Microeconomics II

Lecture 4: Incomplete Information

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November 2016

- Problem 2
- Problem 1: unsure!

Modelling incomplete information

So far, we have studied games in which information was *complete*, although it may have been imperfect (in the sense that not every move in the game is observed by every player). In a game of *incomplete information*, some aspects of the structure of the game itself are not common knowledge among the players. It could be the case that a player does not know which strategies are available to another player, or what his payoffs are.

Early on, such games were thought not to be analyzable. Harsanyi (1967–68) argued that, under certain conditions, all forms of uncertainty about the game could be reduced to uncertainty about other players' payoffs. Hence the standard approach to games of incomplete information is now to consider the associated games of complete but imperfect information which start with a move by Nature, who determines the players' *types*, i.e., their payoff functions.

We first illustrate this with some examples.

Voluntary public goods provision

Palfrey and Rosenthal (1989) discuss voluntary contributions to public goods under incomplete information. We consider a simple example with only two players.

Each player can either contribute or not contribute. If at least one player contributes, the public good is provided, and both players enjoy a utility of 1 from this. If player i contributes, he also sustains a cost $1 > c_i > 0$. If nobody contributes, each gets utility zero. Given the costs of the two players the situation is therefore as in the matrix below.

		Player 2	
		Contribute	Don't
Player 1	Contribute	$1 - c_1, 1 - c_2$	$1 - c_1, 1$
	Don't	$1, 1 - c_2$	$0, 0$

First note that, regardless of what the costs are, there are always two asymmetric pure-strategy equilibria in which one player contributes and the other does not.

Next assume that a player's cost is private information, but each player has the same continuous prior probability distribution F for the cost of the other player. Then player i rationally contributes with probability 1 if we have that

$$1 - c_i > \text{Prob}(\text{Player } j \text{ contributes}),$$

or, equivalently,

$$c_i < 1 - \text{Prob}(\text{Player } j \text{ contributes}) =: c^\star.$$

Since the situation for player j is symmetric, we must have that

$$c^\star = 1 - F(c^\star) \quad \Rightarrow \quad c^\star = 1 - c^\star \Rightarrow c^\star = 1/2$$

in equilibrium. Hence, for example, if F is the uniform distribution on $[0, 1]$, we have $c^\star = .5$.

$$F(c^\star) = \begin{cases} 0 & \text{for } c^\star < 0 \\ \frac{c^\star - 0}{1 - 0} & \text{for } c^\star \in [0,1] \\ 1 & \text{for } c^\star > 1 \end{cases}$$

Cournot duopoly with incomplete information

Consider the same Cournot duopoly as before, but now assume Firm 1's cost is commonly known to be c_1, but Firm 2's cost is private information and commonly known to be c_2^L with probability ρ and c_2^H with probability $1-\rho$, with $c_2^L < c_2^H$.

Let q_2^L and q_2^H be the quantities supplied by Firm 2 when it has low and high cost, respectively. Then Firm 1's expected profit is

$$\mathrm{E}\pi_1 = \rho(a - q_1 - q_2^L - c_1)q_1 + (1-\rho)(a - q_1 - q_2^H - c_1)\, q_1$$

Hence Firm 1's first order condition for a best reply is

$$q_1 = \frac{a - c_1 - (\rho q_2^L + (1-\rho)q_2^H)}{2}.$$

A Firm 2 with low cost has expected profit

$$\pi_2^L = (a - q_1 - q_2^L - c_2^L)q_2^L$$

and therefore first order condition

$$q_2^L = \frac{a - c_2^L - q_1}{2}.$$

Similarly, a high cost type Firm 2 has first order condition

$$q_2^H = \frac{a - c_2^H - q_1}{2}.$$

We assume, for simplicity, that an interior equilibrium exists, i.e., one where Firm 1 and both types of Firm 2 produce positive output. Solving these three equations simultaneously for an equilibrium, we get

$$q_2^{L\star} = \frac{a - 2c_2^L + c_1}{3} - \frac{1-\rho}{6}(c_2^H - c_2^L),$$

$$q_2^{H\star} = \frac{a - 2c_2^H + c_1}{3} + \frac{\rho}{6}(c_2^H - c_2^L),$$

and

$$q_1^\star = \frac{a - 2c_1 + \rho c_2^L + (1-\rho)c_2^H}{3}.$$

From an economic point of view, note that for $0 < \rho < 1$, the low-type Firm 2 produces less than it would if $\rho = 1$ and the high-type Firm 2 produces more than it would if $\rho = 0$.

