HANK Beyond FIRE: Amplification, Forward Guidance, and Belief Shocks

José-Elías Gallegos

Banco de España

March 26, 2025

Monetary policy transmission in the benchmark New Keynesian (NK) framework depends critically on the Full Information Rational Expectations (FIRE) assumption, particularly through indirect general equilibrium (GE) effects. This paper relaxes the FIRE assumption by introducing noisy information into a Heterogeneous-Agents NK model with financial frictions. Noisy information attenuates the amplification of monetary policy shocks by muting GE effects, aligning the model's dynamics with empirical evidence, and resolving the forward guidance puzzle. Noisy information also broadens the determinacy region for monetary policy rules. Finally, I explore the role of belief shocks, showing that transitory "animal spirits" shocks can generate persistent effects, offering new insights into the macroeconomic implications of information frictions.

Keywords: Imperfect Information; New Keynesian; Heterogeneous Agents; Monetary Policy.

JEL Classifications: E31, E43, E52, E71.

C/ Alcalá 48, 28014 Madrid, Spain. E-mail: jose.elias.gallegos@bde.es. I am grateful to Tobias Broer, Alex Kohlhas and Per Krusell for their advice and support, and to Florin Bilbiie and Rajssa Mechelli for discussing my paper. Further, I would like to thank Mattias Almgren, Gualtiero Azzalini, Henrique Basso, Larry Christiano, James Costain, José Luis Dago Elorza, Rubén Domínguez-Díaz, Richard Foltyn, Gaetano Gaballo, John Hassler, Philipp Hochmuth, Markus Kondziella, John Kramer, Kurt Mitman, Markus Peters, Omar Rachedi, Anna Seim, Ulf Söderström, Carlos Thomas, Claire Thurwächter, Karl Walentin, Magnus Ahl and seminar and conference participants at the 54th Annual Conference of the MMF Society, Banco de España, Central Bank Macroeconomic Modeling Workshop 2023, DMM Workshop 2022, ERMAS 2023, IIES, INFER Annual Meeting 2020, Oxford Nucamp Conference, Nordic Junior Macro Seminar Series, Norges Bank Macro Modelling Workshop, Perspectives on Kurtosis in Macroeconomics, SAE 2020 and 2022, 3rd Sailing the Macro Workshop, Swedish Economics Conference 2022, UCL, 11th UECE for useful feedback and comments. I am indebted to Marios Angeletos for the title of the paper, which I took from his MIT graduate course "Macroeconomics beyond FIRE". I thank Zhen Huo for graciously sharing his teaching materials. The views expressed in this paper are those of the author and do not necessarily represent the views of Banco de España or the Eurosystem.

1. Introduction

Inequality and information frictions significantly influence how aggregate shocks impact the economy. On the one hand, higher levels of inequality have been associated with stronger responses of output to monetary policy shocks (Bilbiie 2024; Almgren et al. 2022).[1] On the other hand, information frictions that manifest in the form of forecast underreaction to news (Coibion and Gorodnichenko 2015) dampen the immediate effect of shocks and explain the lagged general equilibrium (GE) response documented in the data (Holm et al. 2021). In this paper, I show that the interaction between inequality and information imperfections reshapes the transmission of monetary policy.

To clearly examine how financial constraints and information frictions interact, I develop a tractable Heterogeneous-Agent New Keynesian (HANK) model based on Bilbiie (2024), extended to include noisy and dispersed information following Angeletos and Huo (2021), disciplined with survey evidence (Coibion and Gorodnichenko 2015). This framework integrates key aspects of microeconomic heterogeneity: cyclical inequality, idiosyncratic risk, and precautionary savings, which together yield heterogeneous marginal propensities to consume.

In the Full Information Rational Expectations (FIRE) benchmark, economies with greater inequality exhibit amplified responses to exogenous shocks due to higher marginal propensities to consume among financially constrained households. This amplification hinges on agents having perfect information about the indirect GE effects.

I relax the FIRE assumption by introducing noisy and dispersed information, where agents receive imperfect, idiosyncratic signals about the economy's state. As a result, they are uncertain about both exogenous fundamentals (e.g., monetary policy shocks) and aggregate variables shaped by others' actions, such as output and inflation. This uncertainty forces agents to form higher-order beliefs—expectations about others' expectations—which in turn weakens the indirect effects driving amplification under FIRE. This reduces the degree of complementarity of actions across agents, dampens GE effects disproportionately, and partially mutes the amplification mechanism that critically relies on them.

Utilizing this setting and disciplining the degree of belief formation frictions with survey evidence, I revisit several key results in the monetary economics literature.

First, I find that noisy information dampens the amplification of monetary policy shocks by weakening indirect GE effects: the multiplier is lowered from 20% to 14%. The

[1] Galí et al. (2007) and Brinca et al. (2016) document the equivalent result for fiscal policy.

arrest of indirect effects is consistent with empirical findings from Holm et al. (2021): GE effects do not contribute to the output response on impact, but account for around 70% in the long run. Finally, the hump-shaped impulse response of output is consistent with empirical evidence from Ramey (2016). Importantly, these dynamics emerge without relying on ad hoc features such as habit formation or price indexation.

Second, the model addresses the forward guidance puzzle (FGP) by reducing the excessive sensitivity of current output to expected future interest rates. This over-sensitivity is prevalent in FIRE models and is further exacerbated by financial frictions in HANK economies. I find that the impact of a forward guidance promise made four quarters ahead on current output is about one-third the size of the impact of the same promise made one quarter ahead. Furthermore, I show that imperfect information effectively relaxes the constraints on monetary policy parameters required for a determinate equilibrium.

Lastly, I investigate the effects of transitory "animal spirits" shocks—perturbations to non-fundamentals that move agents' beliefs without directly affecting the fundamentals in the economy—and show that they can generate persistent effects on output due to agents' inability to discern noise from fundamental shocks. For instance, the output response to a purely transitory "animal spirits" shock has a half-life of approximately four quarters.

Related Literature. This paper builds on the HANK literature. As laid out by Galí et al. (2007); Bilbiie (2008)—as well as richer quantitative models by Gornemann et al. (2016), Werning (2015), Auclert (2019) and Hagedorn et al. (2019)— whether aggregate shocks have larger effects on aggregate consumption, compared to the representative-agent framework, is ambiguous. In a model that combines the tractability of two-agent New Keynesian (TANK) models with the key elements of heterogeneous-agent models, Bilbiie (2024) shows that the output response to shocks is amplified if the income elasticity of constrained agents with respect to aggregate income is larger than one.[2] The amplification is strengthened if a larger fraction of agents is constrained. I contribute to this strand of the literature by demonstrating how noisy information can mute the amplification effects through the dampening and delay of indirect effects.

The belief formation literature extends the FIRE benchmark by introducing frictions in agents' expectations, typically disciplined by survey data. Various approaches that

[2] He refers to this case as cyclical income inequality, for which Almgren et al. (2022); Patterson (2022) find empirical evidence.

deviate from rational expectations include bounded rationality, either through level-K thinking (Farhi and Werning 2019) or cognitive discounting (Gabaix 2020), and diagnostic expectations (Bordalo et al. 2020). Within rational expectations, noisy information captures the gradual propagation of shocks through the imperfect observation of fundamentals, and generates sluggish updating of expectations—consistent with sticky information (Mankiw and Reis 2002)—while additionally attenuating the indirect GE effects by distorting higher-order beliefs (Angeletos and Huo 2021). I build upon this literature and show how noisy information affects the main key results of the monetary economics literature—the hump-shaped response of macroeconomic variables to monetary shocks, the FGP, and the Taylor Principle.

Recent work has explored the interaction between HANK and belief formation frictions. Auclert et al. (2020) introduce sticky information, while Farhi and Werning (2019) and Pfäuti and Seyrich (2022) incorporate bounded rationality through level-K thinking and cognitive discounting, respectively, into heterogeneous-agent settings. These frameworks underscore the role of imperfect expectations in shaping monetary transmission, particularly through their influence on GE effects. My framework builds on this literature by emphasizing the role of information frictions in moderating the timing and strength of indirect effects, aligning with empirical findings from Holm et al. (2021), who document a lagged GE response to monetary shocks in the data. While existing belief formation models primarily affect the magnitude of GE effects, my framework naturally replicates both their delay and attenuation, offering an alternative perspective on the transmission of monetary policy within HANK settings.

Roadmap. The remainder of the paper is organized as follows. Section 2 introduces the theoretical framework, detailing the HANK model with financial frictions and noisy information. In Section 3, I take a brief detour to inspect the transmission mechanism of monetary policy in a reduced-form version of the model. In Section 4, I study quantitatively the implications of the HANK beyond FIRE model on the amplification of monetary policy shocks, the decomposition of direct and indirect effects, the Taylor Principle, the resolution of the FGP, and explore the macroeconomic consequences of belief shocks. Section 5 concludes. Proofs of propositions and corollaries in the main text are relegated to the Appendix A.

2. The Analytical HANK Beyond FIRE Model

The HANK framework described in this section is a reduced-form version of the standard incomplete markets model, based on Bilbiie (2024). Households face an idiosyncratic risk of not being able to access asset markets, instead of risky labor income. This simplifying assumption allows me to solve the model with paper and pencil, and provides the precautionary savings motive that TANK models lack. On top of household heterogeneity concerning their market participation, agents face uncertainty about the state of nature. They receive imperfect and heterogeneous signals, leading to dispersed beliefs and forecast heterogeneity. This gives rise to higher-order beliefs: to forecast aggregate endogenous outcomes such as output and inflation, an agent needs to forecast the actions of other agents, and other agents need to forecast the actions of others, *ad infinitum*.

2.1. Households

The modeling of financial frictions and the households' problem follows closely Bilbiie (2024), and I stress the differences in the setup to accommodate incomplete information. There is a measure-1 continuum of ex-ante identical consumers in the economy, indexed by $i \in \mathcal{I}_c = [0, 1]$, with discount factor β, deriving utility from consumption and disutility from labor supply, and saving in government riskless bonds with real return R_t.

Environment. Financial frictions are exogenous to individual behavior. Following Bilbiie (2024), I assume that households belong to a family whose utilitarian intertemporal welfare its head maximizes facing limits to risk sharing. If the household is financially constrained (state H), it is unable to save and loses access to firm profits, but keeps access to previous-period savings. In contrast, unconstrained households benefit from having access to asset markets (state S).

Each family has two heads, located in each financial state. The family head can transfer all resources across households within the family members in the same financial state, but can only transfer some resources between family members in a different financial state.[3] There is thus full insurance between family members within a state, but limited across states.

[3] I make room for incomplete information by allowing for different families (compared to Bilbiie 2024), with family heads being subject to information frictions.

At the beginning of the period, the family head pools the resources within the family and state. After the aggregate shock is realized, the family head determines the consumption and saving for each household in each state. Then, households change their financial state, taking bonds with them, which allows for incorporating a precautionary savings motive.

The exogenous shock takes the form of a Markov chain. Denote by s the probability of remaining unconstrained and by h the probability of remaining constrained, and denote by $1 - s$ and $1 - h$ the respective transition probabilities. For simplicity, I assume that the Markov process induces a stationary distribution: the share of Hand-to-Mouth (HtM) agents λ is given by $\lambda = (1 - s)/(2 - s - h)$. Welfare maximization implies that the family head pools the resources of their family members in their state at the beginning of the period, and implements symmetric consumption and saving choices for all households in the same family and financial state.

Households' Problem. Denote by $B^X_{i,t+1}$ the per-capita real bonds of households in family i and financial state $X \in \{H, S\}$ at the beginning of period $t + 1$, after the consumption choice, changing state and pooling. Similarly, denote by $Z^X_{i,t+1}$ the per-capita real bonds of households in family i and financial state X at the end of period t, after the consumption choice, but before changing state. The following relations hold: $(1 - \lambda)B^S_{i,t+1} = s(1 - \lambda)Z^S_{i,t+1} + (1 - h)\lambda Z^H_{i,t+1}$ and $\lambda B^H_{i,t+1} = (1 - s)(1 - \lambda)Z^S_{i,t+1} + h\lambda Z^H_{i,t+1}$. Making use of the stationary distribution of agents across states, the two laws of motion can be written as

$$B^S_{i,t+1} = sZ^S_{i,t+1} + (1 - s)Z^H_{i,t+1} \quad \text{and} \quad B^H_{i,t+1} = (1 - h)Z^S_{i,t+1} + hZ^H_{i,t+1}. \tag{1}$$

The program of the family head is

$$\mathcal{W}(B^S_{it}, B^H_{it}) = \max_{\{C^S_{it}, C^H_{it}, Z^S_{i,t+1}, Z^H_{i,t+1}\}} \Big[(1 - \lambda)U(C^S_{it}) + \lambda U(C^H_{it})\Big] + \beta \mathbb{E}_{it} \mathcal{W}(B^S_{i,t+1}, B^H_{i,t+1}),$$

subject to the laws of motion (1) and budget constraints

$$C^S_{it} + Z^S_{i,t+1} = Y^S_{it} + R_{t-1}B^S_{it}, \quad C^H_{it} + Z^H_{i,t+1} = Y^H_{it} + R_{t-1}B^H_{it}, \quad \text{and} \quad Z^S_{i,t+1}, Z^H_{i,t+1} \geq 0, \tag{2}$$

where Y^X_{it} denotes the post-tax income (which for unconstrained households includes dividends from holding firm shares, non-transferable across financial states). The optimality conditions are as follows:

$$
\begin{aligned}
(3) \qquad & U'(C^S_{it}) \geq \beta \mathbb{E}_{it} R_t \left[sU'(C^S_{i,t+1}) + (1-s)U'(C^H_{i,t+1}) \right], \\
& 0 = Z^S_{i,t+1} \left\{ U'(C^S_{it}) - \beta \mathbb{E}_{it} R_t \left[sU'(C^S_{i,t+1}) + (1-s)U'(C^H_{i,t+1}) \right] \right\}, \\
(4) \qquad & U'(C^H_{it}) \geq \beta \mathbb{E}_{it} R_t \left[(1-h)U'(C^S_{i,t+1}) + hU'(C^H_{i,t+1}) \right], \\
& 0 = Z^H_{i,t+1} \left\{ U'(C^H_{it}) - \beta \mathbb{E}_{it} R_t \left[(1-h)U'(C^S_{i,t+1}) + hU'(C^H_{i,t+1}) \right] \right\}.
\end{aligned}
$$

I follow Bilbiie (2024) and focus on equilibria where the constraint H always binds and (4) is a strict inequality,[4] and under a zero liquidity limit (Krusell et al. 2011): the demand for bonds of households in S is well-defined, the supply is zero, and there are no bonds held in equilibrium. These constraints imply that, for the HtM households, $C^H_{it} = Y^H_{it}$.

