau_t^t - n^{-1}\tau_t^{t-1},$$

where τ_t^s is the time t tax on a person born in period s.

9.7. Optimality and the existence of monetary equilibria

Wallace (1980) discusses the connection between nonoptimality of the equilibrium without valued money and existence of monetary equilibria. Abstracting from his assumption of a storage technology, we study how the arguments apply to a pure endowment economy. The environment is as follows. At any date t, the population consists of N_t young agents and N_{t-1} old agents where $N_t = nN_{t-1}$ with $n > 0$. Each young person is endowed with $y_1 > 0$ goods, and an old person receives the endowment $y_2 > 0$. Preferences of a young agent at time t are given by the utility function $u(c_t^t, c_{t+1}^t)$, which is twice differentiable with indifference curves that are convex to the origin. The two goods in the utility function are normal goods, and

$$\theta(c_1, c_2) \equiv u_1(c_1, c_2)/u_2(c_1, c_2),$$

the marginal rate of substitution function, approaches infinity as c_2/c_1 approaches infinity and approaches zero as c_2/c_1 approaches zero. The welfare of the initial old agents at time 1 is strictly increasing in c_1^0, and each one of them is endowed with y_2 goods and $m_0^0 > 0$ units of fiat money. Thus, the beginning-of-period aggregate nominal money balances in the initial period 1 are $M_0 = N_0 m_0^0$.

For all $t \geq 1$, M_t, the post-transfer time t stock of fiat money obeys $M_t = zM_{t-1}$ with $z > 0$. The time t transfer (or tax), $(z-1)M_{t-1}$, is divided equally at time t among the N_{t-1} members of the current old generation. The transfers (or taxes) are fully anticipated and are viewed as lump-sum: they do not depend on consumption and saving behavior. The budget constraints of a young agent born in period t are

$$c_t^t + \frac{m_t^t}{p_t} \leq y_1, \tag{9.7.1}$$

$$c_{t+1}^t \le y_2 + \frac{m_t^t}{p_{t+1}} + \frac{(z-1)}{N_t}\frac{M_t}{p_{t+1}}, \tag{9.7.2}$$

$$m_t^t \ge 0, \tag{9.7.3}$$

where $p_t > 0$ is the time t price level. In a nonmonetary equilibrium, the price level is infinite, so the real values of both money holdings and transfers are zero. Since all members in a generation are identical, the nonmonetary equilibrium is autarky with a marginal rate of substitution equal to

$$\theta_{\text{aut}} \equiv \frac{u_1(y_1, y_2)}{u_2(y_1, y_2)}.$$

We ask two questions about this economy. Under what circumstances does a monetary equilibrium exist? And, when it exists, under what circumstances does it improve matters?

Let $\hat{m}_t$ denote the equilibrium real money balances of a young agent at time t, $\hat{m}_t \equiv M_t/(N_t p_t)$. Substitution of equilibrium money holdings into budget constraints (9.7.1) and (9.7.2) at equality yield $c_t^t = y_1 - \hat{m}_t$ and $c_{t+1}^t = y_2 + n\hat{m}_{t+1}$. In a monetary equilibrium, $\hat{m}_t > 0$ for all t and the marginal rate of substitution $\theta(c_t^t, c_{t+1}^t)$ satisfies

$$\theta(y_1 - \hat{m}_t,\ y_2 + n\hat{m}_{t+1}) = \frac{p_t}{p_{t+1}} > \theta_{\text{aut}}, \quad \forall t \ge 1. \tag{9.7.4}$$

The equality part of (9.7.4) is the first-order condition for money holdings of an agent born in period t evaluated at the equilibrium allocation. Since $c_t^t < y_1$ and $c_{t+1}^t > y_2$ in a monetary equilibrium, the inequality in (9.7.4) follows from the assumption that the two goods in the utility function are normal goods.

Another useful characterization of the equilibrium rate of return on money, p_t/p_{t+1}, can be obtained as follows. By the rule generating M_t and the equilibrium condition $M_t/p_t = N_t\hat{m}_t$, we have for all t,

$$\frac{p_t}{p_{t+1}} = \frac{M_{t+1}}{zM_t}\frac{p_t}{p_{t+1}} = \frac{N_{t+1}\hat{m}_{t+1}}{zN_t\hat{m}_t} = \frac{n}{z}\frac{\hat{m}_{t+1}}{\hat{m}_t}. \tag{9.7.5}$$

We are now ready to address our first question, under what circumstances does a monetary equilibrium exist?

PROPOSITION: $\theta_{\text{aut}} z < n$ is necessary and sufficient for the existence of at least one monetary equilibrium.

PROOF: We first establish necessity. Suppose to the contrary that there is a monetary equilibrium and $\theta_{\text{aut}} z/n \geq 1$. Then, by the inequality part of (9.7.4) and expression (9.7.5), we have for all t,

$$\frac{\hat{m}_{t+1}}{\hat{m}_t} > \frac{z\theta_{\text{aut}}}{n} \geq 1. \tag{9.7.6}$$

If $z\theta_{\text{aut}}/n > 1$, one plus the net growth rate of $\hat{m}_t$ is bounded uniformly above one and, hence, the sequence $\{\hat{m}_t\}$ is unbounded, which is inconsistent with an equilibrium because real money balances per capita cannot exceed the endowment y_1 of a young agent. If $z\theta_{\text{aut}}/n = 1$, the strictly increasing sequence $\{\hat{m}_t\}$ in (9.7.6) might not be unbounded but converge to some constant $\hat{m}_\infty$. According to (9.7.4) and (9.7.5), the marginal rate of substitution will then converge to n/z, which by assumption is now equal to θ_{aut}, the marginal rate of substitution in autarky. Thus, real balances must be zero in the limit, which contradicts the existence of a strictly increasing sequence of positive real balances in (9.7.6).

To show sufficiency, we prove the existence of a unique equilibrium with constant per capita real money balances when $\theta_{\text{aut}} z < n$. Substitute our candidate equilibrium, $\hat{m}_t = \hat{m}_{t+1} \equiv \hat{m}$, into (9.7.4) and (9.7.5), which yields two equilibrium conditions,

$$\theta(y_1 - \hat{m},\ y_2 + n\hat{m}) = \frac{n}{z} > \theta_{\text{aut}}.$$

The inequality part is satisfied under the parameter restriction of the proposition, and we only have to show the existence of $\hat{m} \in [0, y_1]$ that satisfies the equality part. But the existence (and uniqueness) of such a $\hat{m}$ is trivial. Note that the marginal rate of substitution on the left side of the equality is equal to θ_{aut} when $\hat{m} = 0$. Next, our assumptions on preferences imply that the marginal rate of substitution is strictly increasing in $\hat{m}$, and approaches infinity when $\hat{m}$ approaches y_1. ∎

The stationary monetary equilibrium in the proof will be referred to as the $\hat{m}$ equilibrium. In general, there are other nonstationary monetary equilibria when the parameter condition of the proposition is satisfied. For example, in the case of logarithmic preferences and a constant population, recall the continuum of equilibria indexed by the scalar $c > 0$ in expression (9.4.8). But here we choose to focus solely on the stationary $\hat{m}$ equilibrium and its welfare implications. The $\hat{m}$ equilibrium will be compared to other feasible allocations using the Pareto criterion. Evidently, an allocation $C = \{c_1^0; (c_t^t, c_{t+1}^t), t \geq 1\}$ is feasible if

$$N_t c_t^t + N_{t-1} c_t^{t-1} \leq N_t y_1 + N_{t-1} y_2, \quad \forall t \geq 1,$$

or, equivalently,

$$n c_t^t + c_t^{t-1} \leq n y_1 + y_2, \quad \forall t \geq 1. \tag{9.7.7}$$

The definition of Pareto optimality is:

Definition: A feasible allocation C is Pareto optimal if there is no other feasible allocation $\tilde{C}$ such that

$$\begin{aligned} &\tilde{c}_1^0 \geq c_1^0, \\ &u(\tilde{c}_t^t, \tilde{c}_{t+1}^t) \geq u(c_t^t, c_{t+1}^t), \quad \forall t \geq 1, \end{aligned}$$

and at least one of these weak inequalities holds with strict inequality.

We first examine under what circumstances the nonmonetary equilibrium (autarky) is Pareto optimal.

Proposition: $\theta_{\rm aut} \geq n$ is necessary and sufficient for the optimality of the nonmonetary equilibrium (autarky).

Proof: To establish sufficiency, suppose to the contrary that there exists another feasible allocation $\tilde{C}$ that is Pareto superior to autarky and $\theta_{\rm aut} \geq n$. Without loss of generality, assume that the allocation $\tilde{C}$ satisfies (9.7.7) with equality. (Given an allocation that is Pareto superior to autarky but that does not satisfy (9.7.7), one can easily construct another allocation that is Pareto superior to the given allocation, and hence to autarky.) Let period t be the first period when this alternative allocation $\tilde{C}$ differs from the autarkic allocation. The requirement that the old generation in this period is not made worse off, $\tilde{c}_t^{t-1} \geq y_2$, implies that the first perturbation from the autarkic allocation must be $\tilde{c}_t^t < y_1$, with the subsequent implication that $\tilde{c}_{t+1}^t > y_2$. It follows that the consumption of young agents at time $t+1$ must also fall below y_1, and we define

$$\epsilon_{t+1} \equiv y_1 - \tilde{c}_{t+1}^{t+1} > 0. \tag{9.7.8}$$

Now, given $\tilde{c}_{t+1}^{t+1}$, we compute the smallest number c_{t+2}^{t+1} that satisfies

$$u(\tilde{c}_{t+1}^{t+1}, c_{t+2}^{t+1}) \geq u(y_1, y_2).$$

Let $\overline{c}_{t+2}^{t+1}$ be the solution to this problem. Since the allocation $\tilde{C}$ is Pareto superior to autarky, we have $\tilde{c}_{t+2}^{t+1} \geq \overline{c}_{t+2}^{t+1}$. Before using this inequality, though, we want to derive a convenient expression for $\overline{c}_{t+2}^{t+1}$.

Consider the indifference curve of $u(c_1, c_2)$ that yields a fixed utility equal to $u(y_1, y_2)$. In general, along an indifference curve, $c_2 = h(c_1)$, where $h' = -u_1/u_2 = -\theta$ and $h'' > 0$. Therefore, applying the intermediate value theorem to h, we have

$$h(c_1) = h(y_1) + (y_1 - c_1)[-h'(y_1) + f(y_1 - c_1)], \tag{9.7.9}$$

where the function f is strictly increasing and $f(0) = 0$.

Now, since $(\tilde{c}_{t+1}^{t+1}, \overline{c}_{t+2}^{t+1})$ and (y_1, y_2) are on the same indifference curve, we can use (9.7.8) and (9.7.9) to write

$$\overline{c}_{t+2}^{t+1} = y_2 + \epsilon_{t+1}[\theta_{\text{aut}} + f(\epsilon_{t+1})],$$

and after invoking $\tilde{c}_{t+2}^{t+1} \geq \overline{c}_{t+2}^{t+1}$, we have

$$\tilde{c}_{t+2}^{t+1} - y_2 \geq \epsilon_{t+1}[\theta_{\text{aut}} + f(\epsilon_{t+1})]. \tag{9.7.10}$$

Since $\tilde{C}$ satisfies (9.7.7) at equality, we also have

$$\epsilon_{t+2} \equiv y_1 - \tilde{c}_{t+2}^{t+2} = \frac{\tilde{c}_{t+2}^{t+1} - y_2}{n}. \tag{9.7.11}$$

Substitution of (9.7.10) into (9.7.11) yields

$$\begin{aligned} \epsilon_{t+2} &\geq \epsilon_{t+1} \frac{\theta_{\text{aut}} + f(\epsilon_{t+1})}{n} \\ &> \epsilon_{t+1}, \end{aligned} \tag{9.7.12}$$

where the strict inequality follows from $\theta_{\text{aut}} \geq n$ and $f(\epsilon_{t+1}) > 0$ (implied by $\epsilon_{t+1} > 0$). Continuing these computations of successive values of ϵ_{t+k} yields

$$\epsilon_{t+k} \geq \epsilon_{t+1} \prod_{j=1}^{k-1} \frac{\theta_{\text{aut}} + f(\epsilon_{t+j})}{n} > \epsilon_{t+1} \left[\frac{\theta_{\text{aut}} + f(\epsilon_{t+1})}{n}\right]^{k-1}, \text{ for } k > 2,$$

where the strict inequality follows from the fact that $\{\epsilon_{t+j}\}$ is a strictly increasing sequence. Thus, the ϵ sequence is bounded below by a strictly increasing exponential and hence is unbounded. But such an unbounded sequence violates feasibility because ϵ cannot exceed the endowment y_1 of a young agent. It follows that we can rule out the existence of a Pareto superior allocation $\tilde{C}$, and conclude that $\theta_{\rm aut} \geq n$ is a sufficient condition for the optimality of autarky.

