Microeconomics II

Lecture 7: Adverse selection

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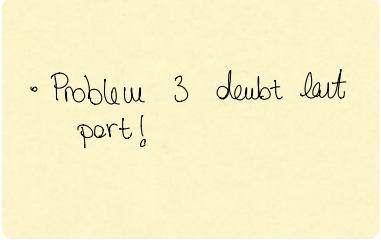


The setup

A principal wishes an agent to produce a quantity q of a good. The principal's valuation (e.g., gross profit) of the quantity q is $S(q)$, with $S(0) = 0$, $S' > 0$, and $S'' < 0$.

The agent's marginal production cost θ is observed only by him, but it is commonly known that it is $\underline{\theta}$ with probability ν and $\overline{\theta}$ with probability $1 - \nu$, with $\underline{\theta} < \overline{\theta}$. (We assume fixed costs are zero.)

efficient ↑ ↳ inefficient

A contract is a pair (q, t) specifying the quantity to be produced by the agent and the transfer to be paid to the agent.

Three stages are customarily identified:

- the *ex ante* stage, when both parties know only the prior distribution of θ,
- the *interim* stage, when the agent knows θ, and
- the *ex post* stage, when everybody knows everything.

We shall consider contracts that are signed at the interim stage.

Since the principal will want different behavior from different types, he would offer them different contracts if he was informed. Hence we can think of the contracting procedure we shall study in at least two different ways. Either the principal offers a menu of different contracts to the agent, out of which the agent selects one. Alternatively, the agent reports his type to the principal, who picks the relevant quantity and transfer from the menu. (The latter would be the direct mechanism associated with the contracting problem.)

The first best

We start by considering what the optimal contracts would look like if the agent's type were observable. The efficient production levels $\underline{q}^\star$ and $\overline{q}^\star$ are given by

$$S'(\underline{q}^\star) = \underline{\theta}$$

and

$$S'(\overline{q}^\star) = \overline{\theta}.$$

We note that because S is strictly concave, this implies $\underline{q}^\star > \overline{q}^\star$. Since the agent must participate voluntarily, we must have that

$$\underline{t} - \underline{\theta}\underline{q} \geq 0$$

and

$$\overline{t} - \overline{\theta}\overline{q} \geq 0.$$

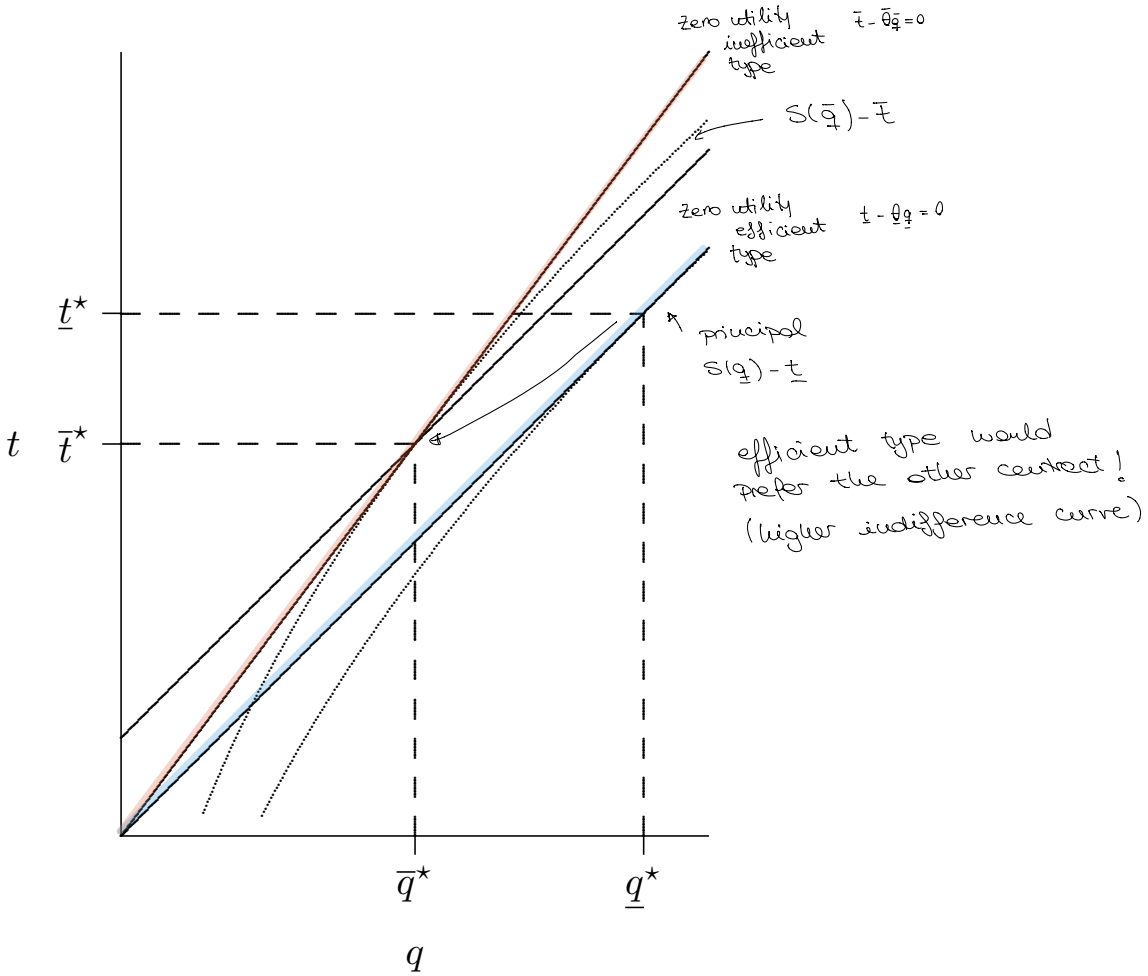
(Again we are assuming a reservation utility of zero.) That is, we must have that

$$\underline{t}^\star = \underline{\theta}\underline{q}^\star$$

and

$$\overline{t}^\star = \overline{\theta}\overline{q}^\star.$$

The figure illustrates the first-best contract. At each point in the menu, an indifference curve of the principal is exactly tangent to the zero-utility indifference curve of the respective type of agent.



$$\left.\begin{array}{ll} \underline{\theta} < \overline{\theta} & \underline{t} = \underline{\theta}\underline{q} \\ \underline{q} < \overline{q} & \overline{t} = \overline{\theta}\overline{q} \end{array}\right\} \Rightarrow \underline{t} < \overline{t}$$

We note that the optimal combination for the inefficient (high type) agent is preferred by both types. Under the high-type contract, the efficient agent would get

$$\overline{t} - \underline{\theta}\overline{q} = \overline{\theta}\overline{q} - \underline{\theta}\overline{q} = \Delta\theta\overline{q} > 0.$$

We therefore realize that the first best can never be implemented under private information, because it is not incentive compatible for the efficient agent.