The rest of the model follows closely the two-agent version of Bilbiie (2008), enlarged to accommodate incomplete information. For both sets of agents, with post-tax incomes given by $Y^H_{it} = W_{it}N^H_{it} + E^H_{it}$ and $Y^S_{it} = W_{it}N^S_{it} + \frac{1}{1-\lambda}D_{it} + E^S_{it}$, the optimal labor supply condition is $U_C(C^X_{it})/U_N(N^X_{it}) = W_{it}$. I assume that utility takes a CRRA form $U(C_i, N_i) = C_i^{1-\sigma}/(1-\sigma) - N_i^{1+\varphi}/(1+\varphi)$, with σ denoting the inverse intertemporal elasticity of substitution and φ the inverse Frisch elasticity. The model is solved in its log-linearized version around a steady state with no consumption inequality among financial states, $C^S_i = C^H_i$.

Auxiliary Shocks. Let W_t, D_t, and E^X_t denote, respectively, the average real wage, the average real dividend from firms' profits, and the average real fiscal transfer received by households in state X, to be defined. Following Angeletos and Lian (2018), I consider a set of auxiliary shocks whose only modeling role is to noise up the information that consumers can extract from their individual information. The real wage, the transfer and the dividend received by consumer i at time t are given by, respectively, $W_{it} = \zeta^W_{it} W_t$, $D_{it} = \zeta^D_{it} D_t$, and $E^X_{it} = \zeta^X_{it} E^X_t$, where $\{\zeta^W_{it}, \zeta^D_{it}, \zeta^S_{it}, \zeta^H_{it}\}$ are i.i.d. across i and t. One can interpret these as shocks to a consumer's labor and financial income.

2.2. Firms

Each good is produced by an intermediate monopolistic firm $j \in \mathcal{I}_f = [0,1]$ that uses a technology linear in labor $Y_{jt} = N_{jt}$.

[4] Bilbiie (2024) motivates this assumption through liquidity or impatience shocks incentivizing H agents to consume more today, or a technological constraint that prevents them from accessing asset markets.

Price-Setting. Nominal price rigidities take the form of a Calvo price-setting friction. In every period, each firm can reset its price with probability $1-\theta$, independent of the time of the last price change. A measure $1-\theta$ of firms can reset their prices in a given period, and the average duration of a price is given by $1/(1-\theta)$.

A firm re-optimizing in period t will choose the price P^*_{jt} that maximizes the current market value of the profits generated while the price remains effective. Formally,

$$P^*_{jt} = \arg\max_{P_{jt}} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_{jt} \left\{ \Lambda_{t,t+k}/P_{t+k} \left[P_{jt} Y_{j,t+k|t} - \mathcal{C}_{t+k}(Y_{j,t+j|t}) \right] \right\},$$

subject to the sequence of demand equations, $Y_{j,t+k|t} = (P_{jt}/P_{t+k})^{-\epsilon} Y_{t+k}$, where ϵ is the constant elasticity of substitution between different good varieties, where $\Lambda_{t,t+k} \equiv \beta^k (C_{t+k}/C_t)^{-\sigma}$ is the stochastic discount factor, $\mathcal{C}_t(\cdot)$ is the (nominal) cost function, and $Y_{j,t+k|t}$ denotes output in period $t+k$ for a firm j that last reset its price in period t.

The firms' FOC is

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_{jt} \left[\Lambda_{t,t+k} Y_{j,t+k|t}/P_{t+k} \left(P_{jt} - \mathcal{M}\Psi_{j,t+k|t} \right) \right] = 0, \tag{5}$$

where $\mathcal{M} = \epsilon/(\epsilon-1)$ denotes the steady-state markup, and $\Psi_{j,t+k|t} \equiv \mathcal{C}'_{t+k}(Y_{j,t+k|t}) = W^n_{jt}$ denotes the nominal marginal cost for firm j, W^n_{jt} is the nominal wage faced by firm j.

Auxiliary Shocks. Equivalently to the household side, I make use of the auxiliary shock $W^n_{jt} = \zeta^F_{jt} W^n_t$ with ζ^F_{it} i.i.d. across i and t.

Production Subsidy. Note that, under flexible prices ($\theta = 0$) $P_{jt} = \mathcal{M} W^n_{jt}$. Aggregating across firms I obtain the standard result that the aggregate price level is greater than the aggregate marginal cost, due to the markup of monopolistic firms. Aggregating the optimal labor supply condition over households, I obtain $C^\sigma_t N^\varphi_t = W_t$. Combining the last two conditions, I can write $N^\varphi_t C^\sigma_t = W_t = (\epsilon-1)/\epsilon < 1$. Output and employment are below their efficient levels, a result of monopolistic competition.

The suboptimality is corrected by implementing a production subsidy that induces marginal cost pricing, so that the model is efficient in equilibrium.[5] The subsidy guar-

[5] With the desired markup defined by $P_{jt} = \mathcal{M}/(1-\tau^s) W^n_{jt}$, the optimal subsidy is $\tau^s = 1/(\epsilon-1)$. The profit function is $D_{jt} = (1+\tau^s) P_{jt} Y_{jt} - W^n_{jt} N_{jt} - T_{jt}$. The subsidy is financed by taxing firms $T_{jt} = \tau^s Y_{jt}$, which gives the aggregate real profits $D_t = Y_t - W_t N_t$.

antees zero consumption inequality in steady-state.

2.3. Fiscal and Monetary Policy

I assume that the government, which conducts fiscal and monetary policy, does not face any information friction. In fiscal terms, the government conducts two policies. First, it sets the NK standard optimal production steady-state inducing marginal cost pricing, which results in zero steady-state profits and an equalization in steady-state consumption across financial states. Second, it conducts a redistribution scheme by taxing profits from unconstrained households and rebating the proceeds to the constrained. In log-linear terms, $e_t^S = \frac{1-\tau_D}{1-\lambda} d_t$ and $e_t^H = \frac{\tau_D}{\lambda} d_t$, where a low-case variable denotes the log deviation from steady-state (e.g., $x_t = \log X_t - \log X$), and τ_D is the tax rate applied on firms' dividends reimbursed to unconstrained households. Furthermore, monetary policy is conducted following a Taylor rule of the form

$$i_t = \phi_\pi \pi_t + \phi_y y_t + \nu_t, \tag{6}$$

$$\nu_t = \rho \nu_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2), \tag{7}$$

where the monetary policy shock ν_t follows an AR(1) process, to match the empirically observed inertia in the interest rate.

Definition 1. *Given an initial price level P_{-1}, a stationary distribution of agents λ, a general equilibrium is a path for prices $\{P_t, W_t, \pi_t, R_t, i_t\}_{t=0}^{\infty}$, aggregates $\{Y_t, C_t, B_t, E_t, D_t\}_{t=0}^{\infty}$, and individual allocation rules $\{C_{it}, P_{jt}\}_{t=0,\forall i,j}^{\infty}$ such that households and firms optimize (given their beliefs), monetary and fiscal policy follow their rules, and goods and bond markets clear.*

2.4. Information Structure

Both types of agents, households and firms, are subject to information frictions: they do not observe the fundamental shock and are uncertain about the state of nature. To form expectations, agents only use the signals informing on the monetary shock, so that each agent's information set is the history of observed private signals.[6]

Every period, each agent receives a dose of private information on the aggregate fundamental. Formally, there is a collection of private Gaussian signals, one per agent and per period. In particular, the period–t signal received by agent l in group g is given

[6] I indirectly assume that the idiosyncratic components in the set of auxiliary shocks introduced in Sections 2.1-2.2 have large variance (relative to the aggregate).

by

$$x_{lgt} = v_t + u_{lgt}, \quad u_{lgt} \sim \mathcal{N}(0, \sigma_g^2), \tag{8}$$

where $g = \{\text{household, firm}\}$, and $\sigma_g \geq 0$ parameterizes the noise in group g. Notice that, by allowing σ_g to differ by g, I accommodate rich information heterogeneity. (For example, firms could on average be more informed than households.)

2.5. Equilibrium Dynamics

The model is solved in its log-linearized version around a steady state with no consumption inequality. I relegate the detailed derivation to Appendix B, and sketch here the main steps.

DIS Curve. To obtain the Dynamic IS curve, I combine the savings laws of motion (1) and the budget constraints (2) to obtain an intertemporal budget constraint. After introducing the consumption Euler equation (3), and combining it with the labor supply conditions, I obtain the consumption function for unconstrained households. Aggregating across households, and making use of the goods and labor market clearing conditions, $C_t = \lambda C_t^H + (1-\lambda)C_t^S$, $N_t = \lambda N_t^H + (1-\lambda)N_t^S$, and $Y_t = C_t = N_t$, I obtain the log-linearized aggregate consumption function,

$$c_t = -\frac{\beta}{\nu}\sum_{k=0}^{\infty}(\beta s)^k \overline{\mathbb{E}}_t^c r_{t+k} + (1-\beta)\overline{\mathbb{E}}_t^c y_t + \beta\phi\sum_{k=0}^{\infty}(\beta s)^k \overline{\mathbb{E}}_t^c y_{t+k+1}, \tag{9}$$

with $\nu \equiv \sigma(1-\lambda\chi)/(1-\lambda)$, $\phi \equiv (1-s)\chi\sigma/\nu + s(1-\beta)$, and where $\overline{\mathbb{E}}_t^c(\cdot) = \int_0^1 \mathbb{E}_{it}(\cdot)\ di$ is the cross-sectional average forecast across households. The composite parameter $\chi \equiv \left[1+\varphi\left(1-\tau^D/\lambda\right)\right]$ is a fundamental element in the model, and can be interpreted as a sufficient statistic for amplification: both in the noisy information and FIRE framework, there is amplification (dampening) of monetary policy if $\chi > 1$ ($\chi < 1$).[7]

Notice that the aggregate consumption function (9) is implied by the following

[7] Aggregate consumption of HtM households is given by $c_t^H = \chi y_t$, and the output response to shocks is amplified if the income elasticity of HtM agents with respect to aggregate income is larger than one.

beauty-contest game,[8]

$$c_{it} = -\beta\nu^{-1}\mathbb{E}_{it}r_t + (1-\beta)\mathbb{E}_{it}y_t + \beta(\delta - s)\mathbb{E}_{it}y_{t+1} + \beta s\mathbb{E}_{it}c_{i,t+1}, \tag{10}$$

where $\delta \equiv 1 + (\chi - 1)(1 - s)/(1 - \lambda\chi)$ measures the compounding at the consumer's Euler condition in the FIRE model (if $\delta > 1$ there is compounding and if $\delta < 1$ there is discounting). As in the textbook NK, the demand curve (9) can be summarized as a single equation; however, under incomplete information, it cannot be collapsed into a first-order stochastic difference equation since the hierarchy of beliefs prevents the Law of Iterated Expectations from holding at the aggregate level.

With perfect information, all agents make the same action $c_{it} = c_t$, and observe the current realization of output $\mathbb{E}_{it}y_t = y_t$. The aggregate FIRE DIS curve is thus given by[9]

$$y_t = -\nu^{-1}\mathbb{E}_t r_t + \delta\mathbb{E}_t y_{t+1}. \tag{11}$$

Phillips Curve. To obtain the Phillips curve, I combine the pricing function (5)—where nominal marginal costs equal the nominal wage—with the aggregate labor supply condition. Following Angeletos and Lian (2018) and Angeletos and Huo (2021), I simplify the model by assuming that firms observe the aggregate prices up to period $t-1$, but do not extract information from them.[10] This allows me to write the Phillips curve in inflation terms.

The (log-linearized) firm-level Phillips curve is given by the beauty-contest game

$$\pi_{jt} = \kappa\theta\mathbb{E}_{jt}y_t + (1-\theta)\mathbb{E}_{jt}\pi_t + \beta\theta\mathbb{E}_{jt}\pi_{j,t+1}, \tag{12}$$

where $\kappa = (1-\theta)(1-\beta\theta)\,(\sigma+\varphi)\,/\theta$. Iterating forward and aggregating across firms, the aggregate Phillips curve can be written as

$$\pi_t = \kappa\theta\sum_{k=0}^{\infty}(\beta\theta)^k\overline{\mathbb{E}}_t^f\, y_{t+k} + (1-\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\overline{\mathbb{E}}_t^f\,\pi_{t+k} \tag{13}$$

where $\overline{\mathbb{E}}_t^f(\cdot) = \int_0^1 \mathbb{E}_{jt}(\cdot)\;dj$ is the cross-sectional average forecast across firms.