To establish necessity, we prove the existence of an alternative feasible allocation $\hat{C}$ that is Pareto superior to autarky when $\theta_{\rm aut} < n$. First, pick an $\epsilon > 0$ sufficiently small so that

$$\theta_{\rm aut} + f(\epsilon) \leq n, \tag{9.7.13}$$

where f is defined implicitly by equation (9.7.9). Second, set $\hat{c}_t^t = y_1 - \epsilon \equiv \hat{c}_1$, and

$$\hat{c}_{t+1}^t = y_2 + \epsilon[\theta_{\rm aut} + f(\epsilon)] \equiv \hat{c}_2, \quad \forall t \geq 1. \tag{9.7.14}$$

That is, we have constructed a consumption bundle $(\hat{c}_1, \hat{c}_2)$ that lies on the same indifference curve as (y_1, y_2), and from (9.7.13) and (9.7.14), we have

$$\hat{c}_2 \leq y_2 + n\epsilon,$$

which ensures that the condition for feasibility (9.7.7) is satisfied for $t \geq 2$. By setting $\hat{c}_1^0 = y_2 + n\epsilon$, feasibility is also satisfied in period 1 and the initial old generation is then strictly better off under the alternative allocation $\hat{C}$. ∎

With a constant nominal money supply, $z = 1$, the two propositions show that a monetary equilibrium exists if and only if the nonmonetary equilibrium is suboptimal. In that case, the following proposition establishes that the stationary $\hat{m}$ equilibrium is optimal.

PROPOSITION: Given $\theta_{\rm aut} z < n$, then $z \leq 1$ is necessary and sufficient for the optimality of the stationary monetary equilibrium $\hat{m}$.

PROOF: The class of feasible stationary allocations with $(c_t^t, c_{t+1}^t) = (c_1, c_2)$ for all $t \geq 1$, is given by

$$c_1 + \frac{c_2}{n} \leq y_1 + \frac{y_2}{n}, \tag{9.7.15}$$

i.e., the condition for feasibility in (9.7.7). It follows that the $\hat{m}$ equilibrium satisfies (9.7.15) at equality, and we denote the associated consumption allocation of an agent born at time $t \geq 1$ by $(\hat{c}_1, \hat{c}_2)$. It is also the case that $(\hat{c}_1, \hat{c}_2)$ maximizes an agent's utility subject to budget constraints (9.7.1) and (9.7.2). The consolidation of these two constraints yields

$$c_1 + \frac{z}{n} c_2 \leq y_1 + \frac{z}{n} y_2 + \frac{z}{n} \frac{(z-1)}{N_t} \frac{M_t}{p_{t+1}}, \tag{9.7.16}$$

where we have used the stationary rate or return in (9.7.5), $p_t/p_{t+1} = n/z$. After also invoking $zM_t = M_{t+1}$, $n = N_{t+1}/N_t$, and the equilibrium condition $M_{t+1}/(p_{t+1}N_{t+1}) = \hat{m}$, expression (9.7.16) simplifies to

$$c_1 + \frac{z}{n}c_2 \leq y_1 + \frac{z}{n}y_2 + (z-1)\hat{m}. \tag{9.7.17}$$

To prove the statement about necessity, Figure 9.7.1 depicts the two curves (9.7.15) and (9.7.17) when condition $z \leq 1$ fails to hold, i.e., we assume that $z > 1$. The point that maximizes utility subject to (9.7.15) is denoted $(\overline{c}_1, \overline{c}_2)$. Transitivity of preferences and the fact that the slope of budget line (9.7.17) is flatter than that of (9.7.15) imply that $(\hat{c}_1, \hat{c}_2)$ lies southeast of $(\overline{c}_1, \overline{c}_2)$. By revealed preference, then, $(\overline{c}_1, \overline{c}_2)$ is preferred to $(\hat{c}_1, \hat{c}_2)$ and all generations born in period $t \geq 1$ are better off under the allocation $\overline{C}$. The initial old generation can also be made better off under this alternative allocation since it is feasible to strictly increase their consumption,

$$\overline{c}_1^0 = y_2 + n(y_1 - \overline{c}_1^1) > y_2 + n(y_1 - \hat{c}_1^1) = \hat{c}_1^0.$$

Thus, we have established that $z \leq 1$ is necessary for the optimality of the stationary monetary equilibrium $\hat{m}$.

To prove sufficiency, note that (9.7.4), (9.7.5) and $z \leq 1$ imply that

$$\theta(\hat{c}_1, \hat{c}_2) = \frac{n}{z} \geq n.$$

We can then construct an argument that is analogous to the sufficiency part of the proof to the preceding proposition. ∎

As pointed out by Wallace (1980), the proposition implies no connection between the path of the price level in an $\hat{m}$ equilibrium and the optimality of that equilibrium. Thus, there may be an optimal monetary equilibrium with positive inflation, for example, if $\theta_{\text{aut}} < n < z \leq 1$; and there may be a nonoptimal monetary equilibrium with a constant price level, for example, if $z = n > 1 > \theta_{\text{aut}}$. What counts is the nominal quantity of fiat money. The proposition suggests that the quantity of money should not be increased. In particular, if $z \leq 1$, then an optimal $\hat{m}$ equilibrium exists whenever the nonmonetary equilibrium is nonoptimal.

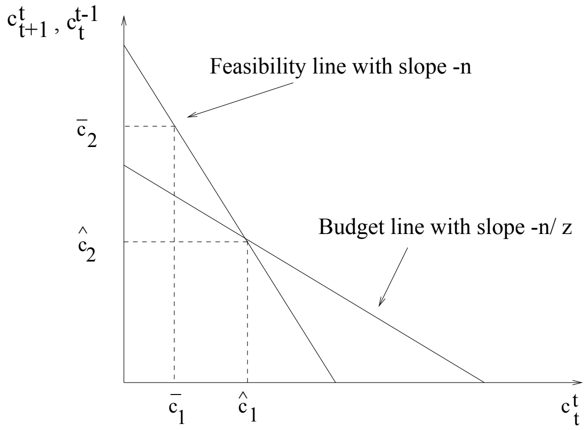


Figure 9.7.1: The feasibility line (9.7.15) and the budget line (9.7.17) when $z > 1$. The consumption allocation in the monetary equilibrium is $(\hat{c}_1, \hat{c}_2)$, and the point that maximizes utility subject to the feasibility line is denoted $(\overline{c}_1, \overline{c}_2)$.

9.7.1. Balasko-Shell criterion for optimality

For the case of constant population, Balasko and Shell (1980) have established a convenient general criterion for testing whether allocations are optimal.[9] Balasko and Shell permit diversity among agents in terms of endowments $[w_t^{th}, w_{t+1}^{th}]$ and utility functions $u^{th}(c_t^{th}, c_{t+1}^{th})$, where w_s^{th} is the time s endowment of an agent named h who is born at t and c_s^{th} is the time s consumption of agent named h born at t. Balasko and Shell assume fixed populations of types h over time. They impose several kinds of technical conditions that serve to rule out possible pathologies. The two main ones are these. First, they assume that indifference curves have neither flat parts nor kinks, and they also rule out indifference curves with flat parts or kinks as limits of sequences of indifference curves for given h as $t \to \infty$. Second, they assume that the aggregate endowments $\sum_h (w_t^{th} + w_t^{t-1,h})$ are uniformly bounded from above and that there exists an $\epsilon > 0$ such that $w_t^{sh} > \epsilon$ for all s, h, and for $t \in \{s, s+1\}$. They consider consumption allocations uniformly bounded away from the axes.

[9] Balasko and Shell credit David Cass (1971) with having authored a version of their criterion.

With these conditions, Balasko and Shell consider the class of allocations in which all young agents at t share a common marginal rate of substitution $1+r_t \equiv u_1^{th}(c_t^{th}, c_{t+1}^{th})/u_2^{th}(c_t^{th}, c_{t+1}^{th})$ and in which all of the endowments are consumed. Then Balasko and Shell show that an allocation is Pareto optimal if and only if

$$\sum_{t=1}^{\infty} \prod_{s=1}^{t} [1+r_s] = +\infty, \tag{9.7.18}$$

that is, if and only if the infinite sum of t-period gross interest rates, $\prod_{s=1}^{t}[1+r_s]$, diverges.

The Balasko-Shell criterion for optimality succinctly summarizes the sense in which low-interest-rate economies are not optimal. We have already encountered repeated examples of the situation that, before an equilibrium with valued currency can exist, the equilibrium without valued currency must be a low-interest-rate economy in just the sense identified by Balasko and Shell's criterion, (9.7.18). Furthermore, by applying the Balasko-Shell criterion, (9.7.18), or generalizations of it that allow for a positive net growth rate of population n, it can be shown that, among equilibria with valued currency, only equilibria with high rates of return on currency are optimal.

9.8. Within-generation heterogeneity

This section describes an overlapping generations model having within-generation heterogeneity of endowments. We shall follow Sargent and Wallace (1982) and Smith (1988) and use this model as a vehicle for talking about some issues in monetary theory that require a setting in which government-issued currency coexists with and is a more-or-less good substitute for private IOUs.

We now assume that within each generation born at $t \geq 1$, there are J groups of agents. There is a constant number N_j of group j agents. Agents of group j are endowed with $w_1(j)$ when young and $w_2(j)$ when old. The saving function of a household of group j born at time t solves the time t version of problem (9.5.3). We denote this savings function $f(R_t, j)$. If we assume that all households of generation t have preferences $U^t(c^t) = \ln c_t^t + \ln c_{t+1}^t$, the saving function is

$$f(R_t, j) = .5\left(w_1(j) - \frac{w_2(j)}{R_t}\right).$$

At $t=1$, there are old people who are endowed in the aggregate with $H=H(0)$ units of an inconvertible currency.

For example, assume that $J=2$, that $(w_1(1), w_2(1)) = (\alpha, 0), (w_1(2), w_2(2)) = (0, \beta)$, where $\alpha > 0, \beta > 0$. The type 1 people are lenders, while the type 2 are borrowers. For the case of log preference we have the savings functions $f(R,1) = \alpha/2, f(R,2) = -\beta/(2R)$.

9.8.1. Nonmonetary equilibrium

A nonmonetary equilibrium consists of sequences (R, s_j) of rates of return R and savings rates for $j = 1, \ldots, J$ and $t \geq 1$ that satisfy (1) $s_{tj} = f(R_t, j)$, and (2) $\sum_{j=1}^J N_j f(R_t, j) = 0$. Condition (1) builds in household optimization; condition (2) says that aggregate net savings equals zero (borrowing equals lending).

For the case in which the endowments, preferences, and group sizes are constant across time, the interest rate is determined at the intersection of the aggregate savings function with the R axis, depicted as R_1 in Figure 9.8.1. No intergenerational transfers occur in the nonmonetary equilibrium. The equilibrium consists of a sequence of separate two-period pure consumption loan economies of a type analyzed by Irving Fisher (1907).

9.8.2. Monetary equilibrium

In an equilibrium with valued fiat currency, at each date $t \geq 1$ the old receive goods from the young in exchange for the currency stock H. For any variable x, $\vec{x} = \{x_t\}_{t=1}^\infty$. An equilibrium with valued fiat money is a set of sequences $\vec{R}, \vec{p}, \vec{s}$ such that (1) $\vec{p}$ is a positive sequence, (2) $R_t = p_t/p_{t+1}$, (3) $s_{jt} = f(R_t, j)$, and (4) $\sum_{j=1}^J N_j f(R_t, j) = \frac{H}{p_t}$. Condition (1) states that currency is valued at all dates. Condition (2) states that currency and consumption loans are perfect substitutes. Condition (3) requires that saving decisions are optimal. Condition (4) equates the net saving of the young (the left side) to the net dissaving of the old (the right side). The old supply currency inelastically.