The rent-extraction—efficiency tradeoff

If separation (or *screening*) of types is desirable under private information, the menu of contracts must additionally satisfy the incentive compatibility constraints

$$\underline{U} := \underline{t} - \underline{\theta}\underline{q} \geq \bar{t} - \underline{\theta}\bar{q}$$

and

$$\overline{U} := \bar{t} - \overline{\theta}\bar{q} \geq \underline{t} - \overline{\theta}\underline{q}.$$

This will not be possible without giving the efficient type a utility greater than zero, i.e., giving him an (*information*) *rent*. Since the efficient type can do anything the inefficient type can do, but at lower cost, he cannot be held down to zero utility. To see this, note that if the efficient type pretends to be the inefficient type he gets

$$\bar{t} - \underline{\theta}\bar{q} = \bar{t} - \overline{\theta}\bar{q} + (\overline{\theta} - \underline{\theta})\bar{q} = \overline{U} + \Delta\theta\bar{q} > 0$$

$\pm\overline{\theta}\bar{q}$

which is positive as long as $\bar{q} > 0$. Hence even if the inefficient type is held down to zero utility, the efficient type can guarantee himself a positive rent by mimicking the inefficient type.

The principal wishes to choose the menu $\{(\underline{t}, \underline{q}), (\overline{t}, \overline{q})\}$ so as to maximize

$$\nu(S(\underline{q}) - \underline{t}) + (1 - \nu)(S(\overline{q}) - \overline{t})$$

subject to the incentive and participation constraints.

Using our definitions of the information rents, we can restate the problem. The principal wishes to choose $\{(\underline{U}, \underline{q}), (\overline{U}, \overline{q})\}$ so as to maximize

$$\nu(S(\underline{q}) - \underline{\theta q}) + (1 - \nu)(S(\overline{q}) - \overline{\theta}\overline{q}) - (\nu\underline{U} + (1 - \nu)\overline{U})$$

subject to the incentive compatibility constraints

$$(1) \qquad \underline{U} \geq \overline{U} + \Delta\theta\overline{q}$$

and

$$(2) \qquad \overline{U} \geq \underline{U} - \Delta\theta\underline{q}$$

and the participation constraints

$$(3) \qquad \underline{U} \geq 0$$

and

$$(4) \qquad \overline{U} \geq 0.$$

How to solve this?

We already understand that the efficient type's participation constraint cannot bind at the optimum, since he can imitate the inefficient type. Furthermore, it is implied by the efficient type's incentive constraint and the inefficient type's participation constraint. Hence it is redundant.

We also guess that the inefficient type's incentive constraint is redundant (we shall verify this later), since we expect problems to arise rather with the efficient type having an incentive to mimic the inefficient type. If this is true, then we are left with the efficient type's incentive constraint and the inefficient type's participation constraint.

Suppose the inefficient type's participation constraint does not bind, i.e., that $\overline{U} = \epsilon > 0$. Then both $\overline{U}$ and $\underline{U}$ could be decreased by ϵ, increasing the principal's expected utility by ϵ. Hence this constraint must bind.

Given this, suppose we had $\underline{U} = \Delta\theta\overline{q} + \epsilon$. Then the principal could lower $\underline{U}$ by ϵ, gaining $\nu\epsilon$. Hence the efficient type's incentive constraint must bind.

That is, we must have

$$\underline{U} = \Delta\theta\overline{q}$$

and

$$\overline{U} = 0.$$

FOC's

$\underline{q}$: $\nu\left[S'(\underline{q}) - \underline{\theta}\right] = 0 \Rightarrow \boxed{S'(\underline{q}) = \underline{\theta}}$

$\overline{q}$: $(1-\nu)\left[S'(\overline{q}) - \overline{\theta}\right] - \nu\Delta\theta = 0 \Rightarrow \boxed{S'(\overline{q}) = \frac{\nu\Delta\theta}{1-\nu} + \overline{\theta}}$

Substituting into the principal's objective function, we now only have to maximize

$$\nu(S(\underline{q}) - \underline{\theta}\underline{q}) + (1-\nu)(S(\overline{q}) - \overline{\theta}\overline{q}) - \nu\Delta\theta\overline{q}$$

with respect to $\underline{q}$ and $\overline{q}$. We note that this is equivalent to maximizing the social value of production minus the expected rent of the efficient agent.

Since the expected rent of the efficient type does not depend on $\underline{q}$, the solution for the efficient type's output is the same as the first-best, i.e., we have that

$$S'(\underline{q}^{\mathrm{SB}}) = \underline{\theta},$$

implying that $\boxed{\underline{q}^{\mathrm{SB}} = \underline{q}^{\star}}$. As for $\overline{q}$, we have that

$$S'(\overline{q}^{\mathrm{SB}}) = \overline{\theta} + \frac{\nu\Delta\theta}{(1-\nu)} > \overline{\theta}.$$

$\underline{q}^{\mathrm{SB}} = \underline{q}^{\star} > \overline{q}^{\star} > \overline{q}^{\mathrm{SB}}$

Since S is strictly concave, this implies that $\boxed{\overline{q}^{\mathrm{SB}} < \overline{q}^{\star}}$.