[8] A beauty contest is a strategic setting where agents choose actions based on both their private information and their expectations of others' actions, balancing the trade-off between fundamental accuracy and coordination incentives (see e.g., Woodford 2001; Morris and Shin 2002; Angeletos and Pavan 2007).
[9] Notice that under amplification ($\delta \geq 1$) the FIRE dynamics become explosive.
[10] See Gallegos (2023) for the alternative in which firms do not observe the history of the price level, p^{t-1}.

Model Dynamics. The economy is described as a pair of across–group dynamic beauty contests between consumers (the spending-income multiplier 10) and firms (the strategic complementarity in price-setting 12), jointly determining the inflation-spending NK multiplier.

The equilibrium dynamics must satisfy the individual-level optimal policy functions (10) and (12), and rational expectation formation must be consistent with the Taylor rule (6), the exogenous monetary shock process (7) and the signal process (8). I show in Proposition 1 that the solution to the fixed point is a VARX(1), where the exogenous component is the monetary policy shock.

PROPOSITION 1. *In equilibrium, the aggregate outcome obeys the law of motion*

$$\boldsymbol{x}_t = A(\vartheta_1, \vartheta_2)\boldsymbol{x}_{t-1} + B(\vartheta_1, \vartheta_2)\nu_t, \tag{14}$$

where $\boldsymbol{x}_t = \begin{bmatrix} y_t & \pi_t \end{bmatrix}^\top$ *is a vector containing output and inflation,* $A(\vartheta_1, \vartheta_2)$ *is a* 2×2 *matrix and* $B(\vartheta_1, \vartheta_2)$ *is a* 2×1 *vector, both defined in Appendix A. These matrices depend on* $(\vartheta_1, \vartheta_2)$*, two scalars defined as the reciprocals of the two largest roots of the polynomial* $\boldsymbol{C}(z) = \{(\beta - z)(z - 1/\rho)(z - \rho) + \sigma_\varepsilon^2/(\rho\sigma_1^2)z[(z - \delta)(1 - \lambda\chi) + z\phi_y(1 - \lambda)/\sigma]\}[(\beta\theta - z)(z - 1/\rho)(z - \rho) + \sigma_\varepsilon^2/(\rho\sigma_2^2)\theta z(z - \beta)] - z^3(1 - z\phi_\pi)\sigma_\varepsilon^4/(\rho^2\sigma_1^2\sigma_2^2)\kappa\theta(1 - \lambda)/\sigma$.

The equilibrium dynamics (14) follow a VARX(1) process. In this framework, ϑ_1 and ϑ_2 govern the effect of information frictions on macroeconomic dynamics. The square coefficient matrix $A(\vartheta_1, \vartheta_2)$ is endogenous to ϑ_1 and ϑ_2 (the reciprocals of the outside roots of its characteristic polynomial), with $A(0, 0) = \mathbf{0}$. That is, when signals are perfectly informative, $\vartheta_1 = \vartheta_2 = 0$. In that case, which is simply the FIRE NK model, the model dynamics are given by $\boldsymbol{x}_t = B(0, 0)\nu_t$. When the signal noise is high enough such that the signals are completely uninformative, ϑ_1 and ϑ_2 reach their maximum value of ρ.

Two aspects are worth discussing. First, the beyond FIRE model produces intrinsic persistence, without assuming habits, adjustment costs, or price indexation.[11] Second, the equilibrium dynamics are less sensitive to monetary policy changes: I show in

[11] Havranek et al. (2017) present a meta-analysis of the different estimates of consumption habits in the macro literature and the available micro-estimates. Macroeconomic models use values around 0.75, whereas micro-estimates suggest a value around 0.4. Groth and Khan (2010) conduct a similar analysis for the investment adjustment frictions case, finding that the microeconomic estimates an order of magnitude below the ones used in the empirical macro literature. Finally, the price-indexation model suggests that every price is changed every period, which is inconsistent with the micro-data estimates provided by Nakamura and Steinsson (2008).

Appendix A that each element in $B(\vartheta_1, \vartheta_2)$ is smaller than each element in $B(0, 0)$—in absolute terms—provided that $\{\vartheta_1, \vartheta_2\} \in [0, \rho]^2$.

3. Inspecting the Mechanism: A Brief Detour

In order to uncover the main mechanisms behind the interaction between financial and information frictions and their effect on output, I first consider a reduced-form economy in which the real interest rate follows an exogenous path, which allows me to abstract from the firms' side.[12] The simplified model allows me to present a sequence of corollaries that lend support to the quantitative findings in Section 4.

I assume that the monetary authority conducts its policy in a way that results in a real interest rate that follows an AR(1) process,

$$r_t = \rho r_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2). \tag{15}$$

In this simplified framework the real interest rate law of motion (15) replaces the Taylor rule (6)-(7).

I also reformulate the information structure. Households receive a signal on the aggregate fundamental, the real interest rate. The period–t private signal received by household i is given by

$$x_{it} = r_t + u_{it}, \quad u_{it} \sim \mathcal{N}(0, \sigma_u^2), \tag{16}$$

where $\sigma_u \geq 0$ parameterizes the noise in the private signal.

In the reduced-form economy, equilibrium dynamics are consistent with the individual-level optimal policy functions (10), and rational expectation formation is consistent with the real interest rate process (15) and the signal process (16). The following proposition posits the implied equilibrium dynamics in the beyond FIRE framework, and is the HANK version of proposition 2 in Angeletos and Huo (2021).

PROPOSITION 2. *In equilibrium, aggregate output obeys the law of motion*

$$y_t = \vartheta\, y_{t-1} - (1 - \vartheta/\rho)\, \psi r_t, \tag{17}$$

where ϑ is a scalar that is given by the reciprocal of the largest root of the polynomial $\mathcal{P}(z) \equiv$

[12] Since the real interest rate is exogenous, output dynamics are orthogonal to inflation, and I do not need to keep track of firms' decisions.

$(\beta s - z)(z - \rho)(z - 1/\rho) - \sigma_\varepsilon^2 \beta/(\sigma_u^2 \rho) z(\delta - z)$, *and* $\psi = [\nu(1 - \rho\delta)]^{-1}$.

As in the full model, $\vartheta \in [0, \rho]$ governs the macreconomic effect of information frictions. The beyond FIRE model ($\vartheta > 0$) produces intrinsic persistence, and output becomes less sensitive to real interest rate changes. With perfectly informative signals, $\vartheta = 0$ and the output dynamics are given by (11).

The following corollary documents the degree of information frictions necessary in the beyond FIRE setup to outweigh the forward-lookingness introduced by precautionary savings.

COROLLARY 1. *The equilibrium dynamics described by* (17) *exist and are unique if* $\lambda < (1 + \varphi\tau_D)/(1 + \varphi)$ *and* $\sigma_u^2 > \sigma_\varepsilon^2 \beta(\delta - 1)/[(1 - \beta s)(1 - \rho)^2]$.

First, the model imposes a limit on the degree of amplification. The first condition requires $1 - \lambda\chi > 0$. Under the parametric space studied in Bilbiie (2024), $\chi = 1.48$, the share of HtM agents should not exceed 68%.[13] The second condition requires a sufficiently high degree of information frictions if there is compounding.[14]

Interaction Between Information Frictions and Inequality. I focus on the amplification case, which requires $\tau_D < \lambda$ and implies both $\chi > 1$ and $\delta > 1$. The following corollary documents the interaction between the two frictions, and their effect on ϑ and output dynamics.

COROLLARY 2. *The information-related parameter* ϑ *is increasing in the share of HtM agents, and decreasing in the persistence of the income process.*

A higher share of HtM agents λ (lower persistence of the income process s) induces higher strategic complementarities across agents (see the third term on the right-hand side in equation 10), making them rely less on their private signal, and amplifying the consequences of a given level of information noise σ_u.

Next, I study how the share of HtM agents and the persistence of the income process affect the dynamics of output. I focus on the variance and the first-order autocorrelation, which can be interpreted as the sensitivity of output to the real interest rate shock and the memory of a given real interest rate shock on output, both being key for the cumulative effect of real interest rate changes on output. The following corollary documents the effect of an increase in the two HANK variables on output dynamics.

[13] In an empirical cross-country comparison, Kaplan et al. (2014) find that the share of HtM agents does not exceed 40% anywhere.
[14] This condition always holds in the case $\delta \leq 1$, irrespective of the degree of information frictions.

COROLLARY 3. (*i*) *Compared to the FIRE economy, an increase in the share of HtM agents (persistence of the income process) increases (decreases) the variance of output by a smaller magnitude.* (*ii*) *In the FIRE economy, the first-order autocorrelation is orthogonal to changes in the share of HtM agents and the persistence of the income process. In the beyond FIRE economy, the first autocorrelation of output is increasing (decreasing) in the share of HtM agents (persistence of the income process).*

Corollary 3 shows that an increase in the share of HtM agents or a decrease in the persistence of the income process amplify the output responses to monetary shocks—but this effect is tempered by information frictions—and increase the persistence of output. To explain these results, I inspect the transmission mechanism.

Higher-Order Beliefs and the Indirect Effects. Higher-order beliefs play a crucial role in shaping the response of aggregate demand to monetary shocks. When agents receive new information, they must not only form expectations about fundamentals but also anticipate how others will react. Due to uncertainty about others' awareness of the shock, individuals systematically underestimate future aggregate income, effectively behaving as if they excessively discount the future. This mechanism dampens the strength of indirect GE effects, weakening the amplification of monetary policy.

To see this, one can rearrange the aggregate DIS curve (9) as

$$y_t = -\frac{\beta}{\nu}\overline{\mathbb{E}}_t^c r_t + (1-\beta)\overline{\mathbb{E}}_t^c y_t + \beta\phi\overline{\mathbb{E}}_t^c y_{t+1} + \beta s\left\{ -\frac{\beta}{\nu}\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t^c r_{t+k+1} + \beta\phi\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t^c y_{t+k+2}\right\}. \tag{18}$$

Next, one can compute $\overline{\mathbb{E}}_t^c y_{t+1}$ by shifting forward (9) one period ahead and taking expectations,

$$\overline{\mathbb{E}}_t^c y_{t+1} - (1-\beta)\overline{\mathbb{E}}_t^c\left[\overline{\mathbb{E}}_{t+1}^c y_{t+1}\right] = -\frac{\beta}{\nu}\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t^c\left[\overline{\mathbb{E}}_{t+1}^c r_{t+k+1}\right] + \beta\phi\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t^c\left[\overline{\mathbb{E}}_{t+1}^c y_{t+k+2}\right], \tag{19}$$

where $\overline{\mathbb{E}}_t^c\left[\overline{\mathbb{E}}_{t+1}^c(\cdot)\right]$ denotes the second-order belief. The distinction between the term in braces in condition (18) and the right-hand side of the previous expression arises from the presence of higher-order beliefs under noisy information. The following corollary presents the consequences of higher-order beliefs.

COROLLARY 4. *Forecasts are more persistent at each increasing order in the hierarchy of beliefs. Furthermore, for a given order in the hierarchy of beliefs, the forecast of endogenous variables is more persistent than the forecast of an exogenous variable.*

In the FIRE framework, amplification arises because the income elasticity of HtM agents to aggregate income is greater than one, making indirect effects particularly important. Beyond FIRE, the information environment generates a wedge between the forecasts of direct and indirect effects, the latter being more anchored due to sluggish higher-order belief updating of endogenous variables. As a result, aggregate dynamics are initially driven by direct effects, while indirect effects are muted initially and gradually gain importance over time.

To show this, I decompose the total response in the DIS curve (9) into a partial equilibrium (direct) and a GE (indirect) component, where the direct effect depends only on first-order beliefs and the indirect effects depends on higher-order beliefs:

$$y_t = \underbrace{-\beta\frac{1-\lambda}{\sigma}\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t^c r_{t+k}}_{\text{direct effect}} \underbrace{+\lambda\chi y_t + (1-\beta)(1-\lambda\chi)\overline{\mathbb{E}}_t^c y_t + \beta\phi(1-\lambda\chi)\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t^c y_{t+k+1}}_{\text{indirect effect}}, \tag{20}$$

where the two rightmost terms of the indirect effects depend on higher-order beliefs, whereas the direct effects depend only on first-order beliefs.

Defining the direct share at time τ as $\alpha_\tau = \text{direct}_\tau/(\text{direct}_\tau + \text{indirect}_\tau)$, the following proposition provides the direct share α_τ beyond FIRE after a real interest rate change at time $t \leq \tau$.

PROPOSITION 3. *Beyond FIRE, the time-varying direct share is given by* $\alpha_\tau = \alpha(\rho^{\tau+1} - \omega^{\tau+1})/(\rho^{\tau+1} - \vartheta^{\tau+1})$*, where* $\alpha \equiv \frac{\beta(1-\lambda\chi)(1-\rho\delta)}{1-\rho s\beta}$ *denotes the long-run direct share, where* $\omega \in (\vartheta, \rho)$ *is the inside root of the polynomial* $\mathcal{D}(z) \equiv (1-\rho z)(\rho - z) - \sigma_\varepsilon^2/\sigma_u^2 z$.

Compared to the FIRE setup in which the direct share is constant across time, noisy information generates a larger direct share initially ($\alpha_\tau > \alpha$ for low τ, with $\alpha_\tau \to \alpha$ from above as $\tau \to \infty$). This arrests the GE effect initially, which is the driver of amplification, as compared to the FIRE framework. The following corollary documents the effect of an increase in the two inequality-related variables on the PE share.