We can determine a stationary equilibrium graphically. A stationary equilibrium satisfies $p_t = p$ for all t, which implies $R = 1$ for all t. Thus, if it exists, a stationary equilibrium solves

$$\sum_{j=1}^{J} N_j f(1,j) = \frac{H}{p} \tag{9.8.1}$$

for a positive price level. (See Figure 9.8.1.) Evidently, a stationary monetary equilibrium exists if the net savings of the young are positive for $R=1$.

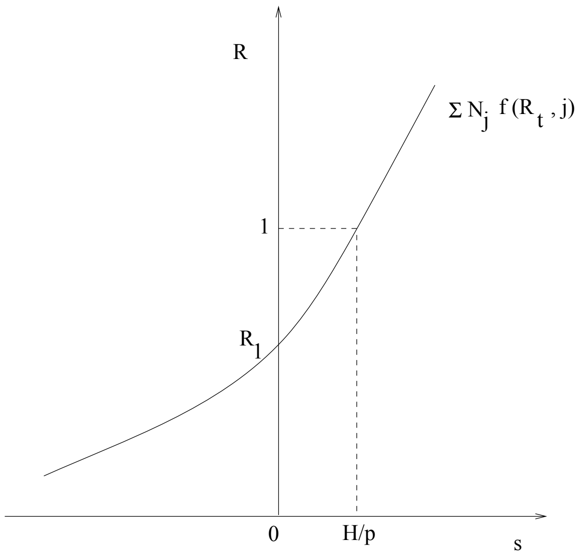


Figure 9.8.1: The intersection of the aggregate savings function with a horizontal line at $R=1$ determines a stationary equilibrium value of the price level, if positive.

For the special case of logarithmic preferences and two classes of young people, the aggregate savings function of the young is time invariant and equal to

$$\sum_j f(R,j) = .5(N_1\alpha - N_2\frac{\beta}{R}).$$

Note that the equilibrium condition (9.8.1) can be written

$$.5N_1\alpha = .5N_2\frac{\beta}{R} + \frac{H}{p}.$$

The left side is the demand for savings or the demand for "currency" while the right side is the supply, consisting of privately issued IOU's (the first term) and government-issued currency. The right side is thus an abstract version of what is called M1, which is a sum of privately issued IOUs (demand deposits) and government-issued reserves and currency.

9.8.3. *Nonstationary equilibria*

Mathematically, the equilibrium conditions for the model with log preferences and two groups have the same structure as the model analyzed previously in equations (9.4.7) and (9.4.8), with simple reinterpretations of parameters. We leave it to the reader here and in an exercise to show that if there exists a stationary equilibrium with valued fiat currency, then there exists a continuum of equilibria with valued fiat currency, all but one of which have the real value of government currency approaching zero asymptotically. A linear difference equation like (9.4.7) supports this conclusion.

9.8.4. *The real bills doctrine*

In nineteenth-century Europe and the early days of the Federal Reserve system in the United States, central banks conducted open market operations not by purchasing government securities but by purchasing safe (risk-free) short-term private IOUs. We now analyze this old-fashioned type of open market operation. We allow the government to issue additional currency each period. It uses the proceeds exclusively to purchase private IOUs (make loans to private agents) in the amount L_t at time t. Such open market operations are subject to the sequence of restrictions

$$L_t = R_{t-1} L_{t-1} + \frac{H_t - H_{t-1}}{p_t} \tag{9.8.2}$$

for $t \geq 1$ and $H_0 = H > 0$ given, $L_0 = 0$. Here L_t is the amount of the time t consumption good that the government lends to the private sector from period t to period $t+1$. Equation (9.8.2) states that the government finances these loans in two ways: first, by rolling over the proceeds $R_{t-1}L_{t-1}$ from the repayment of last period's loans, and second, by injecting new currency in the amount

$H_t - H_{t-1}$. With the government injecting new currency and purchasing loans in this way each period, the equilibrium condition in the loan market becomes

$$\sum_{j=1}^{J} N_j f(R_t, j) + L_t = \frac{H_{t-1}}{p_t} + \frac{H_t - H_{t-1}}{p_t} \tag{9.8.3}$$

where the first term on the right is the real dissaving of the old at t (their real balances) and the second term is the real value of the new money printed by the monetary authority to finance purchases of private IOUs issued by the young at t. The left side is the net savings of the young plus the savings of the government.

Under several guises, the effects of open market operations like this have concerned monetary economists for centuries.[10] We state the following proposition:

Irrelevance of Open Market Operations: Open market operations are irrelevant: all positive sequences $\{L_t, H_t\}_{t=0}^{\infty}$ that satisfy the constraint (9.8.2) are associated with the same equilibrium allocation, interest rate, and price level sequences.

Proof: Evidently, we can write the equilibrium condition (9.8.3) as

$$\sum_{j=1}^{J} N_j f(R_t, j) + L_t = \frac{H_t}{p_t}. \tag{9.8.4}$$

For $t \geq 1$, iterating (9.8.2) once and using $R_{t-1} = \frac{p_{t-1}}{p_t}$ gives

$$L_t = R_{t-1} R_{t-2} L_{t-2} + \frac{H_t - H_{t-2}}{p_t}.$$

Iterating back to time 0 and using $L_0 = 0$ gives

$$L_t = \frac{H_t - H_0}{p_t}. \tag{9.8.5}$$

Substituting (9.8.5) into (9.8.4) gives

$$\sum_{j=1}^{J} N_j f(R_t, j) = \frac{H_0}{p_t}. \tag{9.8.6}$$

[10] One issue concerned the effects on the price level of allowing banks to issue private bank notes. Nothing in our model makes us take seriously that the notes H_t are issued by the government. We can also think of them as being issued by a private bank.

This is the same equilibrium condition in the economy with no open market operations, i.e., the economy with $L_t \equiv 0$ for all $t \geq 1$. Any price level and rate of return sequence that solves (9.8.6) also solves (9.8.3) for any L_t sequence satisfying (9.8.2). ■

This proposition captures the spirit of Adam Smith's real bills doctrine, which states that if the government issues bank notes to purchase safe evidences of private indebtedness, it is not inflationary. Sargent and Wallace (1982) go on to analyze settings in which the money market is initially separated from the credit market by some legal restrictions that inhibit intermediation. Then open market operations are no longer irrelevant because they can partially undo the legal restrictions. Sargent and Wallace show how those legal restrictions can help stabilize the price level at a cost in terms of economic efficiency. Kahn and Roberds (1998) extend the Sargent and Wallace model to study issues about regulating electronic payments systems.

9.9. Gift-giving equilibrium

Michihiro Kandori (1992) and Lones Smith (1992) have used ideas from the literature on reputation (see chapter 23) to study whether there exist history-dependent sequences of gifts that support an optimal allocation. Their idea is to set up the economy as a game played with a sequence of players. We briefly describe a gift-giving game for an overlapping generations economy in which voluntary intergenerational gifts support an optimal allocation. Suppose that the consumption of an initial old person is

$$c_1^0 = y_1^0 + s_1$$

and the utility of each young agent is

$$u(y_i^i - s_i) + u(y_{i+1}^i + s_{i+1}), \quad i \geq 1 \tag{9.9.1}$$

where $s_i \geq 0$ is the gift from a young person at i to an old person at i. Suppose that the endowment pattern is $y_i^i = 1 - \epsilon, y_{i+1}^i = \epsilon$, where $\epsilon \in (0, .5)$.

Consider the following system of expectations, to which a young person chooses whether to conform:

$$s_i = \begin{cases} .5 - \epsilon & \text{if } v_i = \overline{v}; \\ 0 & \text{otherwise.} \end{cases} \tag{9.9.2a}$$

$$v_{i+1} = \begin{cases} \overline{v} & \text{if } v_i = \overline{v} \text{ and } s_i = .5 - \epsilon; \\ \underline{v} & \text{otherwise.} \end{cases} \tag{9.9.2b}$$

Here we are free to take $\overline{v} = 2u(.5)$ and $\underline{v} = u(1-\epsilon) + u(\epsilon)$. These are "promised utilities." We make them serve as "state variables" that summarize the history of intergenerational gift giving. To start, we need an initial value v_1. Equations (9.9.2) act as the transition laws that young agents face in choosing s_i in (9.9.1).

An initial condition v_1 and the rule (9.9.2) form a system of expectations that tells the young person of each generation what he is expected to give. His gift is immediately handed over to an old person. A system of expectations is called an *equilibrium* if for each $i \geq 1$, each young agent chooses to conform.

We can immediately compute two equilibrium systems of expectations. The first is the "autarky" equilibrium: give nothing yourself and expect all future generations to give nothing. To represent this equilibrium within equations (9.9.2), set $v_1 \neq \overline{v}$. It is easy to verify that each young person will confirm what is expected of him in this equilibrium. Given that future generations will not give, each young person chooses not to give.

For the second equilibrium, set $v_1 = \overline{v}$. Here each household chooses to give the expected amount, because failure to do so causes the next generation of young people not to give; whereas affirming the expectation to give passes that expectation along to the next generation, which affirms it in turn. Each of these equilibria is credible, in the sense of subgame perfection, to be studied extensively in chapter 23.

Narayana Kocherlakota (1998) has compared gift giving and monetary equilibria in a variety of environments and has used the comparison to provide a precise sense in which "money" substitutes for "memory".

9.10. Concluding remarks

The overlapping generations model is a workhorse in analyses of public finance, welfare economics, and demographics. Diamond (1965) studied some fiscal policy issues within a version of the model with a neoclassical production. He showed that, depending on preference and productivity parameters, equilibria of the model can have too much capital, and that such capital overaccumulation can be corrected by having the government issue and perpetually roll over unbacked debt.[11] Auerbach and Kotlikoff (1987) formulated a long-lived overlapping generations model with capital, labor, production, and various kinds of taxes. They used the model to study a host of fiscal issues. Rios-Rull (1994a) used a calibrated overlapping generations growth model to examine the quantitative importance of market incompleteness for insuring against aggregate risk. See Attanasio (2000) for a review of theories and evidence about consumption within life-cycle models.

Several authors in a 1980 volume edited by John Kareken and Neil Wallace argued through example that the overlapping generations model is useful for analyzing a variety of issues in monetary economics. We refer to that volume, McCandless and Wallace (1992), Champ and Freeman (1994), Brock (1990), and Sargent (1987b) for a variety of applications of the overlapping generations model to issues in monetary economics.

Exercises

Exercise 9.1 At each date $t \geq 1$, an economy consists of overlapping generations of a constant number N of two-period-lived agents. Young agents born in t have preferences over consumption streams of a single good that are ordered by $u(c_t^t) + u(c_{t+1}^t)$, where $u(c) = c^{1-\gamma}/(1-\gamma)$, and where c_t^i is the consumption of an agent born at i in time t. It is understood that $\gamma > 0$, and that when $\gamma = 1$, $u(c) = \ln c$. Each young agent born at $t \geq 1$ has identical preferences and endowment pattern (w_1, w_2), where w_1 is the endowment when young and w_2 is the endowment when old. Assume $0 < w_2 < w_1$. In addition, there are some initial old agents at time 1 who are endowed with w_2 of the time 1

[11] Abel, Mankiw, Summers, and Zeckhauser (1989) propose an empirical test of whether there is capital overaccumulation in the U.S. economy, and conclude that there is not.

consumption good, and who order consumption streams by c_1^0. The initial old (i.e., the old at $t = 1$) are also endowed with M units of unbacked fiat currency. The stock of currency is constant over time.

a. Find the saving function of a young agent.

b. Define an equilibrium with valued fiat currency.

c. Define a stationary equilibrium with valued fiat currency.

d. Compute a stationary equilibrium with valued fiat currency.

e. Describe how many equilibria with valued fiat currency there are. (You are not being asked to compute them.)

f. Compute the limiting value as $t \to +\infty$ of the rate of return on currency in each of the nonstationary equilibria with valued fiat currency. Justify your calculations.