Finally, we need to check that it was correct to exclude the incentive constraint of the inefficient agent, i.e., that we have that (2)

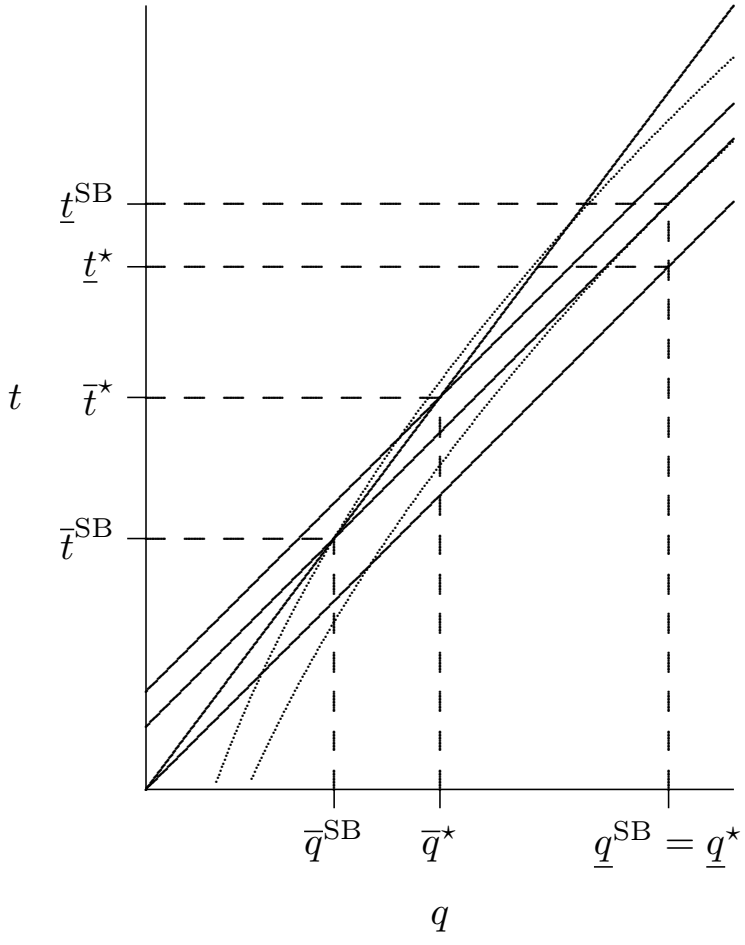
$$\overline{U} \geq \underline{U} - \Delta\theta\underline{q}$$

at our solution. Inserting the solution values, we get

$$0 \geq \Delta\theta\overline{q}^{\mathrm{SB}} - \Delta\theta\underline{q}^{\mathrm{SB}}.$$

Since we have $\underline{q}^{\mathrm{SB}} = \underline{q}^{\star} > \overline{q}^{\star} > \overline{q}^{\mathrm{SB}}$, it does indeed hold.

The figure illustrates what is going on.



The Spence-Mirrlees property

Consider more general utility functions for the agent, of the form $U = t - C(q, \theta)$.

Our argument about the redundancy of the inefficient type's incentive constraint depended on our having $\underline{q}^{\text{SB}} = \underline{q}^\star > \overline{q}^\star > \overline{q}^{\text{SB}}$. This is guaranteed more generally if we have

$$\frac{\partial C}{\partial q \partial \theta} > 0.$$

This condition ensures that indifference curves of different types cross at most once. Hence this property is also known as the *single-crossing* property.

It held trivially in our example with linear indifference curves.

Renegotiation

We have assumed that the principal can commit himself perfectly to the contract that is signed. But in general it seems reasonable that any contract can be renegotiated if both parties stand to benefit. This is troublesome if renegotiation can happen before the contracted-on quantity is produced.

Then if an inefficient type has revealed himself by selecting a contract, the principal would have an incentive to correct the downward distortion in production. He could now ask the agent to produce $\overline{q}^{\star}$ in return for a payment of $\underline{\theta}\overline{q}^{\star}$. The agent would accept, since he is indifferent between this and the original contract.

But given that this will happen, the original incentive scheme is of course invalidated, since now the efficient type will want to mimic the inefficient type, expecting renegotiation to take place.

One solution in this particular setting is to have the agent reveal his type by selecting output himself, so that this decision is irreversible, rather than report his type.

Regulation with a politician-supervisor

We next consider a model (due to Tirole 1986) with a *supervisor* placed in between the principal and the agent. The supervisor gets a signal about the type of the agent, i.e., he engages in monitoring, and can report it to the principal. But the supervisor may be self-interested and have an incentive to collude with the agent.

We shall study this in terms of Laffont's (2000) political economy model, in which the principal is taken to be some political assembly (e.g., voters, Congress, or a benevolent despot) that wishes to maximize social welfare by procuring the production of a public good from a firm. The supervisor is then a politician or regulatory agency.

But with minor modifications you can just as easily think of this as being about different levels of hierarchy in a firm (e.g., owner, manager, and workers), and in fact this interpretation may make more sense.

As before, we assume the firm can be of type $\underline{\theta}$ or $\overline{\theta}$, with probabilities ν and $1-\nu$, respectively. The firms utility is $U := t - \theta q$.

Consumers get utility $S(q)$, with $S' > 0$ and $S'' < 0$, from consuming q units of the public good. The payment to the firm has to be raised from taxes, however, which are associated with deadweight losses. We assume that raising \$1 costs $\$(1+\lambda)$, with $\lambda > 0$. Hence consumer's welfare is

$$S := S(q) - (1+\lambda)t.$$

Social welfare is then

$$W := S + U = S(q) - (1+\lambda)\theta q - \lambda U.$$

The first best solution would therefore be such that

$$S'(\underline{q}^\star) = (1+\lambda)\underline{\theta}$$

and

$$S'(\overline{q}^\star) = (1+\lambda)\overline{\theta}$$

and $\underline{t}^\star = \underline{\theta}\underline{q}^\star$ and $\overline{t}^\star = \overline{\theta}\overline{q}^\star$, leaving no rent to the firm.

Alas, the firm's type is its private information, and hence the political principal must offer incentive compatible contracts. That is, using the same transformation of variables as before, the principal wishes to maximize expected social welfare

$$\nu(S(\underline{q}) - (1+\lambda)\underline{\theta}\underline{q} - \lambda\underline{U}) + (1-\nu)(S(\overline{q}) - (1+\lambda)\overline{\theta}\overline{q} - \lambda\overline{U})$$

subject to the same incentive and participation constraints as before.