COROLLARY 5. *Compared to the FIRE economy, an increase in the share of HtM agents (persistence of the income process) reduces (increases) the direct share by a smaller magnitude.*

Consistent with the previous discussion, an increase in the HANK dimension (either through an increase in the share of HtM agents, or a decrease in the probability of staying in state S) has a smaller effect on the direct share than in the FIRE setup. These findings explain the smaller amplification effect beyond FIRE, measured by the sensitivity of output to the real interest rate.

Having inspected the mechanism in closed form, I now explore these results quantitatively in the fully-fledged model.

4. Quantitative Results

So far I have only considered the demand side of the economy. For the remaining part of the paper, I rely on the full theoretical model presented in Section 2.

In this section, I study the different implications of the HANK beyond FIRE economy by conducting several policy experiments. I exploit the two main frictions, financial and information, and explain their joint interaction and consequences. In particular, I explain the key role of direct vs. indirect effects and how these are affected by financial frictions, I show that the information friction helps reduce the excessive forward-lookingness induced by the precautionary savings motive in HANK. I show that the beyond FIRE framework relaxes the Taylor principle, resolves the FGP, and I obtain the effect of an "animal spirits" shock.

4.1. Calibration

Table 1 reports the model calibration used in the different quantitative analyses throughout the rest of the section: the dynamics of the direct vs. indirect share, the amplification of monetary policy, the Taylor principle and equilibrium determinacy, the FGP and its resolution, and the effect of belief shocks on output dynamics.

The first block contains the standard RANK parameters. The discount factor β, the inverse intertemporal elasticity of substitution σ, the inverse Frisch elasticity φ, and the variance of the monetary shock σ_ε^2 are taken from Bilbiie (2024). The persistence of the monetary shock $\rho = 0.8$ is in the range of the literature, generating inertia in the Taylor rule. The Calvo inaction probability θ is calibrated to generate a flat slope of the Phillips curve, $\kappa = 0.06$, in the mid-range of the empirical literature.[15] (See Hazell et

[15] For instance, the three other papers that combine inequality and belief formation frictions (Farhi and Werning 2019; Auclert et al. 2020; Pfäuti and Seyrich 2022) set θ equal to 0.85, 0.926, and 0.85, respectively.

Parameter	Description	Value	Source
β	Discount factor	0.99	Bilbiie (2024)
σ	Inv. Intertemporal Elas. of Subs.	1	Bilbiie (2024)
φ	Inverse Frisch elasticity	1	Bilbiie (2024)
σ_ε^2	Variance of monetary shock	1	Bilbiie (2024)
θ	Calvo probability	0.85	$\kappa = 0.06$
ϕ_π	Inflation response in Taylor rule	1.5	Christiano et al. (2005)
ϕ_y	Output response in Taylor rule	0.1	Christiano et al. (2005)
ρ	Autocorrelation of monetary shock	0.8	Standard Value
τ_D	Profit tax rate	0.19	Bilbiie (2024)
λ	Share of HtM	0.37	Bilbiie (2024)
s	$\Pr(\text{unconstrained}_{t+1}\|\text{unconstrained}_t)$	0.96	Bilbiie (2024)
σ_1^2	Consumer signal innovation variance	3.66	Coibion and Gorodnichenko (2015)
σ_2^2	Firm signal innovation variance	3.66	Coibion and Gorodnichenko (2015)

TABLE 1. Parameter values.

al. 2022 for a detailed discussion.) The Taylor rule coefficients ϕ_y and ϕ_π are set to the values used in Christiano et al. (2005).

The second block contains the parameters related to household financial heterogeneity. These are taken from Bilbiie (2024) and include the probability of being financially restricted s, the profit tax rate τ_D, and the share of HtM λ. The transition probability s is calibrated to match the first-order autocorrelation of the (annual) income process of an empirical proxy of unconstrained households in Guvenen et al. (2014).[16] The rest of the parameters are jointly calibrated to match both the amplification magnitude of HANK vs. RANK and the share of indirect effects in Kaplan et al. (2018). This calibration jointly implies $\chi = 1.48$ and $\delta = 1.04$.[17]

The third block contains the parameters related to imperfect information, calibrated using survey evidence. Although the framework is flexible to accommodate heterogeneous signals' precision, I restrict attention to households' inflation forecasts and set $\sigma_1 = \sigma_2$ to match the underrevision coefficient of households. Coibion and Gorodnichenko (2015) focus on annual inflation (GDP Deflator) expectations and regress the ex-ante average forecast error, computed as the difference between the realized variable

[16] The first-order autocorrelation of annual income changes is 0.82, which corresponds to a quarterly transition probability $1 - s = 1 - 0.82^{1/4} = 0.04$.
[17] Bilbiie (2024) matches the share of direct effects in the demand-only economy (proposition 3) to 20%, and the degree of amplification ($\psi\sigma(1-\rho)$ in proposition 2) to 1.5. Since the persistence coefficient in different in this paper ($\rho = 0.61$ in Bilbiie 2020, 2024), the implied direct effect share is 31.39% and the degree of amplification is 1.67.

at $t+3$ and the expectation at time t of that variable at $t+3$, $\pi_{t+3,t} - \mathbb{F}_t \pi_{t+3,t}$, on the average forecast revision, defined as the change in the forecast of a variable at time $t+3$ formed at time t minus the forecast of that same variable formed at time $t-1$, $\mathbb{F}_t \pi_{t+3,t} - \mathbb{F}_{t-1} \pi_{t+3,t}$. This underrevision coefficient can be interpreted as a measure of the inertia in households' inflation expectations.

I match the underrevision coefficient that Coibion and Gorodnichenko (2015) obtain using data on forecasts from the Michigan Survey of Consumers, $\widehat{\beta}_\pi = 0.705$.[18] Proposition A1 in Appendix A derives, in closed-form, the model-implied underrevision coefficient, $\beta^{\mathcal{M}}_{c\pi} = \mathbb{C}(\pi_{t+3,t} - \overline{\mathbb{E}}^c_t \pi_{t+3,t}, \overline{\mathbb{E}}^c_t \pi_{t+3,t} - \overline{\mathbb{E}}^c_{t-1} \pi_{t+3,t}) / \mathbb{V}(\overline{\mathbb{E}}^c_t \pi_{t+3,t} - \overline{\mathbb{E}}^c_{t-1} \pi_{t+3,t})$, and shows that it is endogenous to the signals' noise. I calibrate the pair (σ_1, σ_2) by minimizing the squared distance between the model-implied coefficient $\beta^{\mathcal{M}}_{c\pi}$ and the estimated coefficient in Coibion and Gorodnichenko (2015). This implies that $\sigma_1^2 = \sigma_2^2 = 3.660$.

I use this calibration to study quantitatively the different implications of the two frictions. In particular, I show that the Taylor Principle is relaxed in the economy beyond FIRE (with the determinacy region widened), I explore quantitatively the key role of direct vs. indirect effects and how these are affected by financial frictions, I show that the model resolves the FGP, and I obtain the effect of an "animal spirits" shock.

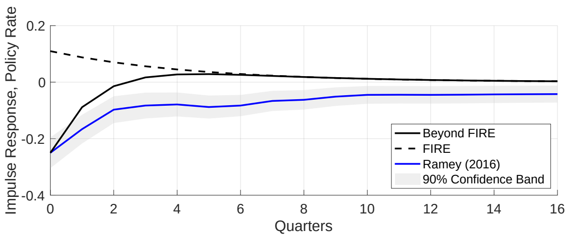
4.2. Response after a Monetary Policy Shock

First, I study the effect of monetary policy shocks on output, exploring how the transmission is affected by both financial and information frictions. I consider an expansionary monetary policy shock that raises the (annualized) nominal interest rate to a peak of 25 basis points in the beyond FIRE framework—this translates to a monetary policy shock of 11.59 basis points. I also consider the dynamics of output under FIRE after a monetary policy shock of the same size, and I compare the dynamics with the empirical results obtained in Ramey (2016).[19]

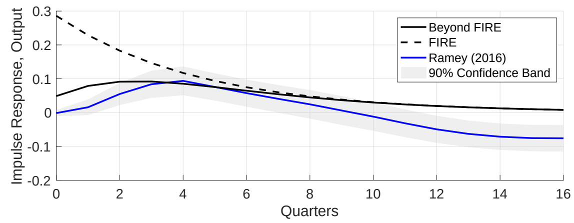
Figure 1A shows the dynamics of the policy rate. The solid black line is the impulse response function (IRF) of the central bank interest rate in the beyond FIRE framework. After the expansionary shock, the interest rate falls, and converges to zero after twelve

[18] Coibion and Gorodnichenko (2015) estimate a variant of the above regression that does not include forecast revisions (the dataset does not permit the calculation) and include oil price changes as an instrument. They show that, for the case of the SPF in which they can perform both estimations, the estimated coefficients are nearly identical.

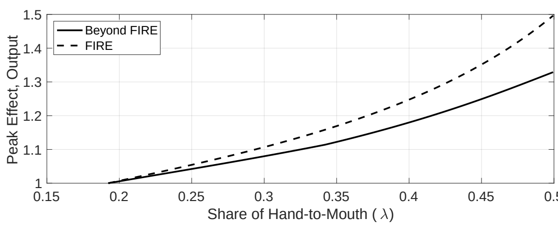
[19] Ramey (2016) considers a monthly VAR with the log of industrial production, the unemployment rate, the log of CPI, the log of a commodity price index, and Romer and Romer (2004) monetary policy shocks, making use of the recursiveness assumption. I transform her monthly impulse responses to quarterly frequency by taking the average over each quarter.



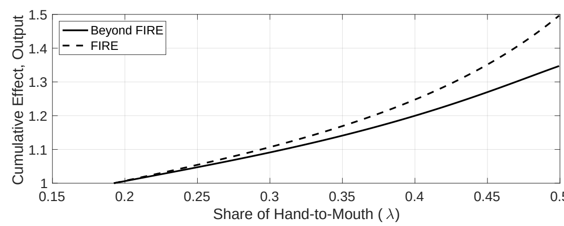
A. Policy rate IRF after a 11.59 b.p. monetary policy shock in the beyond FIRE (solid line) and FIRE (dashed) frameworks. In blue, the empirical IRF in Ramey (2016).



B. Output IRF after a 11.59 b.p. monetary policy shock in the beyond FIRE (solid line) and FIRE (dashed) frameworks. In blue, the empirical IRF of unemployment in Ramey (2016).



C. Amplification multiplier (peak effect) with respect to RANK in the beyond FIRE (solid line) and FIRE (dashed) frameworks.



D. Amplification multiplier (cumulative effect) with respect to RANK in the beyond FIRE (solid line) and FIRE (dashed) frameworks.

FIGURE 1. Policy Rate and Output IRFs (first row) and Amplification (second row).

quarters with a quantitatively small overshooting. The solid blue line is its empirical counterpart in Ramey (2016), inbetween 90% confidence bands. The theoretical model aligns quantitatively with the empirical evidence, although it lacks intrinsic persistence in the policy rule.

I plot the IRF of output after the monetary policy shock in Figure 1B. Output in the noisy information model (solid black line) underreacts and exhibits hump-shaped dynamics, with a peak effect of 0.1 percentage points (p.p.) in the third quarter after the shock. Compared to the empirical VAR—proxied by the inverse unemployment rate—the two dynamics align quantitatively in their hump shape, with excessive overreaction of output in the first quarters in the theoretical model, and less persistence.[20]

The FIRE model—dashed lines in both figures—predicts an even larger response of output, with monotonically decreasing dynamics resulting from the lack of intrinsic persistence. Another qualitative difference between both information setups is the response of the policy rate. Under FIRE, an expansionary monetary policy shock triggers

[20] Noisy information generates hump-shaped dynamics without compromising the individual (monotonically decreasing) responses to income shocks documented in Fagereng et al. (2019).

an immediate positive reaction of output and inflation. The systematic component of the Taylor rule induces a net increase in the policy rate. Beyond FIRE, both output and inflation underreact, resulting in an interest rate easing that aligns with empirical evidence.

Amplification. In Figure 1C I plot the ratio between the peak output response to a monetary policy shock, and the peak output response under RANK, for different HtM shares.[21] Consistent with the analysis in Section 3, HtM households amplify the peak response of output to monetary shocks in both information setups. In the FIRE economy (dashed line), output responds more to monetary policy shocks the larger the share of HtM agents. For the benchmark calibration of the HtM share, $\lambda = 0.37$, the peak output response is 19.88% larger than without financial frictions. Under information frictions (solid line), the amplification effect of HtM agents is still present but partially muted. Consistent with corollary 3, a larger degree of financial frictions leads to a larger peak response of output to monetary shocks, but the multiplier is smaller than in the FIRE case: for the benchmark calibration, the peak output response is 14.45% larger than without financial frictions.

Corollary 3 posits that an increase in the share of HtM increases the first-order autocorrelation of output under noisy information, whereas it has no effect under FIRE. I explore quantitatively the effect of the increase in the share of HtM households on the cumulative output response. In Figure 1D, I plot the ratio between the cumulative output IRF to a monetary policy shock and the cumulative output response under RANK, for different HtM shares. I find that the previous result prevails: HtM agents amplify the cumulative response of output too, but the gap between the cumulative responses is smaller compared to the gap between the peak responses—19.88% under FIRE and 16.37% under noisy information in the baseline calibration of the HtM share—which aligns with the prediction of corollary 3.

To summarize, the transmission mechanism of monetary policy, which operates through the behavior of HtM agents in GE, is partially muted by noisy information.