Exercise 9.2 Consider an economy with overlapping generations of a constant population of an even number N of two-period-lived agents. New young agents are born at each date $t \geq 1$. Half of the young agents are endowed with w_1 when young and 0 when old. The other half are endowed with 0 when young and w_2 when old. Assume $0 < w_2 < w_1$. Preferences of all young agents are as in problem 1, with $\gamma = 1$. Half of the N initial old are endowed with w_2 units of the consumption good and half are endowed with nothing. Each old person orders consumption streams by c_1^0. Each old person at $t = 1$ is endowed with M units of unbacked fiat currency. No other generation is endowed with fiat currency. The stock of fiat currency is fixed over time.

a. Find the saving function of each of the two types of young person for $t \geq 1$.

b. Define an equilibrium without valued fiat currency. Compute all such equilibria.

c. Define an equilibrium with valued fiat currency.

d. Compute all the (nonstochastic) equilibria with valued fiat currency.

e. Argue that there is a unique stationary equilibrium with valued fiat currency.

f. How are the various equilibria with valued fiat currency ranked by the Pareto criterion?

Exercise 9.3 Take the economy of exercise *9.1*, but make one change. Endow the initial old with a tree that yields a constant dividend of $d > 0$ units of the consumption good for each $t \geq 1$.

a. Compute all the equilibria with valued fiat currency.

b. Compute all the equilibria without valued fiat currency.

c. If you want, you can answer both parts of this question in the context of the following particular numerical example: $w_1 = 10, w_2 = 5, d = .000001$.

Exercise 9.4 Take the economy of exercise *9.1* and make the following two changes. First, assume that $\gamma = 1$. Second, assume that the number of young agents born at t is $N(t) = nN(t-1)$, where $N(0) > 0$ is given and $n \geq 1$. Everything else about the economy remains the same.

a. Compute an equilibrium without valued fiat money.

b. Compute a stationary equilibrium with valued fiat money.

Exercise 9.5 Consider an economy consisting of overlapping generations of two-period-lived consumers. At each date $t \geq 1$ there are born $N(t)$ identical young people each of whom is endowed with $w_1 > 0$ units of a single consumption good when young and $w_2 > 0$ units of the consumption good when old. Assume that $w_2 < w_1$. The consumption good is not storable. The population of young people is described by $N(t) = nN(t-1)$, where $n > 0$. Young people born at t rank utility streams according to $\ln(c_t^t) + \ln(c_{t+1}^t)$ where c_t^i is the consumption of the time t good of an agent born in i. In addition, there are $N(0)$ old people at time 1, each of whom is endowed with w_2 units of the time 1 consumption good. The old at $t = 1$ are also endowed with one unit of unbacked pieces of infinitely durable but intrinsically worthless pieces of paper called fiat money.

a. Define an equilibrium without valued fiat currency. Compute such an equilibrium.

b. Define an equilibrium with valued fiat currency.

c. Compute all equilibria with valued fiat currency.

d. Find the limiting rates of return on currency as $t \to +\infty$ in each of the equilibria that you found in part c. Compare them with the one-period interest rate in the equilibrium in part a.

e. Are the equilibria in part c ranked according to the Pareto criterion?

Exercise 9.6 **Exchange rate determinacy**

The world consists of two economies, named $i = 1, 2$, which except for their governments' policies are "copies" of one another. At each date $t \geq 1$, there is a single consumption good, which is storable, but only for rich people. Each economy consists of overlapping generations of two-period-lived agents. For each $t \geq 1$, in economy i, N poor people and N rich people are born. Let $c_t^h(s), y_t^h(s)$ be the time s (consumption, endowment) of a type h agent born at t. Poor agents are endowed with $[y_t^h(t), y_t^h(t+1)] = (\alpha, 0)$; rich agents are endowed with $[y_t^h(t), y_t^h(t+1)] = (\beta, 0)$, where $\beta >> \alpha$. In each country, there are $2N$ initial old who are endowed in the aggregate with $H_i(0)$ units of an unbacked currency and with $2N\epsilon$ units of the time 1 consumption good. For the rich people, storing k units of the time t consumption good produces Rk units of the time $t+1$ consumption good, where $R > 1$ is a fixed gross rate of return on storage. Rich people can earn the rate of return R either by storing goods or by lending to either government by means of indexed bonds. We assume that poor people are prevented from storing capital or holding indexed government debt by the sort of denomination and intermediation restrictions described by Sargent and Wallace (1982).

For each $t \geq 1$, all young agents order consumption streams according to $\ln c_t^h(t) + \ln c_t^h(t+1)$.

For $t \geq 1$, the government of country i finances a stream of purchases (to be thrown into the ocean) of $G_i(t)$ subject to the following budget constraint:

$$G_i(t) + RB_i(t-1) = B_i(t) + \frac{H_i(t) - H_i(t-1)}{p_i(t)} + T_i(t), \tag{1}$$

where $B_i(0) = 0$; $p_i(t)$ is the price level in country i; $T_i(t)$ are lump-sum taxes levied by the government on the *rich* young people at time t; $H_i(t)$ is the stock of i's fiat currency at the end of period t; $B_i(t)$ is the stock of indexed government interest-bearing debt (held by the rich of either country). The government does not explicitly tax poor people, but might tax through an inflation tax. Each government levies a lump-sum tax of $T_i(t)/N$ on each young rich citizen of its own country.

Poor people in both countries are free to hold whichever currency they prefer. Rich people can hold debt of either government and can also store; storage and both government debts bear a constant gross rate of return R.

a. Define an *equilibrium* with valued fiat currencies (in both countries).

b. In a nonstochastic equilibrium, verify the following proposition: if an equilibrium exists in which both fiat currencies are valued, the exchange rate between the two currencies must be constant over time.

c. Suppose that government policy in each country is characterized by specified (exogenous) levels $G_i(t) = G_i, T_i(t) = T_i$, $B_i(t) = 0, \forall t \geq 1$. (The remaining elements of government policy adjust to satisfy the government budget constraints.) Assume that the exogenous components of policy have been set so that an equilibrium with two valued fiat currencies exists. Under this description of policy, show that the equilibrium exchange rate is indeterminate.

d. Suppose that government policy in each country is described as follows: $G_i(t) = G_i, T_i(t) = T_i, H_i(t+1) = H_i(1), B_i(t) = B_i(1)\ \forall t \geq 1$. Show that if there exists an equilibrium with two valued fiat currencies, the exchange rate is determinate.

e. Suppose that government policy in country 1 is specified in terms of exogenous levels of $s_1 = [H_1(t) - H_1(t-1)]/p_1(t)\ \forall t \geq 2$, and $G_1(t) = G_1\ \forall t \geq 1$. For country 2, government policy consists of exogenous levels of $B_2(t) = B_2(1), G_2(t) = G_2 \forall t \geq 1$. Show that if there exists an equilibrium with two valued fiat currencies, then the exchange rate is determinate.

Exercise 9.7 **Credit controls**

Consider the following overlapping generations model. At each date $t \geq 1$ there appear N two-period-lived young people, said to be of generation t, who live and consume during periods t and $(t+1)$. At time $t = 1$ there exist N old people who are endowed with $H(0)$ units of paper "dollars," which they offer to supply inelastically to the young of generation 1 in exchange for goods. Let $p(t)$ be the price of the one good in the model, measured in dollars per time t good. For each $t \geq 1$, $N/2$ members of generation t are endowed with $y > 0$ units of the good at t and 0 units at $(t+1)$, whereas the remaining $N/2$ members of generation t are endowed with 0 units of the good at t and $y > 0$ units when they are old. All members of all generations have the same utility function:

$$u[c_t^h(t), c_t^h(t+1)] = \ln c_t^h(t) + \ln c_t^h(t+1),$$

where $c_t^h(s)$ is the consumption of agent h of generation t in period s. The old at $t = 1$ simply maximize $c_0^h(1)$. The consumption good is nonstorable. The currency supply is constant through time, so $H(t) = H(0)$, $t \geq 1$.

a. Define a competitive equilibrium without valued currency for this model. Who trades what with whom?

b. In the equilibrium without valued fiat currency, compute competitive equilibrium values of the gross return on consumption loans, the consumption allocation of the old at $t = 1$, and that of the "borrowers" and "lenders" for $t \geq 1$.

c. Define a competitive equilibrium with valued currency. Who trades what with whom?

d. Prove that for this economy there does not exist a competitive equilibrium with valued currency.

e. Now suppose that the government imposes the restriction that $l_t^h(t)[1 + r(t)] \geq -y/4$, where $l_t^h(t)[1 + r(t)]$ represents claims on $(t+1)$–period consumption purchased (if positive) or sold (if negative) by household h of generation t. This is a restriction on the amount of borrowing. For an equilibrium without valued currency, compute the consumption allocation and the gross rate of return on consumption loans.

f. In the setup of part e, show that there exists an equilibrium with valued currency in which the price level obeys the quantity theory equation $p(t) = qH(0)/N$. Find a formula for the undetermined coefficient q. Compute the consumption allocation and the equilibrium rate of return on consumption loans.

g. Are lenders better off in economy b or economy f? What about borrowers? What about the old of period 1 (generation 0)?

Exercise 9.8 **Inside money and real bills**

Consider the following overlapping generations model of two-period-lived people. At each date $t \geq 1$ there are born N_1 individuals of type 1 who are endowed with $y > 0$ units of the consumption good when they are young and zero units when they are old; there are also born N_2 individuals of type 2 who are endowed with zero units of the consumption good when they are young and $Y > 0$ units when they are old. The consumption good is nonstorable. At time $t = 1$, there are N old people, all of the same type, each endowed with zero units of the consumption good and H_0/N units of unbacked paper called "fiat currency." The populations of type 1 and 2 individuals, N_1 and N_2, remain constant for all $t \geq 1$. The young of each generation are identical in preferences and maximize the utility function $\ln c_t^h(t) + \ln c_t^h(t+1)$ where $c_t^h(s)$ is consumption in the sth period of a member h of generation t.

a. Consider the equilibrium without valued currency (that is, the equilibrium in which there is no trade between generations). Let $[1+r(t)]$ be the gross rate of return on consumption loans. Find a formula for $[1+r(t)]$ as a function of N_1, N_2, y, and Y.

b. Suppose that N_1, N_2, y, and Y are such that $[1+r(t)] > 1$ in the equilibrium without valued currency. Then prove that there can exist no quantity-theory-style equilibrium where fiat currency is valued and where the price level $p(t)$ obeys the quantity theory equation $p(t) = q \cdot H_0$, where q is a positive constant and $p(t)$ is measured in units of currency per unit good.

c. Suppose that N_1, N_2, y, and Y are such that in the nonvalued-currency equilibrium $1 + r(t) < 1$. Prove that there exists an equilibrium in which fiat currency is valued and that there obtains the quantity theory equation $p(t) = q \cdot H_0$, where q is a constant. Construct an argument to show that the equilibrium with valued currency is not Pareto superior to the nonvalued-currency equilibrium.

d. Suppose that N_1, N_2, y, and Y are such that, in the preceding nonvalued-currency economy, $[1 + r(t)] < 1$, there exists an equilibrium in which fiat currency is valued. Let $\bar{p}$ be the stationary equilibrium price level in that economy. Now consider an alternative economy, identical with the preceding one in all respects except for the following feature: a government each period purchases a constant amount L_g of consumption loans and pays for them by issuing debt on itself, called "inside money" M_I, in the amount $M_I(t) = L_g \cdot p(t)$. The government never retires the inside money, using the proceeds of the loans to finance new purchases of consumption loans in subsequent periods. The quantity of outside money, or currency, remains H_0, whereas the "total high-power money" is now $H_0 + M_I(t)$.

(i) Show that in this economy there exists a valued-currency equilibrium in which the price level is constant over time at $p(t) = \bar{p}$, or equivalently, with $\bar{p} = qH_0$ where q is defined in part c.

(ii) Explain why government purchases of private debt are not inflationary in this economy.

(iii) In many models, once-and-for-all government open-market operations in private debt normally affect real variables and/or price level. What accounts for the difference between those models and the one in this exercise?