For the same the reasons as before, we know that we can ignore the inefficient type's incentive constraint and the efficient type's participation constraint, and that the other constraints must bind. Hence we must have $\overline{U} = 0$ and $\underline{U} = \Delta\theta\overline{q}$ as before. Substituting into the principal's objective function, he now simply wishes to maximize

$$\nu(S(\underline{q}) - (1+\lambda)\underline{\theta}\underline{q} - \lambda\Delta\theta\overline{q}) + (1-\nu)(S(\overline{q}) - (1+\lambda)\overline{\theta}\overline{q})$$

with respect to $\underline{q}$ and $\overline{q}$. Hence we must have

$$S'(\underline{q}^{\mathrm{SB}}) = (1+\lambda)\underline{\theta}$$

and

$$S'(\overline{q}^{\mathrm{SB}}) = (1+\lambda)\overline{\theta} + \lambda\frac{\nu}{1-\nu}\Delta\theta$$

and $\underline{t} = \underline{\theta}\underline{q}^{\mathrm{SB}} + \Delta\theta\overline{q}^{\mathrm{SB}}$ and $\overline{t} = \overline{\theta}\overline{q}^{\mathrm{SB}}$. The same conclusions about the efficiency/rent extraction tradeoff hold as before.

Now suppose a politician or bureaucrat can be hired as a supervisor. The supervisor gets a signal σ about the type of the firm, as follows. The signal only informs about $\underline{\theta}$. If $\theta = \underline{\theta}$, then with probability ξ the signal says so, and with probability $1 - \xi$ it is uninformative. If $\theta = \overline{\theta}$ the signal is always uninformative.

The signal that reveals $\underline{\theta}$ is assumed to be *verifiable*, i.e., it consists of some conclusive evidence.

Assume first that the supervisor is unable to collude with the firm. If his reservation utility is zero, he will be happy to report the content of the signal to the political principal in return for a flat wage of zero.

Then when the supervisor truthfully reports that he knows the type to be $\underline{\theta}$, the principal can select the first-best contract. This happens with probability $\nu\xi$.

If the supervisor reports that he knows nothing, the conditional probability of the type being $\underline{\theta}$ is given by Bayes' law as

$$\hat{\nu} := \frac{\nu(1-\xi)}{\nu(1-\xi) + 1 - \nu} = \frac{\nu(1-\xi)}{1-\nu\xi} < \nu.$$

Hence in this case the principal can offer second-best contracts computed as before, but replacing ν by $\hat{\nu}$.

That is, we will have

$$S'(\underline{q}_{\emptyset}^{\mathrm{SB}}) = (1+\lambda)\underline{\theta}$$

and

$$S'(\overline{q}_{\emptyset}^{\mathrm{SB}}) = (1+\lambda)\overline{\theta} + \lambda\frac{\nu(1-\xi)}{1-\nu}\Delta\theta.$$

Hence the presence of the supervisor improves efficiency, since we must have $\overline{q}_{\emptyset}^{\mathrm{SB}} > \overline{q}^{\mathrm{SB}}$. The rent $\Delta\theta\overline{q}_{\emptyset}^{\mathrm{SB}}$ sometimes given up to the low-type firm is higher, but this now happens less often—with probability $\nu(1-\xi)$ rather than ν.

Now suppose the supervisor and the firm can collude. A firm whose type has been revealed to the supervisor to be $\underline{\theta}$ has an interest in getting the supervisor to report no information. Specifically, the firm is willing to pay up to $\Delta\theta\overline{q}_{\emptyset}^{\mathrm{SB}}$ for this service.

information rent →

Now note that collusion could be avoided if the principal paid the supervisor this amount every time he (verifiably) reported $\underline{\theta}$. But then the principal would just be paying the supervisor exactly the reduction in the rent of the firm, and the supervisor would therefore be useless. Hence a supervisor cannot be of value unless there are some transaction costs in the collusion between the supervisor and the firm.

It is natural to assume, however, that there are such transaction costs, since collusive contracts would not be legal, and therefore difficult to enforce. Assume, therefore, that if the firm offers $\Delta\theta\overline{q}_{\emptyset}^{\mathrm{SB}}$ to the supervisor, it is only worth

$$\frac{\Delta\theta\overline{q}_{\emptyset}^{\mathrm{SB}}}{1+\lambda_c} =: \underbrace{k}_{k\in(0,1)}\Delta\theta\overline{q}_{\emptyset}^{\mathrm{SB}}$$

a fraction of the information rent

to the supervisor, with $\lambda_c > 0$.

So suppose a payment $\underline{s} = k\Delta\theta\overline{q}_{\emptyset}^{\mathrm{SB}}$ is made to the supervisor when he reports $\underline{\theta}$, and the funds for this are raised from taxes. Since the supervisor's utility is included in social welfare, the net cost is just $\lambda\underline{s}$. Since the payment is only made with probability $\nu\xi$, the expected net social cost is $\nu\xi\lambda\underline{s}$.

Expected social welfare is now

$$\begin{aligned}&\nu\xi(S(\underline{q}^\star)+(1+\lambda)\underline{\theta}\underline{q}^\star)\\&+(1-\nu\xi)\left(\frac{\nu(1-\xi)}{1-\nu\xi}(S(\hat{\underline{q}}_\emptyset^{\mathrm{SB}})-(1+\lambda)\underline{\theta}\hat{\underline{q}}_\emptyset^{\mathrm{SB}}-\lambda\Delta\theta\hat{\underline{q}}_\emptyset^{\mathrm{SB}})\right.\\&\left.+\frac{1-\nu}{1-\nu\xi}(S(\hat{\overline{q}}_\emptyset^{\mathrm{SB}})-(1+\lambda)\overline{\theta}\hat{\overline{q}}_\emptyset^{\mathrm{SB}})\right)-\nu\xi\lambda k\Delta\theta\hat{\overline{q}}_\emptyset^{\mathrm{SB}}.\end{aligned}$$

Maximizing with respect to $\hat{\underline{q}}_\emptyset^{\mathrm{SB}}$ and $\hat{\overline{q}}_\emptyset^{\mathrm{SB}}$ yields

$$S'(\hat{\underline{q}}_\emptyset^{\mathrm{SB}})=(1+\lambda)\underline{\theta}$$

and

$$S'(\hat{\overline{q}}_\emptyset^{\mathrm{SB}})=(1+\lambda)\overline{\theta}+\lambda\frac{\nu(1-(1-k)\xi)}{1-\nu}\Delta\theta.$$

We see that $\hat{\overline{q}}_\emptyset^{\mathrm{SB}}<\overline{q}_\emptyset^{\mathrm{SB}}$. This reflects a desire to reduce the stakes of collusion.