Direct vs. Indirect Effects. The amplification effect of HtM agents is present but dampened by information frictions, which mute the GE dimension. To explain this result, I follow the same decomposition of the aggregate DIS curve (20), with the caveat that what I previously denoted as "direct effects" are now a mixture of pure direct effects

[21] The case $\lambda = \tau_D$ induces $\chi = 1$, and dynamics are equivalent to a RANK model.

coming from the monetary shock, the systematic component of the Taylor rule (6), and future inflation. Proposition A2 in Appendix A derives the direct share α_τ, in closed form, in the full HANK economy.

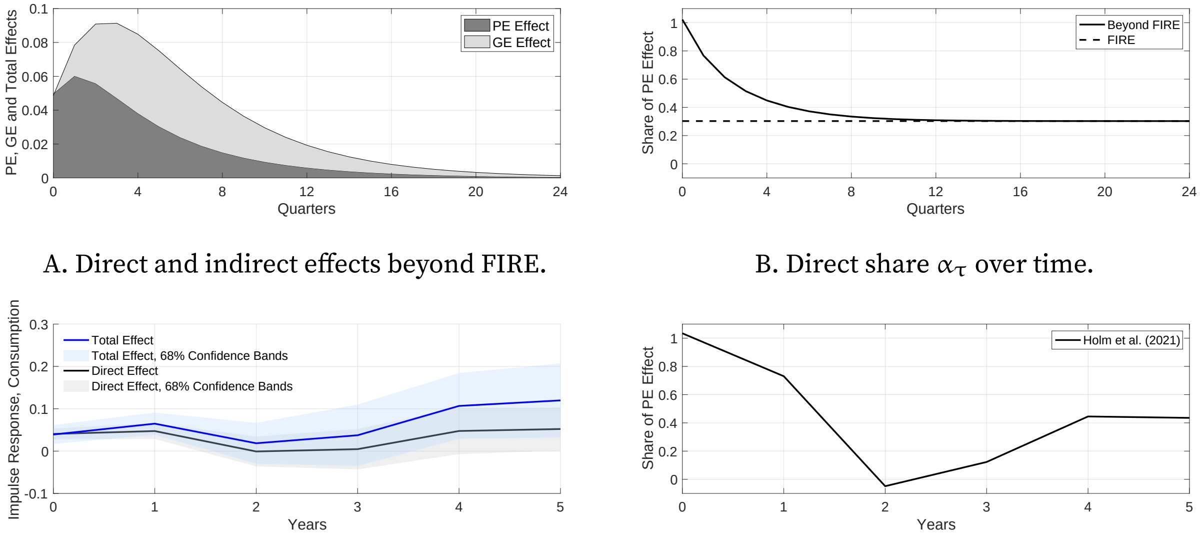
In Figure 2A, I show the decomposition of the output response presented in Figure 1B. Consistent with corollary 5, the indirect share beyond FIRE is lower than in FIRE, and mutes the amplification multiplier coming from HtM households. Indirect effects are arrested in the first periods, while the direct response (dark grey) reacts immediately, explaining the entire effect on impact. Over time, GE effects drive a larger share of the output dynamics. These dynamics explain the dampening of the amplification result—which nourishes from the indirect dimension—and the larger persistence of output. Figure 2B shows the direct share α_τ over time in both theoretical models. While the FIRE framework predicts a constant PE share, the noisy information framework generates a monotonically decreasing PE share that converges to the FIRE share.

These results align quantitatively with the empirical findings in Holm et al. (2021). Using Norwegian administrative data at the annual frequency, the authors decompose the households' consumption responses to monetary shocks into direct and indirect effects by controlling for households' income changes—using lottery winnings as instruments for households' idiosyncratic income shocks. I reproduce their estimated decomposition in Figure 2C, scaling their IRFs to match the size of the monetary policy shock in Figure 1B, and I report the implied PE share in Figure 2D (point estimate).[22]

Quantitatively, the total response of consumption (blue line) aligns with the noisy information model up until the third year after the shock, but exhibits greater persistence. After the third year, the authors estimate a further increase in consumption, which the theoretical model cannot predict. The model does predict a large and monotonically decreasing PE share up until the second year after the monetary policy shock, aligning with the empirical findings. However, the authors report a fall in the PE share to around zero in the second year, which then converges to a value that roughly coincides with the long-run PE share in the noisy information model. The theoretical model fails to predict this bang-bang behavior of the PE share.

The attenuation of indirect effects shares similarities with the mechanism in Farhi and Werning (2019), where level-K thinking weakens the GE feedback by limiting agents' ability to anticipate others' responses. However, the underlying mechanism differs: in level-K the dampening arises from bounded rationality, where agents fail to fully internalize equilibrium interactions. Under noisy information, agents anchor their

[22] Holm et al. (2021) report IRFs of total consumption and direct effects to a 1 p.p. monetary policy shock.



A. Direct and indirect effects beyond FIRE.

B. Direct share α_τ over time.

C. Direct and total effect in Holm et al. (2021).

D. Implied direct share in Holm et al. (2021).

FIGURE 2. Direct and Indirect Effects in the Theoretical Models and in the Data.

expectations to prior beliefs and update them gradually. While both reduce the strength of indirect effects, noisy information generates indirect effects that recover over time, whereas level-K suggests a more permanent weakening of feedback effects.

4.3. The Taylor Principle beyond FIRE

A fundamental criterion for the stability of monetary policy is the Taylor Principle—requiring that the nominal interest rate responds more than one-for-one to deviations in inflation (Taylor 1993; Woodford 2003; Galí 2015). In conventional NK models, this principle is sufficient to rule out self-fulfilling inflationary or deflationary spirals. However, the presence of financial frictions and information frictions can alter the determinacy conditions, potentially expanding or contracting the region in which monetary policy achieves a unique equilibrium. For instance, Gabaix (2020) extends this framework by incorporating bounded rationality, showing that cognitive discounting expands the determinacy region by muting forward-looking dynamics. This insight is particularly relevant in environments where agents do not fully internalize future policy responses, which is a central feature of the noisy information framework introduced in this paper.

In this class of theoretical models, an equilibrium is indeterminate when the current actions are excessively affected by expectations of the future: the Taylor Principle boils down to studying the determinacy of the system (9), (13), (6) and (7). Angeletos and Huo

(2021) show that introducing noisy information is observationally equivalent to having both cognitive discounting à la Gabaix (2020) and intrinsic persistence in the DIS and Phillips curves. One would therefore expect, consistent Gabaix (2020), that introducing noisy information should widen the determinacy region, making the system (9), (13), (6) and (7) stable for a larger set of $\{\phi_\pi, \phi_y\}$ combinations.

I start discussing the FIRE benchmark. In the empirically factual case of amplification, $\delta > 1$ generates compounding in the DIS curve, the model becomes more forward-looking, and the stability region is reduced. Part (*i*) in the following proposition summarizes how financial frictions affect the determinacy region.

PROPOSITION 4. *(i) The FIRE equilibrium exists and is unique if*

$$\phi_\pi + [\nu(1-\beta\delta) + \phi_y]/\kappa > 0, \tag{21}$$

$$\phi_\pi + (1-\beta)[\nu(1-\delta) + \phi_y]/\kappa > 1. \tag{22}$$

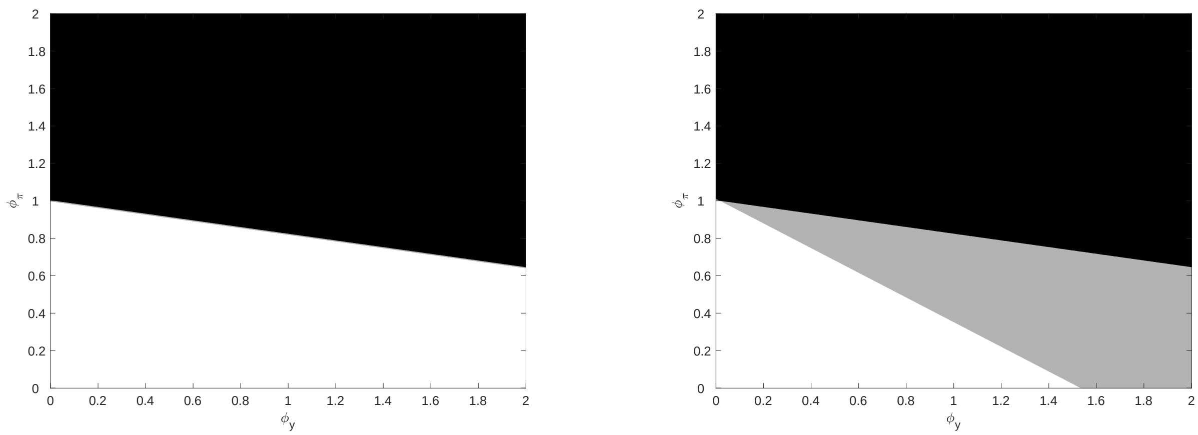
(ii) The beyond FIRE equilibrium exists and is unique if ϑ_1 and ϑ_2 are the reciprocals of the only two outside roots of polynomial $\boldsymbol{C}(z)$, defined in proposition 1.

In the two-agents NK (TANK) case with $s = \delta = 1$, condition (22) implies (21) and the determinacy regions in RANK and TANK coincide. Under HANK (with compounding $\delta > 1$), the determinacy region is reduced: the leftmost terms in (21)-(22) are smaller. As a result, the rightmost element on the left-hand side in both conditions needs to be larger: precautionary savings generate compounding and reduce the determinacy region. A graphical representation of these results is reported in Figure 3A.

In the beyond FIRE framework, equilibrium is determinate if $\{\vartheta_1, \vartheta_2\}$ are the reciprocals of the only outside roots. In Figure 3B I show the determinacy regions both in the FIRE HANK and in the noisy information HANK framework. I find that imperfect information widens the determinacy region as a result of myopia—or extra discounting of the future—micro-founded through sluggish updating of expectations.

The relationship between noisy information and equilibrium determinacy contradicts the findings in the sticky information literature (Mankiw and Reis 2002). For instance Meyer-Gohde and Tzaawa-Krenzler (2023) show that, although sticky information models introduce sluggish expectation updating, the determinacy region is reduced, and $\phi_\pi > 1$ becomes a necessary condition (in RANK, compared to the sticky price model where $\phi_\pi > 1$ is only sufficient).[23] A key difference between both frameworks is

[23] A corollary to this result is that the determinacy disagreement between sticky prices and sticky



A. Black: simultaneous determinacy region in RANK and HANK (FIRE). Gray: additional determinacy region in RANK. White: simultaneous indeterminacy region.

B. Black: simultaneous determinacy region under FIRE and beyond FIRE (HANK). Gray: additional determinacy region beyond FIRE. White: simultaneous indeterminacy region.

FIGURE 3. Determinacy Regions in the Theoretical Models.

that, under noisy information, equilibrium objects depend on expectations about the future (see equations 9 and 13); whereas under sticky information, equilibrium objects depend on past expectations about current variables.

4.4. Forward Guidance

A related but distinct challenge in NK models is the FGP. Forward guidance refers to a central bank's commitment to a future path of monetary policy—typically keeping interest rates lower for longer than implied by current conditions—in order to influence expectations and stimulate economic activity today. Several central banks made use of this tool in the recent financial crisis.[24] The FGP refers to the disproportionately large effects on output predicted by NK models when a central bank announces future interest rate commitments, implying an implausibly strong and persistent impact on current output (Del Negro et al. 2012).

A strand of the literature tries to find an explanation for the FGP from different angles (McKay et al. 2016; Andrade et al. 2019; Hagedorn et al. 2019; Angeletos and Lian 2018), my approach combining those involving financial frictions (Hagedorn et al. 2019) and belief formation frictions (Angeletos and Lian 2018). This framework

information depends on the slope of the long-run Phillips curve. Allowing $\kappa \to \infty$, the determinacy region in both models coincide.

[24] See Angeletos and Sastry (2020) for a more comprehensive treatment.

contributes to both strands of literature by showing that noisy information induces sufficient myopia in expectations, thereby resolving the extreme responses associated with forward guidance policies.

The standard NK model predicts that a forward guidance τ–shock (i.e., a promise at time t to shock the economy in period $\tau \geq t$ by using the real interest rate) has the same effect, regardless of the time in the future it is promised. This is easily verified from the FIRE DIS curve (11) iterated forward, $y_t = -1/\nu \sum_{k=0}^{\infty} \delta^k \mathbb{E}_t r_{t+k}$.

This finding is aggravated in the case of financial constraints, since the precautionary savings motive and amplification induce compounding ($\delta > 1$): any future shock to the real interest rate (a forward guidance shock) has a larger impact on current output. This is what Bilbiie (2024) refers to as the "Catch-22" of forward guidance: a realistic amplification of monetary policy effects aggravates the FGP. Del Negro et al. (2012) study this empirically and find that forward guidance is less effective than what the theoretical model suggests.

In order to study the FGP, it is convenient to write the noisy information model in FIRE terms. Momentarily, consider an alternative ad hoc theoretical model that is extended with intrinsic persistence and myopia, without a proper microfoundation required at this point. Part (*i*) in the following proposition rewrites the DIS curve (9) in FIRE terms, and is a direct application of proposition 11 in Angeletos and Huo (2021). Part (*ii*), the original contribution of this section, proves that under noisy information the FGP no longer occurs—the further into the future forward guidance is implemented, the smaller its effect on current output.