Formalities

Assume a player's type θ_i is drawn from a common prior distribution, with $p(\theta_1, \ldots, \theta_n)$ the probability of a particular type vector over the players. Each player knows only his own type for sure. Let $p(\theta_{\sim i} \mid \theta_i)$ be a player's conditional probability on the others' types given his own. Each player has a pure strategy set S_i as before, but his utility function $u_i(s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)$ may now also depend on the types of the players. Since a player now observes his type, he may make his strategy choice contingent on type. Let $s_i(\theta_i)$ be the pure strategy choice of player i when he observes himself to be of type θ_i.

If the set of types if finite, a (Bayesian) equilibrium in pure strategies of such a game is simply a collection of strategies, one for each type of each player, such that each type of each player plays a best reply given the strategies of the others, or, formally, that

$$s_i(\theta_i) \in \arg\max_{s_i' \in S_i} \sum_{\theta_{\sim i}} p(\theta_{\sim i} \mid \theta_i) u_i(s_i', s_{\sim i}(\theta_{\sim i}), \theta_i, \theta_{\sim i})$$

for all i and θ_i.

types

		P2	
		θ_2^1	θ_2^2
P1	θ_1^1	0.25	0.25
	θ_1^2	0.5	0

if P1 notices he is of type θ_1^2, he knows how P2 is! (θ_2^1)

Purification of mixed equilibria

Harsanyi also showed that mixed-strategy equilibria can be interpreted as pure-strategy equilibria of games with some incomplete information about payoffs, which might be more appealing to people who reject the idea of players actually randomizing their decisions.

Consider the following game, which has some similarities to the market-entry with fixed costs game we studied some time ago.

		Player 2	
		Enter	Don't
Player 1	Enter	$-1, -1$	$1, 0$
	Don't	$0, 1$	$0, 0$

The unique symmetric equilibrium of this game has each player entering with probability 1/2.

every symmetric game has a symmetric equilibrium!

Now instead consider another game, which is exactly like the earlier one except there is some small uncertainty about a player's payoff if he is the only one entering. Specifically, if a player is the only one entering, he now gets $1 + \theta_i$, where θ_i is drawn from the uniform distribution on $[-\varepsilon, \varepsilon]$ for $\varepsilon > 0$.

$\theta_i \in [-\varepsilon, \varepsilon]$!

		Player 2	
		Enter	Don't
Player 1	Enter	$-1, -1$	$1 + \theta_1, 0$
	Don't	$0, 1 + \theta_2$	$0, 0$

This game has an equilibrium in which each player plays the pure strategy

$$s_i(\theta_i) = \begin{cases} \text{Don't} & \text{if } \theta_i < 0 \\ \text{Enter} & \text{otherwise.} \end{cases}$$

To see this, note that if Player 2 plays this strategy, he will enter with probability 1/2. Hence Player 1's expected payoff from entering is

$$.5(-1) + .5(1 + \theta_1)$$

which is non-negative if $\theta_1 \geq 0$. Hence it is a best reply for Player 1 to play the same strategy against Player 2.

As $\varepsilon \to 0$, the game converges to the original, complete information game, and the pure-strategy equilibrium of the incomplete information game converges to the mixed-strategy equilibrium of the original game.

Harsanyi showed more generally that mixed equilibria of generic finite normal-form games can be obtained as the limits of sequences of pure-strategy equilibria of games with payoff uncertainty.

Contests (Wärneryd 2003)

In a contest, players make expenditures in order to increase their probability of winning a fixed prize. Hence any positive expenditure is wasted from an efficiency perspective.

Contests have proved useful in modelling, e.g., rent seeking (going back to Tullock (1980)) and armed warfare (Hirshleifer (1995), Skaperdas (1992)).