Note that everyone expect the supervisor loses from collusion being possible. In particular, the firm would prefer to somehow commit not to collude. ← information rent is less frequent

Problem. *A principal is offering an agent work. If the agent produces q units of output, the principal gets a gross profit of $S(q) = \sqrt{q}$. The agent's utility is $t - \theta q^{3/2}$, where t is the payment he receives and θ is his type. The agent can be one of two types, $\underline{\theta} = 3/4$ and $\overline{\theta} = 1$, which are equally likely. Only the agent knows his type. Find the first best contract. Then find the second best, incentive compatible contract.*

Problem. *A monopolist (the principal) faces a single consumer. The consumer's utility is $u = \theta q - t$, where q is his consumption and t is the transfer to the monopolist. The consumer's reservation utility is zero. The monopolist's production cost is $cq^2/2$. Find the optimal contract when the monopolist knows θ. Next assume θ is private information, but is known to take the value $\underline{\theta}$ with probability ν and $\overline{\theta}$ with probability $1 - \nu$. Assume we have $\underline{\theta} > (1 - \nu)\overline{\theta}$. Find the optimal contract.*

Problem. *Consider again the optimal auction from Lecture 4, but now assume a bidder may have to pay the seller also when he loses. Let $\underline{W}$ and $\underline{L}$ be the payments from a bidder who has announced type $\underline{\theta}$ when he wins and loses, respectively, and define $\overline{W}$ and $\overline{L}$ analogously. A bidder's utility is $u(\theta - W)$ when he wins and $u(-L)$ when he loses, where u is strictly concave. Show that in the optimal (symmetric) auction, we have $\overline{L} < 0$ and $\underline{L} > 0$.*

PROBLEM 1

Three stages are defined:

- Ex-ante stage: both parties know only the prior distribution of θ ($\nu = 1/2$)
- Interim stage: agent observes θ and contracts are signed.
- Ex-post: everybody knows everything.

FIRST BEST: public info

Suppose agent type θ is observable. The efficient production levels $\underline{q}^*$ and $\bar{q}^*$ are given by the intersection of both optimization problems.

$$\left.\begin{aligned} S(q) &= \sqrt{q} \Rightarrow S'(q) = \frac{1}{2\sqrt{q}} \\ C(q) &= \theta q^{3/2} \Rightarrow C'(q) = \frac{3}{2}\theta\sqrt{q} \end{aligned}\right\} S'(q) = C'(q) \Rightarrow \frac{1}{2\sqrt{q}} = \frac{3\theta\sqrt{q}}{2} \Rightarrow \boxed{q^* = \frac{1}{3\theta}}$$

since $\underline{\theta} < \bar{\theta}$, we know that $\underline{q} > \bar{q}$

$$\underline{q} = \frac{1}{3\underline{\theta}} = \frac{1}{3\cdot\frac{3}{4}} = \frac{4}{9} \simeq 0.\widehat{44}$$

$$\bar{q} = \frac{1}{3\bar{\theta}} = \frac{1}{3\cdot 1} = \frac{1}{3} \simeq 0.\widehat{33}$$

Since the agent must participate voluntarily,

$$\underline{t} - \underline{\theta}\,\underline{q}^{3/2} \geq 0$$

$$\bar{t} - \bar{\theta}\,\bar{q}^{3/2} \geq 0$$

(assuming reservation utility of zero). Since the Principal would be better-off by reducing t by $\varepsilon > 0$, we have

$$\underline{t} = \underline{\theta}\,\underline{q}^{\frac{3}{2}} = \frac{3}{4}\left(\frac{4}{9}\right)^{\frac{3}{2}} = \frac{2}{9} \simeq 0.22$$

$$\bar{t} = \bar{\theta}\,\bar{q}^{\frac{3}{2}} = 1\left(\frac{1}{3}\right)^{\frac{3}{2}} = \frac{1}{3\sqrt{3}} \simeq 0.19$$

note that the optimal combination for the inefficient (high type) agent is preferred by both types. Under the high-type contract, the efficient agent would get

$$\bar{t} - \underline{\theta}\,\bar{q}^{\frac{3}{2}} = \bar{\theta}\,\bar{q}^{\frac{3}{2}} - \underline{\theta}\,\bar{q}^{\frac{3}{2}} = \Delta\theta\,\bar{q}^{\frac{3}{2}} = \left(1 - \frac{3}{4}\right)\left(\frac{1}{3}\right)^{\frac{3}{2}} = \frac{1}{12\sqrt{3}} \simeq 0.048 > 0$$

hence, the first-best can <u>never</u> be implemented under private info! Not IC for efficient agent.

SECOND BEST: private info

If separation (screening) of types is desirable, the menu of contracts must additionally satisfy the IC constraints

$$\underline{t} - \underline{\theta}\,\underline{q}^{\frac{3}{2}} \geq \bar{t} - \underline{\theta}\,\bar{q}^{\frac{3}{2}}$$
$$\bar{t} - \bar{\theta}\,\bar{q}^{\frac{3}{2}} \geq \underline{t} - \bar{\theta}\,\underline{q}^{\frac{3}{2}}$$

which would be impossible unless giving the efficient type utility greater than zero (information rent: since the efficient type can do anything the inefficient type can do at a lower cost, he cannot be held down to zero utility). If the efficient pretends to be the inefficient type, he gets

$$\bar{t} - \underline{\theta}\,\bar{q}^{\frac{3}{2}} = \underbrace{\bar{t} - \bar{\theta}\,\bar{q}^{\frac{3}{2}}}_{\bar{u}} + \Delta\theta\,\bar{q}^{\frac{3}{2}} = \bar{u} + \Delta\theta\,\bar{q}^{\frac{3}{2}} > 0 \qquad (1)$$

$\pm\bar{\theta}\bar{q}^{\frac{3}{2}}$

even if the inefficient type is held down to zero ($\bar{u}=0$), the efficient type can guarantee himself a positive rent by mimicking the inefficient type.