PROPOSITION 5. (*i*) *The ad hoc equilibrium dynamics*

$$\boldsymbol{x}_t = \boldsymbol{\omega}_b \boldsymbol{x}_{t-1} + \overline{\delta} \boldsymbol{\omega}_f \mathbb{E}_t \boldsymbol{x}_{t+1} + \overline{\varphi} v_t \tag{23}$$

with $\boldsymbol{\omega}_b = \begin{bmatrix} \omega_{b,11} & \omega_{b,12} \\ \omega_{b,21} & \omega_{b,22} \end{bmatrix}$ *and* $\boldsymbol{\omega}_f = \begin{bmatrix} \omega_{f,11} & \omega_{f,12} \\ \omega_{f,21} & \omega_{f,22} \end{bmatrix}$ *produce identical dynamics to the noisy information model if* $(\boldsymbol{\omega}_b, \boldsymbol{\omega}_f)$ *satisfy*[25]

$$\boldsymbol{\omega}_b = [I - \overline{\delta} \boldsymbol{\omega}_f A]A, \quad \text{and} \quad B - \overline{\varphi} = \overline{\delta} \boldsymbol{\omega}_f (A + \rho) B. \tag{25}$$

[25] In the FIRE model, output and inflation are given by

$$\boldsymbol{x}_t = \overline{\varphi} v_t + \overline{\delta} \mathbb{E}_t \boldsymbol{x}_{t+1} \tag{24}$$

where $\overline{\delta} = \frac{1}{\nu + \phi_y + \kappa \phi_\pi} \begin{bmatrix} \nu\delta & 1 - \phi_\pi \beta \\ \kappa\nu\delta & \kappa + (\nu + \phi_y)\beta \end{bmatrix}$ and $\overline{\varphi} = \frac{1}{\nu + \phi_y + \kappa \phi_\pi} \begin{bmatrix} -1 \\ -\kappa \end{bmatrix}$ are the matrices in condition (25).

Therefore, the DIS curve can be written in FIRE terms as

$$y_t = \omega_{by} y_{t-1} + \omega_{b\pi}\pi_{t-1} - (\nu^{-1} + \omega_{f\pi})\mathbb{E}_t r_t + \omega_{fy}\mathbb{E}_t y_{t+1}, \tag{26}$$

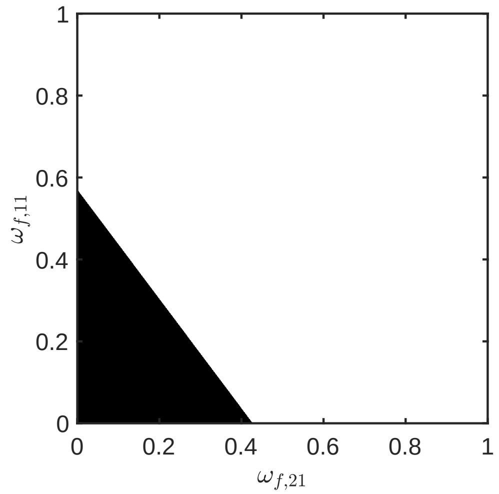
where $\omega_{by} = \frac{(\nu+\phi_y)\omega_{b,11}+\phi_\pi\omega_{b,21}}{\nu}$, $\omega_{b\pi} = \frac{(\nu+\phi_y)\omega_{b,12}+\phi_\pi\omega_{b,22}}{\nu}$, $\omega_{fy} = \frac{\nu\delta\omega_{f,11}+\omega_{f,21}}{\nu}$, *and* $\omega_{f\pi} = \frac{\nu\delta\omega_{f,12}+\omega_{f,22}-1}{\nu}$. *(ii) Consider a situation in which the zero lower bound for nominal interest rates is binding between periods* t *and* τ, *such that* $\tau \geq t$. *Noisy information cures the FGP if one of the roots of the polynomial* $\Omega(x) \equiv \omega_{fy}x^2 - x + \omega_{by}$ *lies outside the unit circle, and the other root lies inside the unit circle. Furthermore, the effect of forward guidance at period* τ *on consumption in period* t *is given by* $FG_{t,t+\tau} = \frac{\partial y_t}{\partial \mathbb{E}_t r_{t+\tau}} = -\frac{\zeta}{\omega_{by}}\left(\frac{1}{\nu} + \omega_{f\pi} + \omega_{b\pi}\zeta^2\frac{\omega_{fy}^2}{\omega_{by}^2}\right)\left(\zeta\frac{\omega_{fy}}{\omega_{by}}\right)^\tau$, *where* $\zeta \in (0, 1)$ *is the only inside root of the polynomial* $\Omega(x)$.

Part (*i*) in proposition 5 posits that, under a certain parameterization of the persistence and myopia coefficients $\{\omega_{by}, \omega_{b\pi}, \omega_{fy}, \omega_{f\pi}\}$, such an ad hoc model produces identical output dynamics to the noisy information framework.[26] A caveat to the above proposition is that these scalars are not unique, although the dynamics are unique. Different weights are consistent with the equilibrium dynamics described by (14).[27] Hence, to study the dynamics in the Phillips curve and the FGP, the theorist is left with one degree of freedom for each equation. For $\{\omega_{f,11}, \omega_{f,21}\} \in [0, 1]^2$, which jointly determine ω_{fy} and then the remaining coefficients through condition (25), I plot in Figure 4A the space in which there is no FGP anymore. Only the dark-shaded region is consistent with (25) and resolves the FGP.

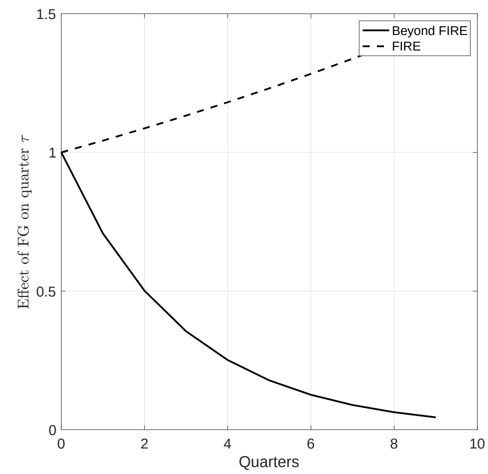
In Figure 4B I plot the impact of a forward guidance shock in period τ on today's output for each τ under FIRE (dashed) and beyond FIRE (solid), normalizing the initial impact to 1 in both models for comparability. In the former (latter) case, the further into the future the forward guidance is implemented, the lesser (larger) the effect.

The attenuation of forward guidance effects in this model is notably stronger than in other frameworks that have attempted to mitigate the FGP. Previous approaches have relied on alternative mechanisms: McKay et al. (2016) introduce household heterogeneity with precautionary savings, which weakens forward guidance effects through income risk considerations, but does not eliminate them entirely. Farhi and Werning (2019) employ level-K thinking, where agents fail to fully internalize general equilibrium

[26] In the benchmark NK model, $\omega_{by} = \omega_{b\pi} = \omega_{f\pi} = 0$ and $\omega_{fy} = \delta$, and the DIS curve is reduced to (11).
[27] Agents' actions can be anchored and myopic with respect to output or inflation, or a combination.



A. FGP resolution region (in black) for different values of the degrees of freedom $\{\omega_{f,11}, \omega_{f,21}\} \in [0, 1]^2$.



B. The Effect of Forward Guidance on current Output. Initial impact of FG normalized to 1 in both models.

FIGURE 4. The Effect of Forward Guidance. Figure 4B: results shown for $\omega_{f,11} = \omega_{f,21} = 0.2$, which imply $\omega_{by} = 0.44$, $\omega_{b\pi} = 1.74$, $\omega_{fy} = 0.49$ and $\omega_{f\pi} = -3.21$.

effects, but predict larger effects of forward guidance. Gabaix (2020) proposes cognitive discounting, measured in reduced form through an additional discount factor $\overline{m} \in (0, 1)$, which discounts future policy effects, but presents the FGP results for the case $\overline{m} = 0.8$. Extending the bounded rationality framework to financial frictions and using survey evidence, Pfäuti and Seyrich (2022) estimate a myopic discount factor between 0.55-0.6, producing results that align quantitatively with those in Figure 4B.

4.5. Beliefs Shock

What is the effect of an "animal spirits" or non-fundamental shock? The macroeconomic dynamics in NK models can also be influenced by belief-driven fluctuations, commonly referred to as belief or "animal spirits" shocks. These non-fundamental shocks arise when agents update expectations based on imperfect signals, leading to self-fulfilling dynamics that can persist even if the underlying shock is transitory. Lorenzoni (2009) formalized this idea in a model of noisy information, showing that agents' misinterpretation of signals can generate prolonged deviations from fundamentals.

In this section, I replace private information with public information.[28] Noisy in-

[28] The private information model does not allow for this exercise, since a shock to an individual signal does not have any effect on aggregate variables.

formation heightens the role of belief shocks by limiting agents' ability to disentangle fundamental innovations from transitory noise. As a result, temporary belief-driven fluctuations can have persistent macroeconomic effects. Formally, agents receive a common and public noisy signal informing them about the monetary policy shock, ν_t. I then study the effect of a non-fundamental—or 'animal spirits"—shock, to such public signal, on output.

Public Information. Consider a collection of public Gaussian signals, one per period and common across agents. In particular, the period–t signal received by all agents, regardless of their group g, is given by

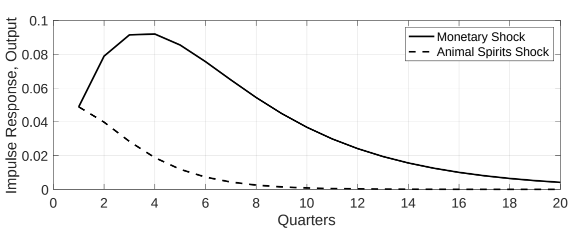
$$z_t = \nu_t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2), \tag{27}$$

where $\sigma_\epsilon \geq 0$ parameterizes the noise in the common signal. The rest of the model is unchanged. The following proposition presents the equilibrium dynamics under public information.

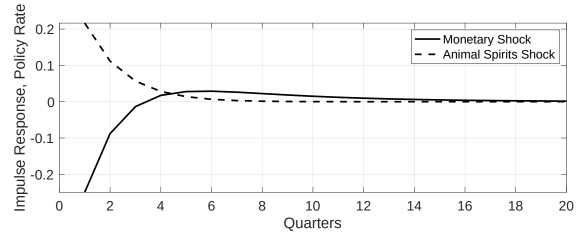
PROPOSITION 6. *In equilibrium, the aggregate outcome obeys the law of motion* $\boldsymbol{x}_t = A(\vartheta_1, \vartheta_2)\boldsymbol{x}_{t-1} + B(\vartheta_1, \vartheta_2)(\nu_t + \epsilon_t)$*, where* $(\vartheta_1, \vartheta_2)$ *are two scalars that are given by the reciprocal of the two largest roots of the polynomial* $\boldsymbol{C}(z)$ *defined in the proposition 1, modulo the replacement of* σ_1 *and* σ_2 *by* σ_ϵ.

The equilibrium dynamics still follow a VARX(1) process, with an additional contemporaneous exogenous shock ϵ_t. This term can be interpreted as a non-fundamental belief or "animal spirits" shock. Figure 5 reports the output and policy rate dynamics, assuming $\sigma_\epsilon = \sigma_1 = \sigma_2$. Both shocks have identical effects on impact on aggregate variables, given that agents cannot completely discern the noise and the fundamental shock from the signal. Even though the belief shock is purely transitory, it produces a persistent and hump-shaped effect on output (see Figure 5A). This is the result of having imperfectly informed agents, who cannot immediately differentiate between a belief shock and a true monetary policy shock.

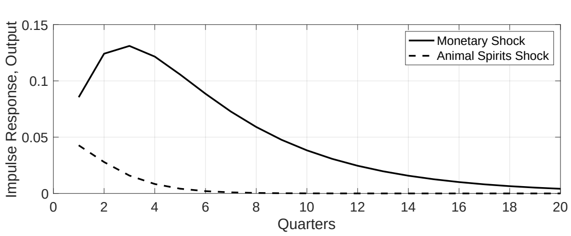
Note also the different response of the policy rate: after the expansionary monetary policy shock, the policy rate decreases, whereas the non-fundamental belief shock raises the interest rate, which in turn reduces the effect of the belief shock (see Figure 5B).



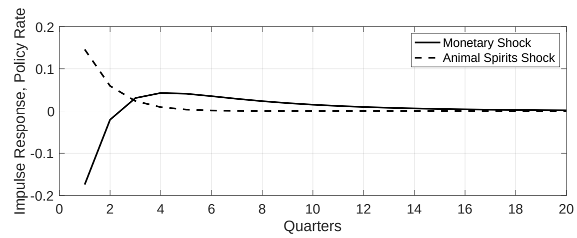
A. Output IRF after a monetary policy shock (solid), and after a belief shock (dashed).



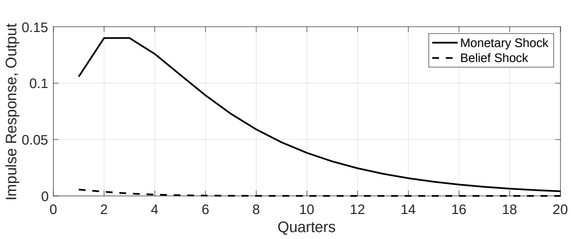
B. Policy rate IRF after a monetary policy shock (solid), and after a belief shock (dashed).



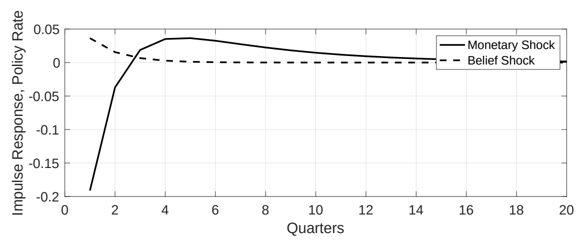
C. Output IRF after a monetary policy shock (solid), and after a belief shock (dashed).