We shall consider the implications of informational asymmetry in a common value contest. The assumption of common values and asymmetric information seems relevant for many applications; one example would be firms lobbying for monopoly privileges in a product market, where one firm is already the incumbent. Another might be a court battle between an investor and an entrepreneur who has defaulted on a debt contract, where only the entrepreneur knows the value of the surplus generated by the project.

Symmetric information

Two risk neutral agents, 1 and 2, compete for a prize of value y, where y is distributed according to the cumulative distribution F with support $[\underline{y}, \infty]$. Let $\tilde{y}$ be the expected value of the prize.

If agent 1 expends x_1 and agent 2 expends x_2, we assume the probability of agent i winning the prize is

$$p_i(x_1, x_2) := \begin{cases} x_i/(x_1+x_2) & \text{if } x_1 + x_2 > 0 \\ 1/2 & \text{otherwise.} \end{cases}$$

in case no one invests!

The expected utility of agent i if neither agent is informed of the value y when making his expenditure decision is then

$$u_i^U(x_1, x_2) := \int_{\underline{y}}^{\infty} p_i(x_1, x_2) y \mathrm{d}F(y) - x_i.$$

The best reply of agent i given the expenditure of the other agent is given by the first-order condition

$$\frac{\partial u_i^U(x_1, x_2)}{\partial x_i} = \frac{x_j}{(x_1+x_2)^2}\tilde{y} - 1 = 0 \text{ for } j \neq i.$$

symmetry

In equilibrium, each agent expends $x^S := \tilde{y}/4$.

Similarly, it is easily seen that if both agents are informed about y when making their expenditures, then for any y they will each expend $x_i := y/4$. Hence the *ex ante* expected individual expenditure is again $\tilde{y}/4$.

what if both know?

$$u_i(x_1, x_2) = P_i(x_1, x_2) y - x_i = \frac{x_i}{x_1 + x_2} y - x_i$$

$$\frac{\partial u_i}{\partial x_i} = \frac{x_j}{(x_1+x_2)^2} y - 1 = 0 \xRightarrow{\text{sym}} \boxed{x^* = \frac{y}{4}}$$

Asymmetric information

Consider next what happens if one agent is informed of y but the other agent is uninformed. The uninformed agent is now potentially subject to an analogue of the winner's curse, and hence cannot rationally estimate the value of the prize at its expectation.

The informed agent's objective function, given the realization y, is now

$$u_I(y) := \frac{x_I(y)}{x_U + x_I(y)} y - x_I(y),$$

where x_U is the expenditure of the uninformed agent and $x_I(y)$ is the expenditure of the informed agent as a function of the value y. We shall sometimes refer to y as the *type* of the informed player.

The first partial derivative of this objective function with respect to $x_I(y)$ is

$$\frac{\partial u_I(y)}{\partial x_I(y)} = \frac{x_U}{(x_U + x_I(y))^2} y - 1,$$

logically, if Uninformed bids more than the value, Informed doesn't enter!

which is negative for all $x_I(y) \geq 0$ if we have $x_U > y$. The informed agent's best reply function, given y, is therefore

$$x_I(y) = \begin{cases} \sqrt{x_U y} - x_U & \text{if } x_U \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Consider next the uninformed agent, whose objective function is now

$$u_U := \int_{\underline{y}}^{\infty} \frac{x_U}{x_U + x_I(y)} y \mathrm{d}F(y) - x_U.$$

It cannot be the case in equilibrium that $x_I(y) = 0$ for all y. This would imply $x_U \geq \bar{y}$. But given that no type of the informed player expends a positive amount, this cannot be a best reply on the part of the uninformed player, since he could lower his expenditure and still win with probability one in all states.

The relevant condition for an optimal expenditure level on the part of the uninformed agent, given $x_I(y)$, is therefore

$$\frac{\partial u_U}{\partial x_U} = \int_{\underline{y}}^{\infty} \frac{x_I(y)}{(x_U + x_I(y))^2} y \mathrm{d}F(y) - 1 = 0.$$

To simplify, suppose there are just two possible values of the prize, y_H and y_L, where we have $0 < y_L < y_H$. Let the probability of y_H be q.

There are two possible types of equilibria.