Exercise 9.9 **Social security and the price level**

Consider an economy ("economy I") that consists of overlapping generations of two-period-lived people. At each date $t \geq 1$ there is born a constant number N of young people, who desire to consume both when they are young, at t, and when they are old, at $(t+1)$. Each young person has the utility function $\ln c_t(t) + \ln c_t(t+1)$, where $c_s(t)$ is time t consumption of an agent born at s. For all dates $t \geq 1$, young people are endowed with $y > 0$ units of a single nonstorable consumption good when they are young and zero units when they are old. In addition, at time $t = 1$ there are N old people endowed in the aggregate with H units of unbacked fiat currency. Let $p(t)$ be the nominal price level at t, denominated in dollars per time t good.

a. Define and compute an equilibrium with valued fiat currency for this economy. Argue that it exists and is unique. Now consider a second economy ("economy II") that is identical to economy I except that economy II possesses a social security system. In particular, at each date $t \geq 1$, the government taxes $\tau > 0$ units of the time t consumption good away from each young person and at the same time gives τ units of the time t consumption good to each old person then alive.

b. Does economy II possess an equilibrium with valued fiat currency? Describe the restrictions on the parameter τ, if any, that are needed to ensure the existence of such an equilibrium.

c. If an equilibrium with valued fiat currency exists, is it unique?

d. Consider the *stationary* equilibrium with valued fiat currency. Is it unique? Describe how the value of currency or price level would vary across economies with differences in the size of the social security system, as measured by τ.

Exercise 9.10 **Seignorage**

Consider an economy consisting of overlapping generations of two-period-lived agents. At each date $t \geq 1$, there are born N_1 "lenders" who are endowed with $\alpha > 0$ units of the single consumption good when they are young and zero units when they are old. At each date $t \geq 1$, there are also born N_2 "borrowers" who are endowed with zero units of the consumption good when they are young and $\beta > 0$ units when they are old. The good is nonstorable, and N_1 and N_2 are constant through time. The economy starts at time 1, at which time there are N old people who are in the aggregate endowed with $H(0)$ units of unbacked, intrinsically worthless pieces of paper called dollars. Assume that α, β, N_1, and N_2 are such that

$$\frac{N_2 \beta}{N_1 \alpha} < 1.$$

Assume that everyone has preferences

$$u[c_t^h(t), c_t^h(t+1)] = \ln c_t^h(t) + \ln c_t^h(t+1),$$

where $c_t^h(s)$ is consumption of time s good of agent h born at time t.

a. Compute the equilibrium interest rate on consumption loans in the equilibrium without valued currency.

b. Construct a *brief* argument to establish whether or not the equilibrium without valued currency is Pareto optimal.

The economy also contains a government that purchases and destroys G_t units of the good in period t, $t \geq 1$. The government finances its purchases entirely by currency creation. That is, at time t,

$$G_t = \frac{H(t) - H(t-1)}{p(t)},$$

where $[H(t) - H(t-1)]$ is the additional dollars printed by the government at t and $p(t)$ is the price level at t. The government is assumed to increase $H(t)$ according to

$$H(t) = zH(t-1), \qquad z \geq 1,$$

where z is a constant for all time $t \geq 1$.

At time t, old people who carried $H(t-1)$ dollars between $(t-1)$ and t offer these $H(t-1)$ dollars in exchange for time t goods. Also at t the government offers $H(t) - H(t-1)$ dollars for goods, so that $H(t)$ is the total supply of dollars at time t, to be carried over by the young into time $(t+1)$.

c. Assume that $1/z > N_2\beta / N_1\alpha$. Show that under this assumption there exists a continuum of equilibria with valued currency.

d. Display the unique stationary equilibrium with valued currency in the form of a "quantity theory" equation. Compute the equilibrium rate of return on currency and consumption loans.

e. Argue that if $1/z < N_2\beta/N_1\alpha$, then there exists no valued-currency equilibrium. Interpret this result. (*Hint:* Look at the rate of return on consumption loans in the equilibrium without valued currency.)

f. Find the value of z that *maximizes* the government's G_t in a stationary equilibrium. Compare this with the largest value of z that is compatible with the existence of a valued-currency equilibrium.

Exercise 9.11 **Unpleasant monetarist arithmetic**

Consider an economy in which the aggregate demand for government currency for $t \geq 1$ is given by $[M(t)p(t)]^d = g[R_1(t)]$, where $R_1(t)$ is the gross rate of return on currency between t and $(t+1)$, $M(t)$ is the stock of currency at t, and $p(t)$ is the value of currency in terms of goods at t (the reciprocal of the price level). The function $g(R)$ satisfies

$$(1) \qquad g(R)(1-R) = h(R) > 0 \qquad \text{for} \quad R \in (\underline{R}, 1),$$

where $h(R) \leq 0$ for $R < \underline{R}, R \geq 1, \underline{R} > 0$ and where $h'(R) < 0$ for $R > R_m$, $h'(R) > 0$ for $R < R_m$ $h(R_m) > D$, where D is a positive number to be defined shortly. The government faces an infinitely elastic demand for its interest-bearing bonds at a constant-over-time gross rate of return $R_2 > 1$. The government finances a budget deficit D, defined as government purchases minus explicit taxes, that is constant over time. The government's budget constraint is

$$(2) \qquad D = p(t)[M(t) - M(t-1)] + B(t) - B(t-1)R_2, \qquad t \geq 1,$$

subject to $B(0) = 0, M(0) > 0$. In equilibrium,

$$(3) \qquad M(t)p(t) = g[R_1(t)].$$

The government is free to choose paths of $M(t)$ and $B(t)$, subject to equations (2) and (3).

a. Prove that, for $B(t) = 0$, for all $t > 0$, there exist two stationary equilibria for this model.

b. Show that there exist values of $B > 0$, such that there exist stationary equilibria with $B(t) = B$, $M(t)p(t) = Mp$.

c. Prove a version of the following proposition: among stationary equilibria, the lower the value of B, the lower the stationary rate of inflation consistent with equilibrium. (You will have to make an assumption about Laffer curve effects to obtain such a proposition.)

This problem displays some of the ideas used by Sargent and Wallace (1981). They argue that, under assumptions like those leading to the proposition stated in part c, the "looser" money is today [that is, the higher $M(1)$ and the lower $B(1)$], the lower the stationary inflation rate.

Exercise 9.12 **Grandmont-Hall**

Consider a nonstochastic, one-good overlapping generations model consisting of two-period-lived young people born in each $t \geq 1$ and an initial group of old people at $t = 1$ who are endowed with $H(0) > 0$ units of unbacked currency at the beginning of period 1. The one good in the model is not storable. Let the aggregate first-period saving function of the young be time-invariant and be denoted $f[1 + r(t)]$ where $[1 + r(t)]$ is the gross rate of return on consumption loans between t and $(t + 1)$. The saving function is assumed to satisfy $f(0) = -\infty$, $f'(1 + r) > 0$, $f(1) > 0$.

Let the government pay interest on currency, starting in period 2 (to holders of currency between periods 1 and 2). The government pays interest on currency at a nominal rate of $[1 + r(t)]p(t + 1)/\bar{p}$, where $[1 + r(t)]$ is the real gross rate of return on consumption loans, $p(t)$ is the price level at t, and $\bar{p}$ is a target price level chosen to satisfy

$$\bar{p} = H(0)/f(1).$$

The government finances its interest payments by printing new money, so that the government's budget constraint is

$$H(t + 1) - H(t) = \left\{ [1 + r(t)] \frac{p(t + 1)}{\bar{p}} - 1 \right\} H(t), \qquad t \geq 1,$$

given $H(1) = H(0) > 0$. The gross rate of return on consumption loans in this economy is $1 + r(t)$. In equilibrium, $[1 + r(t)]$ must be at least as great as the real rate of return on currency

$$1 + r(t) \geq [1 + r(t)]p(t)/\bar{p} = [1 + r(t)] \frac{p(t + 1)}{\bar{p}} \frac{p(t)}{p(t + 1)}$$

with equality if currency is valued,

$$1 + r(t) = [1 + r(t)]p(t)/\bar{p}, \qquad 0 < p(t) < \infty.$$

The loan market-clearing condition in this economy is

$$f[1 + r(t)] = H(t)/p(t).$$

a. Define an equilibrium.

b. Prove that there exists a unique monetary equilibrium in this economy and compute it.

Exercise 9.13 **Bryant-Keynes-Wallace**

Consider an economy consisting of overlapping generations of two-period-lived agents. There is a constant population of N young agents born at each date $t \geq 1$. There is a single consumption good that is not storable. Each agent born in $t \geq 1$ is endowed with w_1 units of the consumption good when young and with w_2 units when old, where $0 < w_2 < w_1$. Each agent born at $t \geq 1$ has identical preferences $\ln c_t^h(t) + \ln c_t^h(t+1)$, where $c_t^h(s)$ is time s consumption of agent h born at time t. In addition, at time 1, there are alive N old people who are endowed with $H(0)$ units of unbacked paper currency and who want to maximize their consumption of the time 1 good.

A government attempts to finance a constant level of government purchases $G(t) = G > 0$ for $t \geq 1$ by printing new base money. The government's budget constraint is

$$G = [H(t) - H(t-1)]/p(t),$$

where $p(t)$ is the price level at t, and $H(t)$ is the stock of currency carried over from t to $(t+1)$ by agents born in t. Let $g = G/N$ be government purchases per young person. Assume that purchases $G(t)$ yield no utility to private agents.

a. Define a stationary equilibrium with valued fiat currency.

b. Prove that, for g sufficiently small, there exists a stationary equilibrium with valued fiat currency.

c. Prove that, in general, if there exists one stationary equilibrium with valued fiat currency, with rate of return on currency $1 + r(t) = 1 + r_1$, then there exists at least one other stationary equilibrium with valued currency with $1+r(t) = 1+r_2 \neq 1+r_1$.

d. Tell whether the equilibria described in parts b and c are Pareto optimal, among allocations among private agents of what is left after the government takes $G(t) = G$ each period. (A proof is not required here: an informal argument will suffice.)

Now let the government institute a forced saving program of the following form. At time 1, the government redeems the outstanding stock of currency $H(0)$, exchanging it for government bonds. For $t \geq 1$, the government offers each young consumer the option of saving at least F worth of time t goods in the form of bonds bearing a constant rate of return $(1+r_2)$. A legal prohibition against private intermediation is instituted that prevents two or more private agents from sharing one of these bonds. The government's budget constraint for $t \geq 2$ is

$$G/N = B(t) - B(t-1)(1+r_2),$$

where $B(t) \geq F$. Here $B(t)$ is the saving of a young agent at t. At time $t=1$, the government's budget constraint is

$$G/N = B(1) - \frac{H(0)}{Np(1)},$$

where $p(1)$ is the price level at which the initial currency stock is redeemed at $t=1$. The government sets F and r_2.

Consider stationary equilibria with $B(t) = B$ for $t \geq 1$ and r_2 and F constant.

e. Prove that if g is small enough for an equilibrium of the type described in part a to exist, then a stationary equilibrium with forced saving exists. (Either a graphical argument or an algebraic argument is sufficient.)

f. Given g, find the values of F and r_2 that maximize the utility of a representative young agent for $t \geq 1$.

g. Is the equilibrium allocation associated with the values of F and $(1+r_2)$ found in part f optimal among those allocations that give $G(t) = G$ to the government for all $t \geq 1$? (Here an informal argument will suffice.)

2. OVERLAPPING GENERATIONS

- Competitive equilibria may not be Pareto optimal
- There may exist a continuum of equilibria
- Fiat money may have positive value

Paul Samuelson 1958 JPE "An exact consumption-loan of interest with or without the social contrivance of money".

Maurice Allais 1947 "Economie et Intérêt"

Demographics :

- Agents live for 2 periods
- At each date $t \geq 1$, there are N_t young agents born
- Population at time t : $N_{t-1} + N_t$. For now, $N_t = N$

Preferences: of agent in generation i

$$u(c_i^i) + u(c_{i+1}^i)$$

Initial old: $u(c_1^0)$

Time / Generation	1	2	3
0	c_1^0		
1	c_1^1	c_2^1	
2		c_2^2	c_3^2
3			c_3^3

Endowment: Stationary endowment

$$y_i^i = 1-\varepsilon \qquad \varepsilon \in [0,1]$$
$$y_{i+1}^i = \varepsilon$$
$$y_t^i = 0 \quad \forall t \neq i, i+1$$

Technology: perishable

Time - 0 trading equilibrium

One change from before : "invisible hand" is allowed to throw away goods in the first period 1.

$$c_1^0 + c_1^1 + \Omega_1 = y_1^0 + y_1^1$$

$\Omega_1 \geq 0$ is thrown away.