D. Policy rate IRF after a monetary policy shock (solid), and a belief shock (dashed).



E. Output IRF after a monetary policy shock (solid), and after a belief shock (dashed).



F. Policy rate IRF after a monetary shock (solid), and a belief shock (dashed).

FIGURE 5. Output and Policy Rate dynamics under Public Information (panels in the first row), Correlated Information (second row) and Simulataneous Public and Private Information (third row) after a 11.59 b.p. monetary policy shock and "animal spirits" shock.

Correlated Information. Consider now an intermediate case in which the private signals are correlated across agents, and this correlation is modulated by the parameter $\gamma \in [0,1]$.[29] The period–t private signal received by agent l in group g is given by $x_{lgt} = v_t + \gamma \epsilon_t + (1-\gamma) u_{lgt}$, with $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $u_{lgt} \sim \mathcal{N}(0, \sigma_g^2)$, where $\sigma_g \geq 0$ and $\sigma_\epsilon \geq 0$ parameterize the noise in group g and public noise, respectively. This general case nests the public information assumption when $\gamma = 1$, and the private information case when $\gamma = 0$. The following proposition presents the equilibrium dynamics under correlated

[29] I thank an anonymous referee for the suggestion to include this case.

information.

PROPOSITION 7. *In equilibrium, the aggregate outcome obeys the law of motion* $\boldsymbol{x}_t = A(\vartheta_1, \vartheta_2)\boldsymbol{x}_{t-1} + B(\vartheta_1, \vartheta_2)(\nu_t + \gamma\epsilon_t)$*, where* $(\vartheta_1, \vartheta_2)$ *are two scalars that are given by the reciprocal of the two largest roots of the polynomial* $\boldsymbol{C}(z)$ *defined in the proposition 1, modulo the replacement of* σ_g *by* $\widetilde{\sigma}_g = [\gamma^2\sigma_\epsilon^2 + (1-\gamma)^2\sigma_g^2]^{1/2}$*, for* $g = \{1, 2\}$*.*

The equilibrium dynamics still follow a VARX(1) process, with the additional contemporaneous exogenous "animal spirits" shock ϵ_t. Compared to the public information case, the impact effect of the non-fundamental shock is dampened by the coefficient γ. A further difference with respect to the private and public information cases is the quantification of $\widetilde{\sigma}_g$. Assuming $\sigma_\epsilon = \sigma_1 = \sigma_2$ as in the public information case, $\widetilde{\sigma}_g^2 = [\gamma^2 + (1-\gamma)^2]\sigma_\epsilon^2 \leq \sigma_\epsilon^2$, holding with equality when $\gamma \in \{0, 1\}$. The signal noise $\widetilde{\sigma}_g$ is minimized at $\gamma = 0.5$, and maximized in the limiting cases of private or public information.

This has an effect on equilibrium dynamics. Figures 5C-5D report the output and policy rate dynamics, assuming $\sigma_\epsilon = \sigma_1 = \sigma_2$ and $\gamma = 0.5$. Compared to the public information case, output reacts more to monetary policy shocks, since agents are better informed. The lesser reaction of output to belief shocks is explained both from the lower information friction and $\gamma < 1$, which modulates the impact effect on output. Consequently, the policy rate underreacts to both shocks, compared to the public information case.

Private and Public Information. What if instead of replacing private with public signals, I allow agents to observe both private and public signals? Assume that, on top of the individual signal (8), agents receive a common and public noisy signal informing them about the monetary policy shock ν_t, (27). The following proposition summarizes the equilibrium dynamics under public information.

PROPOSITION 8. *In equilibrium, the aggregate outcome obeys the law of motion* $\boldsymbol{x}_t = Q_\nu \sum_{k=0}^{\infty} \Lambda^k \Gamma \nu_{t-k} + Q_u \sum_{k=0}^{\infty} \Lambda^k \Gamma \epsilon_{t-k}$*, where* $Q_\nu(\vartheta_1, \vartheta_2)$*,* $Q_u(\vartheta_1, \vartheta_2)$*, and* $\Lambda(\vartheta_1, \vartheta_2)$ *are three* 2×2 *matrices, and* $\Gamma(\vartheta_1, \vartheta_2)$ *is a* 2×1 *vector, defined in Appendix A.*

The first aspect to notice is that the equilibrium dynamics do not follow a VARX(1) process anymore unless $Q_\nu = Q_u$, which is not generally satisfied. The two exogenous shocks have different impact effects, since agents can partly discern them through the two signals.[30] As a result, agents react less to the "animal spirits" shock. I find that the effect of the belief shock on output is smaller than before, output reacts by a larger magnitude to the monetary policy shock, and the resulting dynamics are closer to the standard FIRE dynamics (see Figure 5E). The policy rate responds by less to both shocks, compared to the public information case (see Figure 5F).

To summarize, adding public information reduces information frictions, dampens the effect of any belief shock, and enlarges the effect of monetary policy shocks.

5. Conclusion

I study the transmission of monetary policy in HANK economies. In the FIRE benchmark, the amplified response relies on households' incomes being immediately affected by indirect effects following a monetary policy shock. By relaxing the FIRE assumption, I show that a framework with noisy information yields a different balance between direct and indirect effects compared to standard FIRE models and is consistent with recent empirical evidence. The indirect effects are dampened in the initial periods, reducing the magnitude of the amplification multiplier.

I use the theory to revisit several key results in the monetary economics literature. I find that noisy information produces hump-shaped IRFs that align with empirical evidence, without resorting to ad hoc micro-inconsistent adjustment costs in habits, pricing, or investment adjustment costs. Instead, aggregate sluggishness is micro-founded through expectation formation frictions, for which the literature has found empirical evidence. I also show that noisy information extends the equilibrium determinacy region and is crucial for the solution to the FGP. Finally, I find that purely transitory "animal spirits" shocks can generate large and persistent effects on output.

References

Almgren, Mattias, José-Elías Gallegos, John Kramer, and Ricardo Lima, "Monetary Policy and Liquidity Constraints: Evidence from the Euro Area," *American Economic Journal: Macroeconomics*, October 2022, *14* (4), 309–40.

Andrade, Philippe, Gaetano Gaballo, Eric Mengus, and Benoît Mojon, "Forward Guidance and Heterogeneous Beliefs," *American Economic Journal: Macroeconomics*, July 2019, *11* (3), 1–29.

Angeletos, George-Marios and Alessandro Pavan, "Efficient Use of Information and Social Value of Information," *Econometrica*, 2007, *75* (4), 1103–1142.

[30] By introducing an additional signal, I am effectively reducing the degree of information friction that agents face: if there is an exogenous shock to the common signal, private signals will be unaffected.

——— **and Chen Lian**, “Forward Guidance without Common Knowledge,” *American Economic Review*, September 2018, *108* (9), 2477–2512.

——— **and Karthik A Sastry**, “Managing Expectations: Instruments Versus Targets*,” *The Quarterly Journal of Economics*, 12 2020, *136* (4), 2467–2532.

——— **and Zhen Huo**, “Myopia and Anchoring,” *American Economic Review*, April 2021, *111* (4), 1166–1200.

Auclert, Adrien, “Monetary Policy and the Redistribution Channel,” *American Economic Review*, June 2019, *109* (6), 2333–67.

———**, Matthew Rognlie, and Ludwig Straub**, “Micro Jumps, Macro Humps: Monetary Policy and Business Cycles in an Estimated HANK Model,” Working Paper 26647, National Bureau of Economic Research January 2020.

Bilbiie, Florin O., “Limited asset markets participation, monetary policy and (inverted) aggregate demand logic,” *Journal of Economic Theory*, 2008, *140* (1), 162–196.

——— , “The New Keynesian Cross,” *Journal of Monetary Economics*, 2020, *114*, 90–108.

——— , “Monetary Policy and Heterogeneity: An Analytical Framework,” *The Review of Economic Studies*, 2024.

Bordalo, Pedro, Nicola Gennaioli, Yueran Ma, and Andrei Shleifer, “Overreaction in Macroeconomic Expectations,” *American Economic Review*, September 2020, *110* (9), 2748–82.

Brinca, Pedro, Hans A. Holter, Per Krusell, and Laurence Malafry, “Fiscal multipliers in the 21st century,” *Journal of Monetary Economics*, 2016, *77*, 53–69.

Christiano, Lawrence J., Martin Eichenbaum, and Charles L. Evans, “Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy,” *Journal of Political Economy*, 2005, *113* (1), 1–45.

Coibion, Olivier and Yuriy Gorodnichenko, “Information Rigidity and the Expectations Formation Process: A Simple Framework and New Facts,” *American Economic Review*, August 2015, *105* (8), 2644–78.

Fagereng, Andreas, Martin Blomhoff Holm, Benjamin Moll, and Gisle Natvik, “Saving Behavior Across the Wealth Distribution: The Importance of Capital Gains,” Working Paper 26588, National Bureau of Economic Research December 2019.

Farhi, Emmanuel and Iván Werning, “Monetary Policy, Bounded Rationality, and Incomplete Markets,” *American Economic Review*, November 2019, *109* (11), 3887–3928.

Gabaix, Xavier, “A Behavioral New Keynesian Model,” *American Economic Review*, August 2020, *110* (8), 2271–2327.

Galí, Jordi, J. David López-Salido, and Javier Vallés, “Understanding the Effects of Government Spending on Consumption,” *Journal of the European Economic Association*, 03 2007, *5* (1), 227–270.

Gallegos, José-Elías, “Inflation Persistence, Noisy Information and the Phillips Curve,” Technical Report 2023.

Galí, Jordi, *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications*, 2nd ed., Princeton, NJ: Princeton University Press, 2015.

Gornemann, Nils, Keith Quester, and Makoto Nakajima, “Doves for the Rich, Hawks for the Poor? Distributional Consequences of Monetary Policy,” Technical Report 2016.

Groth, Charlotta and Hashmat Khan, "Investment Adjustment Costs: An Empirical Assessment," *Journal of Money, Credit and Banking*, 2010, *42* (8), 1469–1494.

Guvenen, Fatih, Serdar Ozkan, and Jae Song, "The Nature of Countercyclical Income Risk," *Journal of Political Economy*, 2014, *122* (3), 621–660.

Hagedorn, Marcus, Jinfeng Luo, Iourii Manovskii, and Kurt Mitman, "Forward guidance," *Journal of Monetary Economics*, 2019, *102*, 1–23.

Hamilton, James D., *Time Series Analysis*, Princeton University Press, 1994.

Havranek, Tomas, Marek Rusnak, and Anna Sokolova, "Habit formation in consumption: A meta-analysis," *European Economic Review*, 2017, *95*, 142–167.

Hazell, Jonathon, Juan Herreño, Emi Nakamura, and Jón Steinsson, "The Slope of the Phillips Curve: Evidence from U.S. States," *The Quarterly Journal of Economics*, 2022, *137* (3), 1299–1344.

Holm, Martin Blomhoff, Pascal Paul, and Andreas Tischbirek, "The Transmission of Monetary Policy under the Microscope," *Journal of Political Economy*, 2021, *129* (10), 2861–2904.

Kaplan, Greg, Benjamin Moll, and Giovanni L. Violante, "Monetary Policy According to HANK," *American Economic Review*, March 2018, *108* (3), 697–743.

___ **, Giovanni L. Violante, and Justin Weidner**, "The Wealthy Hand-to-Mouth," *Brookings Papers on Economic Activity*, 2014, pp. 77–138.

Krusell, Per, Toshihiko Mukoyama, and Anthony A. Smith, "Asset prices in a Huggett economy," *Journal of Economic Theory*, 2011, *146* (3), 812–844. Incompleteness and Uncertainty in Economics.

Lorenzoni, Guido, "A Theory of Demand Shocks," *American Economic Review*, December 2009, *99* (5), 2050–84.

Mankiw, N. Gregory and Ricardo Reis, "Sticky Information versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve," *The Quarterly Journal of Economics*, 2002, *117* (4), 1295–1328.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson, "The Power of Forward Guidance Revisited," *American Economic Review*, October 2016, *106* (10), 3133–58.

Meyer-Gohde, Alexander and Mary Tzaawa-Krenzler, "Sticky Information and the Taylor Principle," Technical Report 2023.

Morris, Stephen and Hyun Song Shin, "Social Value of Public Information," *American Economic Review*, December 2002, *92* (5), 1521–1534.

Nakamura, Emi and Jón Steinsson, "Five Facts about Prices: A Reevaluation of Menu Cost Models*," *The Quarterly Journal of Economics*, 11 2008, *123* (4), 1415–1464.

Negro, Marco Del, Marc Giannoni, and Christina Patterson, "The forward guidance puzzle," Technical Report 2012.

Patterson, Christina, "The Matching Multiplier and the Amplification of Recessions," Technical Report 2022.

Pfäuti, Oliver and Fabian Seyrich, "A Behavioral Heterogeneous Agent New Keynesian Model," Working Paper February 2022.

Ramey, V.A., "Macroeconomic Shocks and Their Propagation," in John B. Taylor and Harald Uhlig, eds., *John B. Taylor and Harald Uhlig, eds.*, Vol. 2 of *Handbook of Macroeconomics*, Elsevier, 2016, pp. 71–162.

Romer, Christina D. and David H. Romer, "A New Measure of Monetary Shocks: Derivation and Implications," *American Economic Review*, 2004, *94* (4), 1055–1084.