Case 1: Both informed types are active. Suppose we have $x_I(y_L) > 0$ and $x_I(y_H) > 0$. The uninformed agent's first order condition for a best reply expenditure is then

$x_I(y_L) = \sqrt{x_U y_L} - x_U$

$$(1-q)\frac{x_I(y_L)}{(x_U + x_I(y_L))^2}y_L + q\frac{x_I(y_H)}{(x_U + x_I(y_H))^2}y_H - 1 =$$
$$(1-q)\frac{1}{\sqrt{x_U}}\sqrt{y_L} + q\frac{1}{\sqrt{x_U}}\sqrt{y_H} - 2 = 0,$$

so equilibrium expenditure is

Jensen's inequality

$$x_U = \frac{(q\sqrt{y_H} + (1-q)\sqrt{y_L})^2}{4}. \quad < \frac{\tilde{y}}{4}$$

We must have $x_U < y_L$ for this to be an equilibrium, i.e., that

$$q < \frac{\sqrt{y_L/y_H}}{1 - \sqrt{y_L/y_H}} =: \hat{q}.$$

Case 2: Only the highest informed type is active.
Suppose we have $x_I(y_L) = 0$. The uninformed agent's first-order condition then reduces to

$$q\frac{x_I(y_H)}{(x_U + x_I(y_H))^2}y_H - 1 = q\frac{1}{\sqrt{x_U}}\sqrt{y_H} - 1 - q = 0,$$

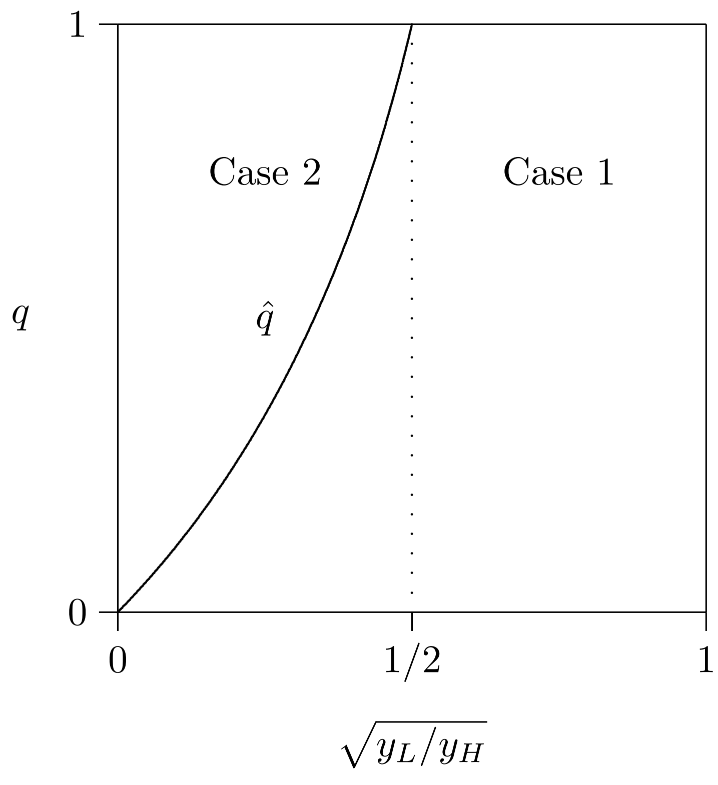
so we have that

$$x_U = \left(\frac{q}{1+q}\right)^2 y_H.$$

In order for this to be consistent with the lowest type expending nothing, we must have $x_U \geq y_L$, i.e., that

$$q \geq \hat{q}.$$

The figure illustrates both types of equilibrium.



Note that in the Case 2 equilibrium, the uninformed player's probability of winning the object is

$$1 - q + q\frac{x_U}{x_U + x_I(y_H)} = 1 - q + \frac{q^2}{1+q} > 1/2.$$

Let $q \to 0$ and $y_L/y_H \to 0$, so that the condition for the existence of equilibrium is satisfied. Then the probability of the uninformed player winning approaches 1.

It can be shown more generally that the uninformed player always wins with a strictly higher probability than does the informed player.