The Principal wishes to choose the contract menu $\{(\underline{t},\underline{q}),(\bar{t},\bar{q})\}$ to max

$$\nu\left(S(\underline{q}) - \underline{t}\right) + (1-\nu)\left(S(\bar{q}) - \bar{t}\right)$$

subject to IC's and PC's. However, let me restate the problem. The Principal wishes to choose $\{(\underline{u},\underline{q}),(\bar{u},\bar{q})\}$ to max

$$\max \quad \nu\left(S(\underline{q}) - \underline{\theta}\,\underline{q}^{\frac{3}{2}}\right) + (1-\nu)\left(S(\bar{q}) - \bar{\theta}\,\bar{q}^{\frac{3}{2}}\right) - \left[\nu\,\underline{u} + (1-\nu)\,\bar{u}\right]$$

s.t.
$$\underline{t} - \underline{\theta}\,\underline{q}^{\frac{3}{2}} \geq \bar{t} - \underline{\theta}\,\bar{q}^{\frac{3}{2}} \Rightarrow \underline{t} - \underline{\theta}\,\underline{q}^{\frac{3}{2}} \geq \bar{t} - \bar{\theta}\,\bar{q}^{\frac{3}{2}} + \Delta\theta\,\bar{q}^{\frac{3}{2}} \Rightarrow \underline{u} \geq \bar{u} + \Delta\theta\,\bar{q}^{\frac{3}{2}} \qquad (IC1)$$

$$\bar{t} - \bar{\theta}\,\bar{q}^{\frac{3}{2}} \geq \underline{t} - \bar{\theta}\,\underline{q}^{\frac{3}{2}} \Rightarrow \bar{t} - \bar{\theta}\,\bar{q}^{\frac{3}{2}} \geq \underline{t} - \underline{\theta}\,\underline{q}^{\frac{3}{2}} - \Delta\theta\,\underline{q}^{\frac{3}{2}} \Rightarrow \bar{u} \geq \underline{u} - \Delta\theta\,\underline{q}^{\frac{3}{2}} \qquad (IC2)$$

$$\underline{t} - \underline{\theta}\,\underline{q}^{\frac{3}{2}} \geq 0 \Rightarrow \underline{u} \geq 0 \qquad (PC1)$$

$$\bar{t} - \bar{\theta}\,\bar{q}^{\frac{3}{2}} \geq 0 \Rightarrow \bar{u} \geq 0 \qquad (PC2)$$

I already showed in (1) that (PC1) cannot bind at the optimum. So we know $\underline{u} > 0$. Furthermore, it is already implied by (IC1) and (PC2)

$$\left.\begin{array}{l}\underline{u} \geq \bar{u} + \Delta\theta\,\bar{q}^{\frac{3}{2}} \\ \bar{u} \geq 0\end{array}\right\} \underline{u} \geq \underbrace{\bar{u}}_{\geq 0} + \underbrace{\Delta\theta\,\bar{q}^{\frac{3}{2}}}_{>0} > 0 \Rightarrow \cancel{\underline{u} > 0}$$

I also guess that (IC2) is redundant. Hence, we are only left with (IC1) and (PC2). Let me now prove that they bind at an optimum.

Suppose (PC2) does not bind: $\bar{u} = \varepsilon > 0$. Then both $\bar{u}$ and $\underline{u}$ could be decreased by ε,

$$\bar{u} \downarrow > 0$$
$$\underline{u} \downarrow \geq \bar{u} \downarrow + \Delta\theta\,\bar{q}^{\frac{3}{2}}$$

increasing the Principal's expected utility $\downarrow$. Hence, (PC2) must bind: $\bar{u} = 0$.

Suppose that (IC1) does not bind:

$$\underline{u} > \underbrace{\overline{u}}_{0} + \Delta\theta\,\overline{q}^{\frac{3}{2}} \implies \underline{u} = \Delta\theta\,\overline{q}^{\frac{3}{2}} + \varepsilon \downarrow$$

then the Principal would reduce $\underline{u}$ by ε $\downarrow$. Hence, (IC1) must bind : $\underline{u} = \Delta\theta\,\overline{q}^{\frac{3}{2}}$.
Substituting into the objective function

$$\nu\left(S(\underline{q}) - \underline{\theta}\,\underline{q}^{\frac{3}{2}}\right) + (1-\nu)\left(S(\overline{q}) - \overline{\theta}\,\overline{q}^{\frac{3}{2}}\right) - \left[\nu\,\Delta\theta\,\overline{q}^{\frac{3}{2}} + (1-\nu)0\right]$$

wrt $\underline{q}$, $\overline{q}$ (≡ maximizing the social value of production minus the expected rent of the efficient agent). Taking the FOC's,

$\underline{q}$: $\nu\left(S'(\underline{q}) - \frac{3}{2}\underline{\theta}\sqrt{\underline{q}}\right) = 0 \implies S'(\underline{q}) = \frac{3}{2}\underline{\theta}\sqrt{\underline{q}} \implies \frac{1}{2\sqrt{\underline{q}}} = \frac{3}{2}\underline{\theta}\sqrt{\underline{q}} \implies \underline{q}^{SB} = \frac{1}{3\underline{\theta}} = \frac{4}{9} \simeq 0.\widehat{44}$