Find equilibrium prices $\{q_t^0\}_{t=1}^{\infty}$ and consumption allocation $\{c_t^i\}_{\forall i, t\geq 1}$

Generation i

$$\max \quad u(c_i^i) + u(c_{i+1}^i)$$

$$\text{s.t.} \quad \underbrace{\sum_{t=1}^{\infty} q_t^0 \, c_t^i}_{q_i^0} \leq \underbrace{\sum_{t=1}^{\infty} q_t^0 \, y_t^i}_{q_i^0} \quad \leftarrow \text{in terms of prices of goods when young}$$

FOC

$$u'(c_i^i) = \mu^i q_i^0$$

$$u'(c_{i+1}^i) = \mu^i q_{i+1}^0$$

$$c_t^i = 0 \qquad \forall t \neq i, i+1$$

Define $\alpha_i = \dfrac{q_{i+1}^0}{q_i^0}$

Old agent at $t=1$

$$\max \quad u(c_1^0)$$

$$\text{s.t.} \quad q_1^0 c_1^0 \leq q_1^0 y_1^0 \quad \Longrightarrow \quad c_1^0 = y_1^0 \overset{!}{=} \varepsilon$$

Feasibility (market clearing) : $c_i^i + c_i^{i-1} \leq y_i^i + y_i^{i-1} \overset{!}{=} 1$

Case 1 : $\varepsilon \geq 0.5$ (a lot when old) $\longrightarrow$ unique competitive equilibrium that is Pareto optimal

Case 2 : $\varepsilon < 0.5$ (a lot when young) - 2 stationary equilibria
- continuum of non-stationary equilibria

Def (stationary equilibrium) :

$$c_i^i = c_y \quad , \quad c_{i+1}^i = c_o \qquad \forall i \geq 1$$

c_1^0 might be different

Guess and verify

① $c_i^i = c_{i+1}^i = 0.5$

From FOC's, $\alpha_i = \dfrac{q_{i+1}^0}{q_i^0} = \dfrac{u'(c_{i+1}^i)}{u'(c_i^i)} = 1$

Goods thrown?

$$\Omega_1 = 1 - \frac{1}{2} - \varepsilon = \frac{1}{2} - \varepsilon > 0$$

② $c_i^i = 1-\varepsilon$, $c_{i+1}^i = \varepsilon$

We know that $\Omega_1 = 0$ (everyone eats their endowment)

From FOC's,

$$\alpha_t = \frac{q_{t+1}^0}{q_t^0} = \frac{u'(\varepsilon)}{u'(1-\varepsilon)}$$

Non-stationary equilibria

David Gale offer-curve approach to equilibrium computation

Generation i: $\quad \max \; u(c_i^i) + u(c_{i+1}^i)$

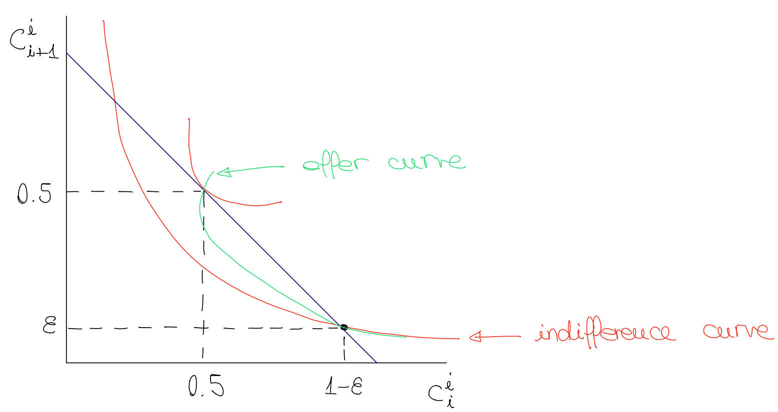
$$\text{s.t.} \quad c_i^i + \alpha_i c_{i+1}^i \leq y_i^i + \alpha_i y_{i+1}^i \qquad (\star)$$

FOC

$$\frac{u'(c_{i+1}^i)}{u'(c_i^i)} = \alpha_i \qquad (\star\star)$$

Offer curve: vary $\alpha_i \in (0, \infty)$ and obtain (c_i^i, c_{i+1}^i) for given α_i

Offer curve solves $(\star)$ s.t. equality and $(\star\star)$. Solution: $\Psi(c_i^i, c_{i+1}^i)$



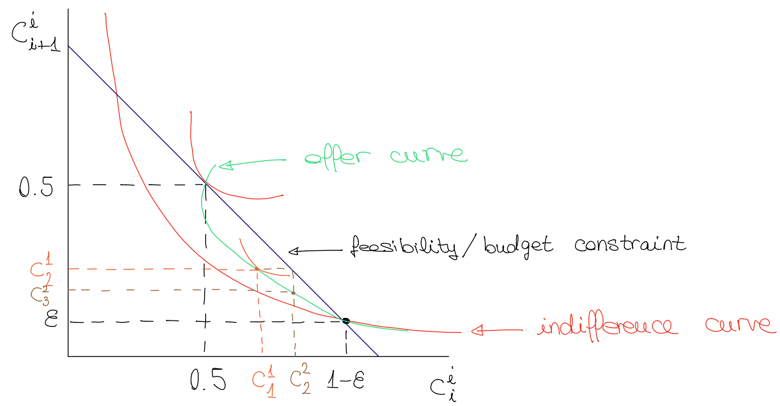
Dynamic equilibrium calculations

a) Start with any $c_1^1 \in \left(\frac{1}{2}, 1-\varepsilon\right)$

Is there any price α_1 for which c_1^1 is the optimal choice ?

Yes, read it from the offer curve $\Omega_1 = 1 - c_1^1 - \varepsilon > 0$

one can see that this equilibrium converges to the endowment (autarky) equilibrium



b) Compute c_2^2 from $c_2^1 + c_2^2 = 1$.

c) Is there any price α_2 for which c_2^2 is the optimal choice of generation 2?

Recover time-0 trading prices

$$q_1^0 = 1$$

$$q_2^0 = \alpha_1 q_1^0 = \frac{u'(c_2^1)}{u'(c_1^1)}$$

$$q_3^0 = \alpha_2 q_2^0 = \frac{u'(c_3^2)}{u'(c_2^2)} q_2^0$$

Logarithmic preferences

$$\max \quad \log c_i^i + \log c_{i+1}^i$$

$$\text{s.t.} \quad c_i^i + \alpha_i c_{i+1}^i \leq y_i^i + \alpha_i y_{i+1}^i \tag{1}$$

FOC:

$$c_i^i : \quad \frac{1}{c_i^i} = \mu^i \tag{2}$$

$$c_{i+1}^i : \quad \frac{1}{c_{i+1}^i} = \mu^i \alpha_i \tag{3}$$

$$\Bigg\} \quad c_{i+1}^i = \frac{c_i^i}{\alpha_i}$$

From (1)-(3)

$$c_i^i = \frac{1}{2}\left(y_i^i + \alpha_i \, y_{i+1}^i \right)$$

$$c_{i+1}^i = \frac{1}{2\alpha_i}\left(y_i^i + \alpha_i \, y_{i+1}^i \right)$$

Feasibility

$$c_i^i + c_i^{i-1} = y_i^i + y_i^{i-1} = 1 \qquad \forall i \geq 2$$

$$\Rightarrow \frac{1}{2}\left[1-\varepsilon+\alpha_i\varepsilon\right] + \frac{1}{2\alpha_{i-1}}\left[1-\varepsilon+\alpha_{i-1}\varepsilon\right] = 1$$

Equilibrium difference equation

$$\alpha_i \, \varepsilon + \frac{1-\varepsilon}{\alpha_{i-1}} - 1 = 0$$

Find two stationary equilibria

$$\alpha_i = \alpha_{i-1} = \alpha$$

$$\alpha^2 - \frac{\alpha}{\varepsilon} + \left(\frac{1}{\varepsilon}-1\right) = 0 \Longrightarrow \alpha = \frac{\frac{1}{\varepsilon} \pm \sqrt{\left(\frac{1}{\varepsilon}\right)^2 - 4\left(\frac{1}{\varepsilon}-1\right)}}{2} = \frac{\frac{1}{\varepsilon} \pm \cancel{\sqrt{}}\left(\frac{1}{\varepsilon}-2\right)^{\cancel{2}}}{2} = \begin{cases} 1 \\ \frac{1}{\varepsilon}-1 \end{cases}$$

Sequential trading and add a government

τ_t^i lump-sum tax on agent i in period t

b_{t+1} government bonds issued at time t and pay $R_t b_{t+1}$ at time $t+1$

R_t one-period interest rate between t and $t+1$

(in book $\frac{B_{t+1}}{R_t} = b_{t+1}$)

Government budget constraint

$$b_{t+1} + \tau_t^{t-1} + \tau_t^t = R_{t-1} b_t$$

Consider tax and deficit policy

$$\tau_1^0 \leq 0$$

$$\tau_t^t = \tau_{t+1}^t = 0 \qquad \forall t \geq 1$$

Initial condition $b_1 = 0$

Consider an equilibrium with valued government debt

$$\Longrightarrow \quad b_2 + \tau_1^o = 0$$

$$b_{t+1} = R_{t-1} b_t \qquad \forall t \geq 2$$

Bond market clearing

$$\text{Demand} = \text{Supply}$$

young

$$y_t^t - c_t^t = b_{t+1}$$

Log preferences : $c_i^i = \frac{1}{2}\left[y_i^i + \alpha_i y_{i+1}^i \right]$; $\alpha_i = \frac{1}{R_i}$

In this case, demand = supply $\Rightarrow$

$$\Rightarrow \quad 1-\varepsilon - \frac{1}{2}\left[1-\varepsilon + \frac{\varepsilon}{R_t}\right] = b_{t+1} \Longrightarrow (1-\varepsilon)R_t - \varepsilon = 2R_t b_{t+1}$$

<u>Equilibrium conditions</u>

$$b_2 = -\tau_1^o \geq 0 \qquad (1) \qquad t=1$$

$$b_{t+1} = R_{t-1} b_t \qquad (2) \qquad \forall t \geq 2$$

$$R_t = \frac{\varepsilon}{1-\varepsilon-2b_{t+1}} \qquad (3) \qquad \forall t \geq 1$$

One solution to this system

$$b_2 = -\tau_1^o = 0$$

$$b_{t+1} = R_{t-1} \cdot 0 = 0 \qquad \forall t \geq 2$$

$$R_t = \frac{\varepsilon}{1-\varepsilon} = \frac{u'(c_t^t)}{u'(c_{t+1}^t)} = \frac{u'(1-\varepsilon)}{u'(\varepsilon)}$$

If $\varepsilon \geq 0.5$, then there are no other equilibria (efficient)

If $\varepsilon < 0.5$ $\left(\frac{\varepsilon}{1-\varepsilon} < 1\right)$, then there exists a continuum of equilibria

<u>Suppose that $\frac{\varepsilon}{1-\varepsilon} < 1$</u>

Guess $R_t = 1$ $\forall t \geq 1$ and verify

From (3) $\quad R_t = 1 = \dfrac{\varepsilon}{1-\varepsilon-2b_{t+1}} \qquad \forall t \geq 1$

$$\Longrightarrow \quad b_{t+1} = \frac{1-2\varepsilon}{2} \overset{\equiv \bar{b}}{>} 0$$

From (2),

$$b_{t+1} = R_{t-1}\, b_t = b_t \qquad \forall t \geq 2$$

From (1),

$$b_2 = -\tau_1^o > 0$$

Pick $-\tau_1^o \in \left(0, \frac{1-2\varepsilon}{2}\right)$ (recall that $b_2 = -\tau_1^o$, $b_{t+1} = b_t = \frac{1}{2} - \varepsilon \ \forall t \geq 2 \Rightarrow -\tau_1^o = \frac{1}{2} - \varepsilon$)

$$\Longrightarrow \quad b_2 = -\tau_1^o$$

since $-\tau_1^o < \frac{1}{2} - \varepsilon$ where $R_1 = \frac{\varepsilon}{1-\varepsilon+2\tau_1^o}$

$$\Longrightarrow \quad \frac{\varepsilon}{1-\varepsilon} < R_1 < 1$$

$\tau_1^o < 0$

$$\Longrightarrow \quad b_3 = R_1 b_2 < b_2$$

$$\Longrightarrow \quad R_2 < R_1$$

$$\Longrightarrow \quad \dots$$

R_t is a monotone decreasing sequence, bounded by $\frac{\varepsilon}{1-\varepsilon}$

$$(b_{t+1}, R_t) \xrightarrow[t\to\infty]{} \left(0, \frac{\varepsilon}{1-\varepsilon}\right)$$

(if $\varepsilon < 0.5$).