Optimal auctions

Here is a little bit of mechanism design, or contract, theory. There is a seller with one unit of a good, and two potential buyers. The buyers have valuations of the good that are independent draws from the same distribution. A buyer's valuation is $\underline{\theta}$ with probability $\underline{p}$ and $\overline{\theta}$ with probability $\overline{p} = 1 - \underline{p}$, with $\underline{\theta} < \overline{\theta}$.

The seller wants to maximize his expected revenue by auctioning off the good in an optimal fashion. We therefore depart slightly from the noncooperative framework by assuming that the buyers can *ex ante* commit to follow the rules of the auction proposed by the seller, i.e., they can sign a binding contract to be governed by the mechanism offered by the seller. Once they have accepted the mechanism, they are in a game designed by the seller. We are thus looking for the optimal game form from the point of view of the seller.

A familar mechanism would be the first-price auction, in which the players simultaneously make bids and the highest bidder gets the object and pays his bid. But more generally the seller could construct a game in which the players have more extensive strategy sets, e.g., they could make bids, but also wiggle their ears, etc. In such a setting, familiar auction types are not necessarily optimal.

Suppose the buyers have some pure strategy sets S_1 and S_2 under a mechanism. The mechanism then also specifies

- the probability $x_i(s_1, s_2)$ that buyer i gets the good, and
- the transfer payment $T_i(s_1, s_2)$ from the buyer to the seller.

Since participation in the mechanism is voluntary, each type of each buyer must have non-negative expectation in equilibrium. Suppose $\sigma_1^\star$ and $\sigma_2^\star$ are equilibrium strategies under the mechanism. For each type $\theta_1 \in \{\underline{\theta}, \overline{\theta}\}$ and each $s_1 \in S_1$ that has positive probability under $\sigma_1^\star$, we must have that

$$\mathrm{E}_{\theta_2}\mathrm{E}_{\sigma_2^\star(\theta_2)}(\theta_1 x_1(s_1, s_2) - T_1(s_1, s_2)) \geq 0$$

expectation conditional on P2

for it to be rational for buyer 1 to accept the mechanism. This is buyer 1's *individual rationality* constraint. A similar one must hold for buyer 2.

Since $\sigma_1^\star$ and $\sigma_2^\star$ are equilibrium strategies, we must also for every θ_1, s_1 in the support of $\sigma_1^\star$, and s_1' have that

$$\begin{aligned}\mathrm{E}_{\theta_2}\mathrm{E}_{\sigma_2^\star(\theta_2)}(\theta_1 x_1(s_1, s_2) - T_1(s_1, s_2)) \geq \\ \mathrm{E}_{\theta_2}\mathrm{E}_{\sigma_2^\star(\theta_2)}(\theta_1 x_1(s_1', s_2) - T_1(s_1', s_2)).\end{aligned}$$

no incentive to lie about their type

This is buyer 1's *incentive compatibility* constraint. A similar one holds for buyer 2.

We now note that given the game and the equilibrium strategies, what happens in the end is a function only of the buyers' types. That is, the seller could instead of having the buyers actually play the game just ask them to report their types and promise to implement the outcome that would have happened in equilibrium of the more complicated game. That is, if we define

$$\tilde{x}_i(\hat{\theta}_1, \hat{\theta}_2) := \mathrm{E}_{(\sigma_1^\star(\hat{\theta}_1), \sigma_2^\star(\hat{\theta}_2))} x_i(s_1, s_2)$$

and

$$\tilde{T}_i(\hat{\theta}_1, \hat{\theta}_2) := \mathrm{E}_{(\sigma_1^\star(\hat{\theta}_1), \sigma_2^\star(\hat{\theta}_2))} T_i(s_1, s_2).$$

Since the strategies they would have played are equilibrium strategies, and they have positive expectation, they would accept a simplified mechanism defined by $\tilde{x}$ and $\tilde{T}$ where they only report their types, and they would in fact truthfully report their types.

This is an instance of the more general *revelation principle*, which says that any mechanism has an equivalent direct-revelation mechanism, i.e., one in which the agents only report their types. This is convenient, since we can then study only direct-revelation mechanisms and still find the overall optimal one.