$\overline{q}$: $(1-\nu)\left(S'(\overline{q}) - \frac{3}{2}\overline{\theta}\sqrt{\overline{q}}\right) - \nu\frac{3}{2}\Delta\theta\sqrt{\overline{q}} = 0 \implies (1-\nu)\left(\frac{1}{2\sqrt{\overline{q}}} - \frac{3}{2}\overline{\theta}\sqrt{\overline{q}}\right) - \nu\frac{3}{2}\Delta\theta\sqrt{\overline{q}} = 0 \implies$

$$\implies \overline{q}^{SB} = \frac{1-\nu}{3[\nu\Delta\theta + (1-\nu)\overline{\theta}]} = \frac{4}{15} \simeq 0.2\widehat{66}$$

notice that $\underline{q}^{SB} = \underline{q}^{FB} > \overline{q}^{FB} > \overline{q}^{SB}$,

$$\frac{4}{9} = \frac{4}{9} > \frac{1}{3} > \frac{4}{15}$$

Finally, we need to check if my guess ((IC2) redundant) was correct

$$\underbrace{\overline{u}}_{0} \geq \underline{u} - \Delta\theta\left(\underline{q}^{SB}\right)^{\frac{3}{2}} \implies 0 \geq \Delta\theta\left(\overline{q}^{SB}\right)^{\frac{3}{2}} - \Delta\theta\left(\underline{q}^{SB}\right)^{\frac{3}{2}} \implies 0 \geq \Delta\theta\left[\left(\overline{q}^{SB}\right)^{\frac{3}{2}} - \left(\underline{q}^{SB}\right)^{\frac{3}{2}}\right] \implies$$

$$\implies 0 \geq (1 - 3/4)\left(\left(\tfrac{4}{15}\right)^{\frac{3}{2}} - \left(\tfrac{4}{9}\right)^{\frac{3}{2}}\right) \implies 0 \geq \frac{1}{4}(0.138 - 0.296) \;\checkmark$$

From (PC2) with equality

$$\overline{t} = \overline{\theta}\,\overline{q}^{\frac{3}{2}} = \frac{8}{15\sqrt{15}} \simeq 0.138$$

From (IC1) with equality

$$\underline{t} = \underline{\theta}\,\underline{q}^{\frac{3}{2}} + \overline{t} - \overline{\theta}\,\overline{q}^{\frac{3}{2}} + \Delta\theta\,\overline{q}^{\frac{3}{2}} \overset{(PC2)}{=} \underline{\theta}\,\underline{q}^{\frac{3}{2}} + \Delta\theta\,\overline{q}^{\frac{3}{2}} = \frac{2}{9} + \frac{2}{15\sqrt{15}} \simeq 0.2\widehat{2} + 0.034 = 0.256$$

PROBLEM 2

a) $u = \theta q - t$

$\tilde{u} = 0$

$c(q) = \frac{cq^2}{2}$

$u'(c) = c'(q) \implies \theta = cq \implies q = \frac{\theta}{c}$; $\boxed{\underline{q} = \underline{\theta}/c}$ $\boxed{\overline{q} = \overline{\theta}/c}$

b) Firm maximizes,

$$\max_{\underline{q},\bar{q}} \quad \nu\left(\underline{t} - \frac{c\underline{q}^2}{2}\right) + (1-\nu)\left(\bar{t} - \frac{c\bar{q}^2}{2}\right)$$

$$\text{s.t.} \quad \underline{\theta}\,\underline{q} - \underline{t} \geq \underline{\theta}\,\bar{q} - \bar{t} \tag{5}$$

$$\bar{\theta}\,\bar{q} - \bar{t} \geq \bar{\theta}\,\underline{q} - \underline{t} \tag{6}$$

$$\underline{\theta}\,\underline{q} - \underline{t} \geq 0 \tag{7}$$

$$\bar{\theta}\,\bar{q} - \bar{t} \geq 0 \tag{8}$$

[same explanation of previous exercise makes (5)&(8) redundant, and (6)-(7) bind (although notice in this case eqs that are redundant and bind are different)]

$$\bar{\theta}\,\bar{q} - \bar{t} = \bar{\theta}\,\underline{q} - \underline{t} = \bar{\theta}\,\underline{q} + \underline{\theta}\,\underline{q} - \underline{\theta}\,\underline{q} - \underline{t} \Rightarrow \bar{\theta}\bar{q} - \bar{t} = \underline{\theta}\,\underline{q} - \underline{t} + \Delta\theta\,\underline{q}$$

$\pm\,\underline{\theta}\,\underline{q}$

$\underline{\theta}\,\underline{q} - \underline{t} = 0$

Rewriting the monopolist objective function

$$\max_{\underline{q},\bar{q}} \quad \nu\left(\underline{\theta}\,\underline{q} - \frac{c\underline{q}^2}{2}\right) + (1-\nu)\left(\bar{\theta}\bar{q} - \Delta\theta\,\underline{q} - \frac{c\bar{q}^2}{2}\right)$$

taking the FOC's

$$\underline{q}: \quad \nu(\underline{\theta} - c\underline{q}) - (1-\nu)\Delta\theta = 0 \Rightarrow \boxed{\underline{q}^* = \frac{\nu\underline{\theta} - (1-\nu)\Delta\theta}{c\nu}}$$

$$\bar{q}: \quad (1-\nu)(\bar{\theta} - c\bar{q}) = 0 \Rightarrow \boxed{\bar{q}^* = \bar{\theta}/c}$$

$$\underline{t}^* = \underline{\theta}\,\underline{q}^* = \frac{\underline{\theta}}{c\nu}\left[\nu\underline{\theta} - (1-\nu)\Delta\theta\right]$$

$$\bar{t}^* = \bar{\theta}\bar{q}^* - \Delta\theta\,\underline{q}^* = \frac{\bar{\theta}^2}{c} - \frac{\Delta\theta}{c\nu}\left[\nu\underline{\theta} - (1-\nu)\Delta\theta\right]$$

$\boxed{\bar{q}^{FB} = \bar{q}^{SB}}$, $\boxed{\underline{q}^{FB} > \underline{q}^{SB}}$

PROBLEM 3

The seller wishes to maximize the expected (per bidder) profit

$$\underline{p}\left[\underline{x}\underline{w} + (1-\underline{x})\underline{L}\right] + \bar{p}\left[\bar{x}\bar{w} + (1-\bar{x})\bar{L}\right]$$

subject to

$$\underline{x}\,u(\underline{\theta}-\underline{w}) + (1-\underline{x})\,u(-\underline{L}) \geq \bar{x}\,u(\underline{\theta}-\bar{w}) + (1-\bar{x})\,u(-\bar{L}) \tag{9}$$

$$\bar{x}\,u(\bar{\theta}-\bar{w}) + (1-\bar{x})\,u(-\bar{L}) \geq \underline{x}\,u(\bar{\theta}-\underline{w}) + (1-\underline{x})\,u(-\underline{L}) \tag{10}$$

$$\underline{x}\,u(\underline{\theta}-\underline{w}) + (1-\underline{x})\,u(-\underline{L}) \geq u(0) \tag{11}$$

$$\bar{x}\,u(\bar{\theta}-\bar{w}) + (1-\bar{x})\,u(-\bar{L}) \geq u(0) \tag{12}$$

where $\underline{x}, \bar{x}$ are the expectations of the probability of getting the good.