<u>Suppose $\varepsilon \geq 0.5$</u>, i.e, $\frac{\varepsilon}{1-\varepsilon} \geq 1$

Try $-\tau_1^o > 0$

$$\Longrightarrow \text{ by (1)} \quad b_2 = -\tau_1^o > 0$$

satisfied if $\varepsilon \geq 0.5$!

$$\Longrightarrow \text{ by (3)} \quad R_1 > 1 \quad \longleftarrow \quad R_1 = \frac{\varepsilon}{1-\varepsilon+2\tau_1^o} > 1 \Rightarrow -\tau_1^o > \frac{1}{2} - \varepsilon$$

$$\Longrightarrow \quad b_3 = R_1 b_2 > b_2$$

$$\Longrightarrow \quad R_2 > R_1 > 1$$

$$\Longrightarrow \quad \dots$$

$$y_t^t - c_t^t = \overbrace{\frac{1}{2}\left[1 - \varepsilon - \frac{\varepsilon}{R_t}\right]}^{\text{Demand}} = \overbrace{b_{t+1}}^{\text{Supply}}$$

LHS bounded by $\frac{1-\varepsilon}{2}$ RHS $\longrightarrow \infty$

$\therefore$ there exists no equilibrium with $-\tau_1^o > 0$

Money

Interpret equilibrium with valued unbacked government debt as a fiat money equilibrium

M_t stock of dollars at time t

P_t nominal price of the good at time t

$1/P_t$ price of money in term of goods at time t

$$\frac{M_t}{P_t} = b_{t+1} \qquad R_t = \frac{\text{real value of money at } t+1}{\text{real value of money at } t} = \frac{1/P_{t+1}}{1/P_t} = \frac{P_t}{P_{t+1}}$$

Optimization problem by agent t, who purchases currency in period t, m_t^t

$$\left.\begin{aligned} c_t^t + \frac{m_t^t}{P_t} &\leq y_t^t \qquad (10) \\ c_{t+1}^t &= y_{t+1}^t + \frac{m_t^t}{P_{t+1}} \qquad (11) \end{aligned}\right\} \quad c_t^t + c_{t+1}^t \frac{P_{t+1}}{P_t} \leq y_t^t + y_{t+1}^t \frac{P_{t+1}}{P_t}$$

Same as before

$$\frac{P_{t+1}}{P_t} = \alpha_t = \frac{1}{R_t} = \frac{q_{t+1}^0}{q_t^0}$$

Government budget constraint

Before: $b_{t+1} + \tau_t^{t-1} + \tau_t^t = R_{t-1} b_t$

After: $\frac{M_t}{P_t} + \tau_t^{t-1} + \tau_t^t = \frac{P_{t-1}}{P_t}\frac{M_{t-1}}{P_{t-1}} \Rightarrow \frac{M_t - M_{t-1}}{P_t} + \tau_t^{t-1} + \tau_t^t = 0$

Seignorance "inflation tax".

Equilibrium conditions

$$\frac{M_1}{P_1} = -\tau_1^0 \qquad (1^M)$$

$$\frac{M_t}{P_t} = \frac{P_{t-1}}{P_t}\frac{M_{t-1}}{P_{t-1}} \qquad (2^M) \qquad \Rightarrow M_t = M_{t-1} \equiv M_1$$

$$\frac{P_t}{P_{t+1}} = \frac{\varepsilon}{1-\varepsilon-2\frac{M_t}{P_t}} \qquad (3^M)$$

Substitute (2^M) $[M_t = M_{t-1}]$ into (3^M)

$$\left[1-\varepsilon-2\frac{M_1}{P_t}\right]\frac{P_t}{P_{t+1}}=\varepsilon \longrightarrow 2M_1=(1-\varepsilon)P_t-\varepsilon P_{t+1}$$

first-order difference equation with invariant forcing term

Using the approach from Maths III,

$$P_{t+1}=\frac{1-\varepsilon}{\varepsilon}P_t-\frac{2M_1}{\varepsilon} \Rightarrow P_t=\frac{1-\varepsilon}{P_{t-1}}-\frac{2M_1}{\varepsilon}$$

we know that $x_t=ax_{t-1}+b \Rightarrow x_t=a^t\left(x_0-\frac{b}{1-a}\right)+\frac{b}{1-a}$. Hence,

$$P_t=\left(\frac{1-\varepsilon}{\varepsilon}\right)^t\underbrace{\left(P_0+\frac{2M_1}{2\varepsilon-1}\right)}_{k}-\frac{2M_1}{2\varepsilon-1}$$

Lag operator: $Lx_t=x_{t-1}$

$L^{-1}x_t=x_{t+1}$

$$2M_1=\left[1-\varepsilon+\varepsilon L^{-1}\right]P_t \Rightarrow \frac{2M_1}{1-\varepsilon}=\left[1-\frac{\varepsilon}{1-\varepsilon}L^{-1}\right]P_t \Rightarrow P_t=\frac{2M_1}{(1-\varepsilon)\left[1-\frac{\varepsilon}{1-\varepsilon}\right]}+k\left(\frac{1-\varepsilon}{\varepsilon}\right)^t$$

for $k\in[0,\infty)$

check the mathematician

Don't follow $\rightarrow$ $$\left[1-\frac{\varepsilon}{1-\varepsilon}L^{-1}\right]k\left(\frac{1-\varepsilon}{\varepsilon}\right)^t=k\left(\frac{1-\varepsilon}{\varepsilon}\right)^t-\frac{\varepsilon}{1-\varepsilon}k\left(\frac{1-\varepsilon}{\varepsilon}\right)^{t+1}=0$$?

There exists a continuum of monetary equilibria indexed by $k\in[0,\infty)$.

There exists a stationary equilibrium with $P_t=P$ $\forall t\geq 1$, i.e, for $k=0$

Different values of $k \longleftrightarrow \frac{M_1}{P_1}$

$b_2\in(0,\bar{b})$

$$P=\frac{1-\varepsilon}{\varepsilon}P-\frac{2M_1}{\varepsilon} \Rightarrow P=\frac{2M_1}{1-2\varepsilon}$$

introducing in P_0 (into k), $k=0$!

Deficit finance

Government wants to finance $g>0$ expenditures in every period without taxing

Government budget constraint

$$b_{t+1}+\tau_t^{t-1}+\tau_t^t=R_{t-1}b_t+g_t$$

Recall earlier policy

$$-\tau_1^0=b_2\in[0,\bar{b}]$$

$$\tau_t^t=\tau_{t+1}^t=0 \qquad \forall t\geq 1$$

$$g_t=0$$

$$\left.\begin{array}{l} b_{t+1} = R_{t-1} b_t \\ \text{Bond market} \\ f(R_t) = b_{t+1} \\ \text{Saving function} \end{array}\right\} \quad \begin{array}{l} \text{Equilibrium} \\ f(R_t) = R_{t-1} f(R_{t-1}) \\ 0 = R_{autarky} \cdot 0 \end{array}$$

Equilibrium

if $b_2 = \bar{b} \implies R_t = 1$

$\uparrow$ $c_t^t = c_{t+1}^t = \frac{1}{2}$ (consumption smoothing)

if $b_2 \in [0, \bar{b}) \implies R_t < 1 \qquad (b_{t+1}, R_t) \longrightarrow (0, \underline{R})$ $R_{autarky}$

The debt is "amortized" over time when $R_t < 1$

$$\text{"amortization"} = b_t - b_{t+1} = b_t - R_{t-1} b_t = (1 - R_{t-1}) b_t > 0$$

<u>Now</u> $\hat{g}_t = g > 0 \quad \forall t \geq 1$

$$\hat{\tau}_t^t = \hat{\tau}_{t+1}^t = 0 \quad \forall t \geq 1$$

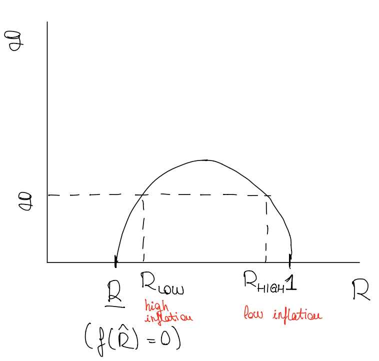
$$b_{t+1} = R_{t-1} b_t + g$$

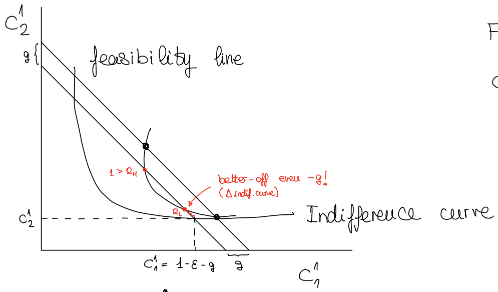
Suppose you have found a stationary equilibrium

$\hat{b} = \hat{R}\hat{b} + g$ government b.c.

$f(\hat{R}) = \hat{b}$ market clearing in bond market

$$\underbrace{\phantom{f(\hat{R}) = \hat{b}}}\quad f(\hat{R}) = \hat{R} f(\hat{R}) + g \implies f(\hat{R})[1 - \hat{R}] = g \longrightarrow \hat{b} = \frac{g}{1 - \hat{R}}$$





Feasibility

$$c_t^t + c_t^{t-1} = 1 - g$$

Recall 2 eqs per $g \longrightarrow$ other eq also

$$b_{t+1} \longrightarrow \dot{b}_{LOW}$$

$$R_t \longrightarrow \dot{R}_{LOW}$$

to start in R_{HIGH}, give initial old $\quad -\tau_1^0 = \hat{b}_{HIGH} - g$

How can the government attain R_H?