Let $\underline{X}$, $\overline{X}$, $\underline{T}$, and $\overline{T}$ be the expectations of the probability of getting the good and the transfer for buyers of low and high type. The individual rationality and incentive compatibility constraints are then

$$\underline{\theta X} - \underline{T} \geq 0, \tag{IR$_1$}$$

$$\overline{\theta X} - \overline{T} \geq 0, \tag{IR$_2$}$$

$$\underline{\theta X} - \underline{T} \geq \underline{\theta}\overline{X} - \overline{T}, \tag{IC$_1$}$$

and

$$\overline{\theta X} - \overline{T} \geq \overline{\theta}\underline{X} - \underline{T}. \tag{IC$_2$}$$

We now want to find X and T such that the seller's expected profit

per - buyer

$$\mathrm{E}u_0 := \underline{p}\underline{T} + \overline{p}\overline{T}$$

is maximized subject to the IR and IC constraints.

It can be shown that the only binding constraints at the optimum are IR_1 and IC_2. Hence the transfer payments are determined as

$$\underline{T} = \underline{\theta X}$$

and

$$\overline{T} = \overline{\theta}(\overline{X} - \underline{X}) + \underline{\theta X}.$$

Substituting these values into the seller's profit gives

$$\mathrm{E}u_0 = (\underline{\theta} - \overline{p}\overline{\theta})\underline{X} + \overline{p}\overline{\theta X}.$$

Suppose we have $\underline{\theta} \le \overline{p}\overline{\theta}$. Then $\mathrm{E}u_0$ is decreasing in $\underline{X}$. Hence the optimal solution is to set $\underline{X} = 0$ and $\overline{X}$ as large as possible. By symmetry, if both buyers are of high type they must each win with probability 1/2. Hence we must have $\overline{X} = \underline{p} + \overline{p}(1/2)$. The optimal mechanism is therefore to sell to nobody if both report low types, to sell only to the high type if only one reports the high type, and to sell to each with probability one-half if both report the high type.

Suppose instead we have $\underline{\theta} > \overline{p}\overline{\theta}$. Then $\mathrm{E}u_0$ is strictly increasing in both $\underline{X}$ and $\overline{X}$. But by symmetry, a player's *ex ante* probability of winning must not exceed 1/2; hence we must have

$$\underline{p}\underline{X} + \overline{p}\overline{X} = 1/2. \tag{$*$}$$

Solving for $\underline{X}$ and substituting in $\mathrm{E}u_0$ gives

$$\mathrm{E}u_0 = \frac{1}{2\underline{p}}(\underline{\theta} - \overline{p}\overline{\theta}) + \frac{\overline{p}}{\underline{p}}(\overline{\theta} - \underline{\theta})\overline{X}.$$

Hence we again have $\overline{X} = \underline{p} + \overline{p}(1/2)$. From ($*$) we have $\underline{X} = \underline{p}/2$. Thus in this case the optimal mechanism awards the good to a high type if he is the only one; in all other cases the mechanism awards the good with equal probability to each player.

Problem. *Consider the following two payoff matrices.*

		Player 2	
		L	R
Player 1	T	$1, 1$	$0, 0$
	B	$0, 0$	$0, 0$

I

		Player 2	
		L	R
Player 1	T	$0, 0$	$0, 0$
	B	$0, 0$	$2, 2$

II

Now consider a game in which Nature first determines whether the payoffs are as in I or as in II, with both being equally likely. Player 1 then observes which is the case, but Player 2 does not. The players next choose actions simultaneously, and get payoffs according to the relevant matrix. Find all the pure-strategy equilibria of this game.