We can show that (9) & (12) are redundant, and (10) - (11) hold with equality. Hence, we have

$$\max_{\substack{\underline{w},\underline{L}\\ \bar{w},\bar{L}}} \quad \underline{p}\left[\underline{x}\underline{w} + (1-\underline{x})\underline{L}\right] + \bar{p}\left[\bar{x}\bar{w} + (1-\bar{x})\bar{L}\right]$$

$$\text{s.t.} \quad \bar{x}\,u(\bar{\theta}-\bar{w}) + (1-\bar{x})\,u(-\bar{L}) = \underline{x}\,u(\bar{\theta}-\underline{w}) + (1-\underline{x})\,u(-\underline{L}) \tag{IC}$$

$$\underline{x}\,u(\underline{\theta}-\underline{w}) + (1-\underline{x})\,u(-\underline{L}) = u(0) \tag{PC}$$

Lagrangian,

$$\mathcal{L} = \underline{p}\left[\underline{x}\underline{w} + (1-\underline{x})\underline{L}\right] + \bar{p}\left[\bar{x}\bar{w} + (1-\bar{x})\bar{L}\right] + \lambda\left[\bar{x}\,u(\bar{\theta}-\bar{w}) + (1-\bar{x})\,u(-\bar{L}) - \underline{x}\,u(\bar{\theta}-\underline{w}) - (1-\underline{x})\,u(-\underline{L})\right] + \mu\left[\underline{x}\,u(\underline{\theta}-\underline{w}) + (1-\underline{x})\,u(-\underline{L}) - u(0)\right]$$

FOC's

$$\underline{w}: \ \underline{p}\,\underline{x} + \lambda\left[\underline{x}\,u'(\bar{\theta}-\underline{w})\right] + \mu\left[-\underline{x}\,u'(\underline{\theta}-\underline{w})\right] = 0 \tag{13}$$

$$\underline{L}: \ \underline{p}(1-\underline{x}) + \lambda\left[(1-\underline{x})\,u'(-\underline{L})\right] + \mu\left[-(1-\underline{x})\,u'(-\underline{L})\right] = 0 \tag{14}$$

$$\bar{w}: \ \bar{p}\,\bar{x} + \lambda\left[-\bar{x}\,u'(\bar{\theta}-\bar{w})\right] = 0 \tag{15}$$

$$\bar{L}: \ \bar{p}(1-\bar{x}) + \lambda\left[-(1-\bar{x})\,u'(-\bar{L})\right] = 0 \tag{16}$$

$$(15) \Rightarrow \bar{p} = \lambda u'(\bar{\theta}-\bar{w}) \rightarrow \lambda > 0 \quad \text{since } \bar{p} > 0 \text{ and } u'(\cdot) > 0 \Rightarrow \text{(IC) binds} \tag{17}$$

$$(16) \Rightarrow \bar{p} = \lambda u'(-\bar{L}) \tag{18}$$

$$(17) = (18) \Rightarrow u'(-\bar{L}) = u'(\bar{\theta}-\bar{w}) \Rightarrow \bar{\theta}-\bar{w} = -\bar{L} \leftarrow \text{the high type is perfectly insured}$$

(14) $\Rightarrow \mu = \frac{\underline{p}}{u'(-\underline{L})} + \lambda \underset{(18)}{\Longrightarrow} \mu = \frac{\underline{p}}{u'(-\underline{L})} + \frac{\overline{p}}{u'(-\overline{L})} \Rightarrow \mu > 0 \Rightarrow$ (PC) binds (19)

Rewritting (IC)

$$\overline{x}\underbrace{u(\overline{\theta}-\overline{W})}_{u(-\overline{L})} + (1-\overline{x})u(-\overline{L}) = \underline{x}u(\overline{\theta}-\underline{W}) + (1-\underline{x})u(-\underline{L}) \Rightarrow$$

$$\Rightarrow u(-\overline{L}) = \underline{x}\underbrace{u(\overline{\theta}-\underline{W})}_{\overline{\theta}-\underline{W} > \underline{\theta}-\underline{W} \overset{u'(\cdot)>0}{\Rightarrow} u(\overline{\theta}-\underline{W}) > u(\underline{\theta}-\underline{W})} + (1-\underline{x})u(-\underline{L}) > \underline{x}u(\underline{\theta}-\underline{W}) + (1-\underline{x})u(-\underline{L}) \underset{(PC)}{=} u(0)$$

$\overline{\theta} > \underline{\theta}$

$$\Rightarrow u(-\overline{L}) > u(0) \Rightarrow -\overline{L} > 0 \Rightarrow \boxed{\overline{L} < 0}$$

Rewritting (13)

$$\underline{p} + \lambda u'(\overline{\theta}-\underline{W}) - \mu u'(\underline{\theta}-\underline{W}) = 0$$

(18) - (19)

$$\underbrace{\underline{p}\left[\frac{u'(-\underline{L}) - u'(\underline{\theta}-\underline{W})}{u'(-\underline{L})}\right]}_{\text{(A)}} + \underbrace{\overline{p}\left[\frac{u'(\overline{\theta}-\underline{W}) - u'(\underline{\theta}-\underline{W})}{u'(-\overline{L})}\right]}_{\text{(B)}} = 0$$

and $u'(-\overline{L})$??

We know, by strict concavity, that (B) < 0 (since $\overline{\theta} > \underline{\theta} \rightarrow u'(\overline{\theta}-\underline{W}) < u'(\underline{\theta}-\underline{W})$).
Hence, (A) > 0,

$$u'(-\underline{L}) > u'(\underline{\theta}-\underline{W}) \overset{\text{concavity}}{\Rightarrow} \underline{\theta}-\underline{W} > -\underline{L} \Rightarrow \underline{L} > \underline{W}-\underline{\theta} \overset{?}{>} 0 \Rightarrow \boxed{\underline{L} > 0}$$

and $u'(-\underline{L})$??