Policy $\quad -\tau_1^0 = \hat{b}_H - g$

$$\tau_t^t = \tau_{t+1}^t = 0 \quad \forall t \geq 1$$

Verify that this is an equilibrium

(1) $\quad b_2 + \tau_1^0 = g$

(2) $\quad b_{t+1} = R_{t-1} b_t + g$

(3) $\quad f(R_t) = b_{t+1}$

From (1): $\quad b_2 = -\tau_1^0 + g = b_H$

From (3) in $t=1$: $\quad f(R_1) = \hat{b}_H \implies R_1 = \hat{R}_H$

From (2): $\quad b_3 = \hat{R}_H \hat{b}_H + g = \hat{R}_H \dfrac{g}{1-\hat{R}_H} + g = \hat{b}_H \quad \checkmark$

Interpret stationary equilibrium with valued unbacked government debt as a fiat money equilibrium

Seignorage (real)

$$\frac{M_t - M_{t-1}}{P_t} = \frac{M_t - M_{t-1}}{M_t} \cdot \frac{M_t}{P_t}$$

Impose equilibrium: $\quad \underbrace{\dfrac{M_t - M_{t-1}}{M_t}}_{\text{inflation tax rate}} \; \underbrace{f\left(\dfrac{P_t}{P_{t+1}}\right)}_{\text{tax base}}$

Money market at time t

$$\left(\frac{M_t}{P_t}\right)^{demand} = f\left(\frac{P_t}{P_{t+1}}\right) = \left(\frac{M_t}{P_t}\right)^{supply} = \underbrace{\frac{M_{t-1}}{P_t}}_{\substack{\text{supply by}\\ \text{old agents}}} + \frac{M_t - M_{t-1}}{P_t}$$

Government budget constraint

$$\frac{M_1 - M_0}{P_1} + \tau_1^o = g \qquad \text{Initial condition} \quad M_0 = 0$$

$$\frac{M_t - M_{t-1}}{P_t} = g \longrightarrow \frac{M_t}{P_t} = \frac{M_{t-1}}{P_t}\frac{P_{t-1}}{P_{t-1}} + g$$

Impose equilibrium

$$\underbrace{f\left(\frac{P_t}{P_{t+1}}\right)}_{\frac{M_t}{P_t}} = \underbrace{f\left(\frac{P_{t-1}}{P_t}\right)}_{\frac{M_{t-1}}{P_{t-1}}} \frac{P_{t-1}}{P_t} + g$$

In a stationary equilibrium

$$\frac{P_t}{P_{t+1}} = \frac{P_{t-1}}{P_t} = R$$

$$f(R) = f(R)\,R + g \longrightarrow (1-R)f(R) = g$$

$$1 - R = 1 - \frac{1}{1+\pi} = \frac{\pi}{1+\pi} \qquad \text{"inflation tax rate"}$$

Fiscal theory of the price level

Many macroeconomic models do not have determinate predictions for the path of inflation: even for a given specification of $\{M_t\}$, many paths of inflation are consistent with equilibrium. According to FTPL, fiscal policy can be used to select which path occurs (Woodford, 1995)

Recall equilibrium conditions

$$\left.\begin{aligned} f\left(\frac{P_t}{P_{t+1}}\right) &= \frac{M_t}{P_t} \quad \forall t \geq 1 \\ \frac{M_t - M_{t-1}}{P_t} &= g \quad \forall t \geq 2 \end{aligned}\right\} \quad (1-R)f(R) = g \text{ determines } R = \frac{P_t}{P_{t+1}}$$

$$\frac{M_1}{P_1} + \tau_1^o = g$$

Recall $\frac{M_1}{P_1} = f(R) \implies P_1 = \frac{M_1}{f(R)}$

Assume logarithmic preferences

$$f(R_t) = (1-\varepsilon) - \frac{1}{2}\left(1-\varepsilon + \frac{\varepsilon}{R_t}\right)$$

Money market

$$f\left(\frac{P_t}{P_{t+1}}\right) = \frac{M_t}{P_t}$$

First-order difference equation

$$\left(1 - \frac{\varepsilon}{1-\varepsilon} L^{-1}\right) P_t = \frac{2}{1-\varepsilon} M_t$$

Scaler $\lambda \in (0,1)$,

$$1 + \lambda + \lambda^2 + \cdots = \frac{1}{1-\lambda}$$

$$1 + \lambda L + \lambda^2 L^2 + \cdots = \frac{1}{1-\lambda L}$$

don't follow $\rightarrow$ $P_t = \frac{2}{1-\varepsilon} \sum_{i=0}^{\infty} \left(\frac{\varepsilon}{1-\varepsilon}\right)^i M_{t+i} + K \left(\frac{1-\varepsilon}{\varepsilon}\right)^t$

Assume $M_{t+1} = z M_t$, $z > 1$.

Prop: $\frac{u'(1-\varepsilon)}{u'(\varepsilon)} \cdot z < 1$ is a necessary and sufficient condition for the existence of, at least, one monetary equilibrium (Wallis, 1980) p.336

In our case $\frac{\varepsilon}{1-\varepsilon} z < 1$.

$$P_t = \frac{\frac{2}{1-\varepsilon} M_t}{1 - \frac{\varepsilon}{1-\varepsilon} z} + K\left(\frac{1-\varepsilon}{\varepsilon}\right)^t = \frac{2 M_t}{1-\varepsilon-\varepsilon z} + K\left(\frac{1-\varepsilon}{\varepsilon}\right)^t$$

(permissible K are $K \geq 0$)

$$\frac{M_t}{P_t} = \frac{M_t}{\frac{2M_t}{1-\varepsilon-\varepsilon z} + K\left(\frac{1-\varepsilon}{\varepsilon}\right)^t} = \frac{1}{\frac{2}{1-\varepsilon-\varepsilon z} + K\left(\frac{1-\varepsilon}{\varepsilon}\right)^t \frac{1}{M_t}} = \frac{1}{\frac{2}{1-\varepsilon-\varepsilon z} + K\left(\frac{1-\varepsilon}{\varepsilon z}\right)^t \frac{z}{M_1}}$$

$M_t = z^{t-1} M_1$ $\qquad$ >1 by prop.

we must impose $K = 0$. Otherwise $\frac{M_t}{P_t} \to 0$

Stationary equilibrium with deficit finance

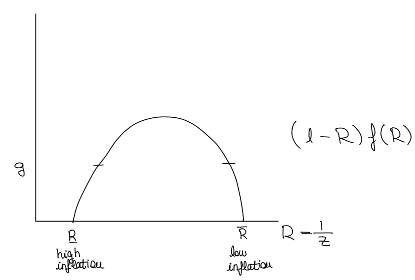
$$P_t = \frac{2M_t}{1-\varepsilon-\varepsilon z}$$

$$\frac{M_t}{P_t} = \frac{1-\varepsilon-\varepsilon z}{2}$$

Real return: $R_t = \frac{P_t}{P_{t+1}} = \frac{M_t}{M_{t+1}} = \frac{1}{z}$

Seigniorage in real terms

$$\frac{M_t - M_{t-1}}{P_t} = \frac{M_t}{P_t} - \frac{M_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t} = \left(1-\frac{P_{t-1}}{P_t}\right)\frac{1-\varepsilon-\varepsilon z}{2} = \left(1-\frac{1}{z}\right) f\left(\frac{1}{z}\right)$$



Overlapping generations in a production economy

Preference $u(c_i^i) + \beta u(c_{i+1}^i)$

Technology same as in the infinitely-lived agent economy with production

Endowment each young agent is endowed with 1 unit of time (used for leisure or work)

old agent at time 1 is endowed with the initial capital stock K_1

Demographics $N_{t+1} = n N_t$, $n>0$

$$C_t + K_{t+1} \leq F(K_t, L_t) + (1-\delta)K_t$$

$$C_t = c_t^t N_t + c_t^{t-1} N_{t-1}$$

Assume CRS technology: $F(K,L) = L\,F\left(\frac{K}{L}, 1\right) = L f(k)$
where $k = \frac{K}{L}$

Def: Given the initial condition that each old agent at time 1 owns $\frac{K_1}{N_0}$ units of capital, a sequential trading equilibrium is an allocation $\{c_t^t, c_{t+1}^t, s_t^t, K_{t+1}, L_t\}_{\forall t\geq 1}$ and c_1^0; and price $\{w_t, r_t\}_{\forall t\geq 1}$ such that

- Households maximize. For all $t\geq 1$, given $\{w_t, r_{t+1}\}$, $(c_t^t, c_{t+1}^t, s_t^t)$ solve

$$\left.\begin{aligned} &\max_{\{c_t^t, c_{t+1}^t, s_t^t\}} && u(c_t^t) + \beta\, u(c_{t+1}^t) \\ &\text{s.t.} && c_t^t + s_t^t \leq w_t\, 1 \\ & && c_{t+1}^t \leq (1 + r_{t+1} - \delta)\, s_t^t \\ & && c_t^t,\ c_{t+1}^t \geq 0 \end{aligned}\right\} \quad \begin{aligned} &\text{savings function} \\ &s(w_t, 1 + r_{t+1} - \delta) \end{aligned}$$

and for the initial old: given r_1, c_1^0 solves

$$\begin{aligned} &\max_{c_1^0} && u(c_1^0) \\ &\text{s.t.} && c_1^0 \leq (1 + r_1 - \delta)\frac{K_1}{N_0} \\ & && c_1^0 \geq 0 \end{aligned}$$

- Firms maximize. For all $t\geq 1$, given (r_t, w_t), (K_t, L_t) solve

$$\left.\begin{aligned} &\max_{K_t, L_t} && F(K_t, L_t) - r_t K_t - w_t L_t \\ &\text{s.t.} && K_t, L_t \geq 0 \end{aligned}\right\} \quad \begin{aligned} r_t &= F_K(K_t, L_t) \\ w_t &= F_L(K_t, L_t) \end{aligned}$$

- Markets clear

(Goods market) $\quad N_t^t c_t^t + N_t^{t-1} c_t^{t-1} + K_{t+1} = F(K_t, L_t) + (1-\delta) K_t$

(Asset market) $\quad N_t s_t^t = K_{t+1}$

(Labor market) $\quad N_t = L_t = 1$

Def: A stationary (or steady-state) equilibrium $(c_1^*, c_2^*, s^*, k^*, r^*, w^*)$ such that the allocation $\{\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t, \hat{K}_{t+1}, \hat{L}_t\}_{\forall t\geq 1}$ and c_1^0; and price $\{\hat{r}_t, \hat{w}_t\}_{\forall t\geq 1}$ defined by

$$\hat{c}_t^t = c_1^*, \quad \hat{c}_{t+1}^t = c_2^*, \quad \hat{K}_{t+1} = N_{t+1} k^*, \quad \hat{L}_t = N_t, \quad \hat{r}_t = r^*, \quad w_t = w^*$$

are an equilibrium for given initial condition $\frac{K_1}{N_0} = n k^*$

A steady state equilibrium may not be optimal

Prop: Suppose a competitive equilibrium converges to a steady-state satisfying

$$f'(k^*) < n+\delta-1$$

Then the equilibrium is not Pareto optimal

Goods market clearing condition in the original equilibrium

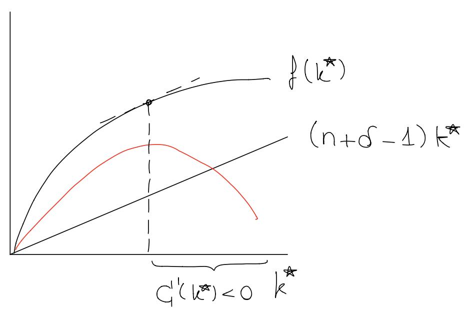
$$N_t \hat{c}_t^t + N_{t-1}\hat{c}_t^{t-1} + \underbrace{\hat{K}_{t+1}}_{N_{t+1}\hat{k}_{t+1}} = \underbrace{F(\hat{K}_t, \hat{L}_t)}_{N_t f(\hat{k}_t)} + (1-\delta)\underbrace{\hat{K}_t}_{N_t \hat{k}_t}$$

Divide by N_t

$$\hat{c}_t^t + \frac{\hat{c}_t^{t-1}}{n} + n\hat{k}_{t+1} = f(\hat{k}_t) + (1-\delta)\hat{k}_t$$

converge to a steady-state

$$c_1^* + \frac{c_2^*}{n} = f(k^*) + (1-n-\delta)k^* \equiv G(k^*)$$



$$G'(k^*) = f'(k^*) - (n+\delta-1) < 0$$

Construct a Pareto dominating allocation

At date t, reduce K_{t+1} to $\tilde{k}$ and keep the capital stock at $\tilde{k}$ forever, where $\tilde{k}$ is chosen to satisfy

$$f'(\tilde{k}) = n+\delta-1$$

Note $\tilde{k} < k^*$ since $f(\cdot)$ is a strictly concave production function

$$\tilde{c}_t^t + \frac{\tilde{c}_t^{t-1}}{n} + n\tilde{k} = f(k^*) + (1-\delta)k^* - nk^* + nk^* = G(k^*) + (k^* - \tilde{k})n > G(k^*)$$

don't follow

$i > t$

$$\tilde{c}_t^t + \frac{\tilde{c}_t^{t-1}}{n} + n\tilde{k} = f(\tilde{k}) + (1-\delta)\tilde{k}$$

$$\tilde{c}_t^t + \frac{\tilde{c}_t^{t-1}}{n} = f(\tilde{k}) - (n+\delta-1)\tilde{k} \equiv G(\tilde{k})$$

Note that our argument hinges upon an infinite horizon. If there was a last period T

$$\tilde{c}_T^T + \frac{\tilde{c}_T^{T-1}}{n} + n0 = f(\tilde{k}) + (1-\delta)\tilde{k} < f(k^*) + (1-\delta)k^*$$