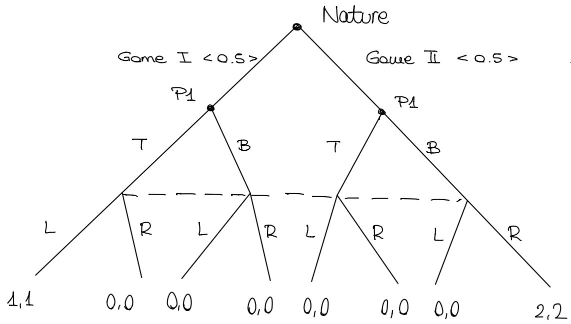
Problem. *Consider again the voluntary public goods provision game with incomplete information that we have discussed, but now let there be $n > 2$ players, and assume the public good is provided only if m players contribute. Find an equilibrium condition. Show that for $m \geq 2$, there is always an equilibrium in which nobody contributes.*

Problem. *Consider again the Cournot duopoly game with incomplete information that we have studied. When is there an equilibrium such that the high-cost type of Firm 2 produces nothing, and what does it look like?*

Problem. *Suppose there are two consumers and one public good. The consumers simultaneously contribute an amount of money $x_i \in [0, a_i]$ for this good, where a_i is the wealth of consumer i. The resulting quantity of public good is $y(x_1, x_2) = x_1 + x_2$. The preferences of consumer i is given by the (Cobb-Douglas) utility function $u_i(x_1, x_2) = y(x_1, x_2)(a_i - x_i)$. It is common knowledge that $a_1 = 1$, but a_2 is known only to consumer 2. Player 1 assigns probability 1/2 to the event $a_2 = a^L = 0.5$ and probability 1/2 to the event $a_2 = a^H = 1.5$. The probabilities are common knowledge. Find the unique equilibrium of this game.*

PROBLEM 1

The game has the following extensive form



Notice that P1 has 4 strategies: TT, TB, BT and BB. P2 has only 2 strategies, L and R. The game can be written in normal form

P1 \ P2	L	R
TT	0.5, 0.5	0, 0
TB	0.5, 0.5	1, 1
BT	0, 0	0, 0
BB	0, 0	1, 1

Pure NE $\equiv \{ \langle TT, L \rangle, \langle TB, R \rangle, \langle BB, R \rangle \}$

PROBLEM 2

PROBLEM 3

F1's cost c_1 is common knowledge, while F2's cost is only privately known. However it is common knowledge that it is low with probability p.

F1 solves

$$\max_{q_1} \ \mathbb{E}\pi_1 = \left[p(a - q_1 - q_2^L - c_1) + (1-p)(a - q_1 - q_2^H - c_1) \right] q_1$$

$$\frac{\partial \mathbb{E}\pi_1}{\partial q_1} = p(a - q_1 - q_2^L - c_1) + (1-p)(a - q_1 - q_2^H - c_1) + \left[-p - (1-p) \right] q_1 = 0$$

$$\Rightarrow \quad q_1 = \frac{a - c_1 - \left[p q_2^L + (1-p) q_2^H \right]}{2} \qquad (1)$$

F2 of type $i \in \{L, H\}$ solves

$$\max_{q_2^i} \ \pi_2^i = (a - q_1 - q_2^i - c_2^i) q_2^i$$

$$\frac{\partial \pi_2^i}{\partial q_2^i} = a - q_1 - q_2^i - c_2^i + (-q_2^i) = 0 \Rightarrow q_2^i = \frac{a - c_2^i - q_1}{2} \qquad (2)$$

We are looking for an equilibrium where $q_2^H = 0$. That occurs when

$$q_2^H \leq 0 \Rightarrow \frac{a - c_2^H - q_1}{2} \leq 0 \Rightarrow c_2^H \geq a - q_1$$

and where $q_2^L \geq 0$,

$$q_2^L \geq 0 \Rightarrow \frac{a - c_2^L - q_1}{2} \geq 0 \Rightarrow c_2^L \leq a - q_1$$

Introducing (2) into (1), and knowing that $q_2^H = 0$

$$q_1 = \frac{a - c_1 - \left[\rho\left(\frac{a - c_2^L - q_1}{2}\right) + (1-\rho)0\right]}{2} \Rightarrow 2q_1 = a - c_1 - \frac{\rho}{2}(a - c_2^L) + \frac{\rho}{2}q_1 \Rightarrow$$

$$\Rightarrow q_1\underbrace{\left(2 - \frac{\rho}{2}\right)}_{\frac{4-\rho}{2}} = a - c_1 - \frac{\rho}{2}a + \frac{\rho}{2}c_2^L \Rightarrow q_1\left(\frac{4-\rho}{2}\right) = a\underbrace{\left(1 - \frac{\rho}{2}\right)}_{\frac{2-\rho}{2}} - c_1 + \frac{\rho}{2}c_2^L \Rightarrow$$

$$\Rightarrow q_1\frac{(4-\rho)}{2} = a\frac{(2-\rho)}{2} - c_1 + \frac{\rho}{2}c_2^L \Rightarrow \boxed{q_1^* = \frac{a(2-\rho) - 2c_1 + \rho c_2^L}{4-\rho}}$$

Recall that, for $F2^H$ not to participate

$$c_2^H \geq a - q_1 = a - \frac{a(2-\rho) - 2c_1 + \rho c_2^L}{4-\rho} = \frac{2(a+c_1) - \rho c_2^L}{4-\rho} \Rightarrow \boxed{c_2^H \geq \frac{2(a+c_1) - \rho c_2^L}{4-\rho}} \quad (3)$$

Equivalently, for $F2^L$ to produce

$$c_2^L \leq a - q_1 = \frac{2(a+c_1) - \rho c_2^L}{4-\rho} \Rightarrow c_2^L\underbrace{\left(1 + \frac{\rho}{4-\rho}\right)}_{\frac{4-\rho+\rho}{4-\rho}} \leq \frac{2(a+c_1)}{4-\rho} \Rightarrow c_2^L\frac{4}{4-\rho} \leq \frac{2(a+c_1)}{4-\rho} \Rightarrow$$

$$\Rightarrow 4c_2^L \leq 2(a+c_1) \Rightarrow \boxed{c_2^L \leq \frac{a+c_1}{2}} \quad (4)$$

Hence, we need to satisfy (3) - (4).

PROBLEM 4

Notice that a_2^i, where $i \in \{L, H\}$ is private info to P2. The maximization problem for P1 is

$$\max_{x_1 \in [0, a_1]} u_1 = p(x_1 + x_2^L)(a_1 - x_1) + (1-p)(x_1 + x_2^H)(a_1 - x_1)$$

$$\frac{du_1}{dx_1} = p(a_1 - x_1) - p(x_1 + x_2^L) + (1-p)(a_1 - x_1) - (1-p)(x_1 + x_2^H) = 0 \Rightarrow$$

$$\Rightarrow p(a_1 - 2x_1 - x_2^L) + (1-p)(a_1 - 2x_1 - x_2^H) \Rightarrow$$

$$\Rightarrow 2px_1 + 2(1-p)x_1 = p(a_1 - x_2^L) + (1-p)(a_1 - x_2^H) \Rightarrow$$

$$\Rightarrow 2x_1 = a_1 - px_2^L - (1-p)x_2^H \Rightarrow x_1 = \frac{a_1 - px_2^L - (1-p)x_2^H}{2} = \frac{1}{2} - \frac{x_2^L}{4} - \frac{x_2^H}{4}$$

The problem for P2 is

$$\max_{x_2^i \in [0, a_2^i]} u_2^i = (x_1 + x_2^i)(a_2^i - x_2)$$

$$\frac{\partial u_2^i}{\partial x_2^i} = (a_2^i - x_2^i) - (x_1 + x_2^i) = 0 \implies a_2^i - x_1 = 2x_2^i \implies x_2^i = \frac{a_2^i - x_1}{2}$$

$$x_2^L = \frac{1}{4} - \frac{x_1}{2}$$

$$x_2^H = \frac{3}{4} - \frac{x_1}{2}$$

Combining the 3 equations

$$x_1 = \frac{1}{2} - \frac{1}{16} + \frac{x_1}{8} - \frac{3}{16} + \frac{x_1}{8} = \frac{8 - 1 - 3 + 2x_1 + 2x_1}{16} = \frac{4 + 4x_1}{16} = \frac{1 + x_1}{4} \implies$$

$$\implies 4x_1 = 1 + x_1 \implies x_1(4-1) = 1 \implies \boxed{x_1 = 1/3}$$

$$x_2^L = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} \implies \boxed{x_2^L = 1/12}$$

$$x_2^H = \frac{3}{4} - \frac{1}{6} = \frac{9-2}{12} \implies \boxed{x_2^H = 7/12}$$