Taylor, John B., "Discretion versus Policy Rules in Practice," in "Carnegie-Rochester Conference Series on Public Policy," Vol. 39, North-Holland, 1993, pp. 195–214.

Werning, Iván, "Incomplete Markets and Aggregate Demand," Working Paper 21448, National Bureau of Economic Research August 2015.

Woodford, Michael, "Imperfect Common Knowledge and the Effects of Monetary Policy," Working Paper 8673, National Bureau of Economic Research December 2001.

—, *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton, NJ: Princeton University Press, 2003.

Appendix A. Proofs of Propositions

Proof of Proposition 1. The best response of agent l in group g is specified as follows

$$a_{lgt} = \varphi_g \mathbb{E}_{lgt}\xi_t + \beta_g \mathbb{E}_{lgt} a_{lgt+1} + \sum_{j=1}^{2} \gamma_{gj} \mathbb{E}_{lgt} a_{jt} + \sum_{j=1}^{2} \alpha_{gj} \mathbb{E}_{lgt} a_{jt+1} \tag{A.1}$$

where a_{-gt} is the aggregate action of the other group at time t. Let $\boldsymbol{a}_t = (a_{gt})$ be a column vector collecting the aggregate actions of all groups (e.g., the vector of aggregate consumption and aggregate inflation), let $\boldsymbol{\varphi} = (\varphi_g)$ be a column vector containing the value of φ_g across groups, let $\boldsymbol{\beta} = \text{diag}(\beta_g)$ be a 2×2 diagonal matrix of discount factors, with off-diagonal elements equal to 0, let $\boldsymbol{\gamma}$ be a 2×2 matrix collecting the (contemporaneous) interaction parameters γ_{gj}, let $\boldsymbol{\alpha} = (\alpha_{gk})$ be a 2×2 matrix collecting the (future) interaction parameters α_{gj}, and finally let $\boldsymbol{\delta} \equiv \boldsymbol{\beta} + \boldsymbol{\alpha}$,

$$\boldsymbol{a}_t = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}, \quad \boldsymbol{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

Notice that (A.1) is equivalent to (10) and (12), respectively, if $a_{1t} = y_t$, $a_{2t} = \pi_t$, $\xi_t = \nu_t$, $\overline{\mathbb{E}}_{1t}(\cdot) = \overline{\mathbb{E}}^c_t(\cdot)$, $\overline{\mathbb{E}}_{2t}(\cdot) = \overline{\mathbb{E}}_{ft}(\cdot)$ and the following parametric restrictions are satisfied: $\varphi_1 = -\beta/\nu$, $\beta_1 = \beta s$, $\gamma_{11} = 1 - \beta(1 + \phi_y/\nu)$, $\gamma_{12} = -\beta\phi_\pi/\nu$, $\alpha_{11} = \beta(\delta - s)$, $\alpha_{12} = \beta/\nu$, $\varphi_2 = 0$, $\beta_2 = \beta\theta$, $\gamma_{21} = \kappa\theta$, $\gamma_{22} = 1 - \theta$, and $\alpha_{21} = \alpha_{22} = 0$.

I now turn to solve the expectation terms. I can write the fundamental representation of the signal process as a system containing (7) and (8), which admits the following state-space representation

$$\text{(A.2)} \qquad \boldsymbol{Z}_t = \boldsymbol{F}\boldsymbol{Z}_{t-1} + \boldsymbol{\Phi}\boldsymbol{s}_{lgt}, \qquad x_{lgt} = \boldsymbol{H}\boldsymbol{Z}_t + \boldsymbol{\Psi}_g\boldsymbol{s}_{lgt}$$

with $\boldsymbol{F} = \rho$, $\boldsymbol{\Phi} = \begin{bmatrix}\sigma_\varepsilon & 0\end{bmatrix}$, $\boldsymbol{Z}_t = \xi_t$, $\boldsymbol{s}_{lgt} = \begin{bmatrix}\varepsilon_t^\xi & u_{lgt}\end{bmatrix}^\intercal$, $\boldsymbol{H} = 1$, and $\boldsymbol{\Psi}_g = \begin{bmatrix}0 & \sigma_{gu}\end{bmatrix}$. It is convenient to rewrite the uncertainty parameters in terms of precision: define $\tau_\varepsilon \equiv \frac{1}{\sigma_\varepsilon^2}$, $\tau_{gu} \equiv \frac{1}{\sigma_{gu}^2}$, and $\tau_g = \frac{\tau_{gu}}{\tau_\varepsilon}$. The signal system can be written as

$$\text{(A.3)} \quad x_{lgt} = \frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t^\xi + \sigma_{gu}u_{lgt} = \begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & \tau_{gu}^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}\varepsilon_t^\xi \\ u_{lgt}\end{bmatrix} = \boldsymbol{M}_g(L)\boldsymbol{s}_{lgt}, \qquad \boldsymbol{s}_{lgt} \sim \mathcal{N}(0, I)$$

The Wold theorem states that there exists another representation of the signal process (A.3), $x_{lgt} = \boldsymbol{B}_g(L)\boldsymbol{w}_{lgt}$ such that $\boldsymbol{B}_g(z)$ is invertible and $\boldsymbol{w}_{lgt} \sim (0, \boldsymbol{V}_g)$ is white noise. Hence, I can write the following equivalence:

$$\text{(A.4)} \qquad x_{lgt} = \boldsymbol{M}_g(L)\boldsymbol{s}_{lgt} = \boldsymbol{B}_g(L)\boldsymbol{w}_{lgt}$$

In the Wold representation of x_{lgt}, observing $\{x_{lgt}\}$ is equivalent to observing $\{\boldsymbol{w}_{lgt}\}$, and $\{x_{lg}^t\}$ and $\{\boldsymbol{w}_{lg}^t\}$ contain the same information. Furthermore, note that the Wold representation has the property that both processes share the autocovariance generating function, $\rho_{xx}^g(z) = \boldsymbol{M}_g(z)\boldsymbol{M}_g^\intercal(z^{-1}) = \boldsymbol{B}_g(z)\boldsymbol{V}_g\boldsymbol{B}_g^\intercal(z^{-1})$. Given the state-space representation of the signal process (A.2), optimal expectations of the exogenous fundamental take the form of a Kalman filter $\mathbb{E}_{lgt}\xi_t = \lambda_g\mathbb{E}_{it-1}\xi_{t-1} + \boldsymbol{K}_g x_{lgt}$, where $\lambda_g = (I - \boldsymbol{K}_g\boldsymbol{H})\boldsymbol{F}$, and $\boldsymbol{K}_g$ is given by

$$\text{(A.5)} \qquad \boldsymbol{K}_g = \boldsymbol{P}_g\boldsymbol{H}^\intercal\boldsymbol{V}_g^{-1}$$

$$\text{(A.6)} \qquad \boldsymbol{P}_g = \boldsymbol{F}[\boldsymbol{P}_g - \boldsymbol{P}_g\boldsymbol{H}^\intercal\boldsymbol{V}_g^{-1}\boldsymbol{H}\boldsymbol{P}_g]\boldsymbol{F} + \boldsymbol{\Phi}\boldsymbol{\Phi}^\intercal$$

I still need to find the unknowns $\boldsymbol{B}_g(z)$ and $\boldsymbol{V}_g$. Propositions 13.1-13.4 in Hamilton (1994) provide us with these objects. Unknowns $\boldsymbol{B}_g(z)$ and $\boldsymbol{V}_g$ satisfy $\boldsymbol{B}_g(z) = I + \boldsymbol{H}(I - \boldsymbol{F}z)^{-1}\boldsymbol{F}\boldsymbol{K}_g$ and $\boldsymbol{V}_g = \boldsymbol{H}\boldsymbol{P}_g\boldsymbol{H}^\intercal + \boldsymbol{\Psi}_g\boldsymbol{\Psi}_g^\intercal$. I can write (A.6) as

$$\text{(A.7)} \qquad \boldsymbol{P}_g^2 + \boldsymbol{P}_g[(1-\rho^2)\sigma_{gu}^2 - \sigma_\varepsilon^2] - \sigma_\varepsilon^2\sigma_{gu}^2 = 0$$

from which I can infer that $\boldsymbol{P}_g$ is a scalar. Denote $k_g = \boldsymbol{P}_g^{-1}$ and rewrite (A.7) as $k_g =$

$\frac{\tau_\varepsilon}{2}\left\{1-\rho^2-\tau_g \pm \sqrt{\left[\tau_g-(1-\rho^2)\right]^2+4\tau_g}\right\}$.

I also need to find $\boldsymbol{K}_g$. Now that I have found $\boldsymbol{P}_g$ in terms of model primitives, I can obtain $\boldsymbol{K}_g$ using condition (A.5), $\boldsymbol{K}_g = \frac{1}{1+k_g\sigma_{gu}^2}$. I can finally write λ_g as $\lambda_g = \frac{k_g\sigma_{gu}^2\rho}{1+k_g\sigma_{gu}^2} = \frac{1}{2}\left[\frac{1}{\rho}+\rho+\frac{\tau_g}{\rho} \pm \sqrt{\left(\frac{1}{\rho}+\rho+\frac{\tau_g}{\rho}\right)^2-4}\right]$. One can show that one of the roots $\lambda_{g,[1,2]}$ lies inside the unit circle, and the other lies outside as long as $\rho \in (0,1)$, which guarantees that the Kalman expectation process is stationary and unique. I set λ_g to the root that lies inside the unit circle (the one with the '–' sign). Notice that I can also write $\boldsymbol{V}_g$ in terms of λ_g, $\boldsymbol{V}_g = k^{-1}+\sigma_{gu}^2 = \frac{\rho}{\lambda_g\tau_{gu}}$, where I have used the identity $k_g = \frac{\lambda_g\tau_{gu}}{\rho-\lambda_g}$. Finally, I can obtain $\boldsymbol{B}_g(z) = 1+\frac{\rho z}{(1-\rho z)(1+k\sigma_{gu}^2)} = \frac{1-\lambda_g z}{1-\rho z}$ and therefore one can verify that $\boldsymbol{B}_g(z)\boldsymbol{V}_g\boldsymbol{B}_g^\intercal(z^{-1}) = \boldsymbol{M}_g(z)\boldsymbol{M}_g^\intercal(z^{-1})$ implies $(\rho-\lambda_g)(1-\rho\lambda_g) = \lambda_g\tau_g$.

Let us now move to the forecast of endogenous variables. Consider a variable $f_t = A(L)\boldsymbol{s}_{lgt}$. Applying the Wiener-Hopf prediction filter, I can obtain the forecast as $\mathbb{E}_{lgt}f_t = \left[A(L)\boldsymbol{M}^\intercal(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+\boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{lgt}$, where $[\cdot]_+$ denotes the annihilator operator.

I need to find the $A(z)$ polynomial for each of the forecasted variables. Let us start from the exogenous fundamental ξ_t to verify that the Kalman and Wiener-Hopf filters result in the same forecast. I can write the fundamental as $\xi_t = \begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & 0\end{bmatrix}\boldsymbol{s}_{it} = A_\xi(L)\boldsymbol{s}_{it}$. Let me now move to the endogenous variables. Guess that agent $i\times g$'s policy function satisfies $a_{lgt} = h_g(L)x_{lgt}$. The aggregate outcome in group g can then be expressed as $a_{gt} = \int a_{lgt}\,di = \int h_g(L)x_{lgt}\,di = h_g(L)\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t = \begin{bmatrix}h_g(L)\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & 0\end{bmatrix}\boldsymbol{s}_{l1t} = A_g(L)\boldsymbol{s}_{lgt}$. Similarly, the own and average future actions can be written as $a_{g,t+1} = \frac{A_g(L)}{L}\boldsymbol{s}_{lgt}$ and $a_{igt+1} = a_{ig,t+1} = h_g(L)x_{ig,t+1} = \begin{bmatrix}\tau_\varepsilon^{-\frac{1}{2}}\frac{h_g(L)}{L(1-\rho L)} & \tau_{gu}^{-\frac{1}{2}}\frac{h_g(L)}{L}\end{bmatrix}\boldsymbol{s}_{lgt} = A_{ig}(L)\boldsymbol{s}_{lgt}$. I now obtain the forecasts,

$$
\begin{aligned}
\mathbb{E}_{lgt}\xi_t &= \left[A_\xi(L)\boldsymbol{M}_g^\intercal(L^{-1})\boldsymbol{B}_g(L^{-1})^{-1}\right]_+\boldsymbol{V}_g^{-1}\boldsymbol{B}_g(L)^{-1}x_{lgt} = \left[\frac{L}{(1-\rho L)(L-\lambda_g)}\right]_+\frac{\lambda\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt} \\
&= \left[\frac{\phi_1(L)}{L-\lambda_g}\right]_+\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt} = \frac{\phi_1(L)-\phi_1(\lambda_g)}{L-\lambda_g}\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}, \quad \phi_1(z) = \frac{z}{1-\rho z}
\end{aligned}
\tag{A.8}
$$

$$
= \frac{\lambda_g\tau_g}{\rho(1-\rho\lambda_g)}\frac{1}{1-\lambda_g L}x_{lgt} = \left(1-\frac{\lambda_g}{\rho}\right)\frac{1}{1-\lambda_g L}x_{lgt} = G_{1g}(L)x_{lgt}
$$

$$
\mathbb{E}_{lgt}a_{k,t+1} = \left[\frac{A_k(L)}{L}\boldsymbol{M}_g^\intercal(L^{-1})\boldsymbol{B}_g(L^{-1})^{-1}\right]_+\boldsymbol{V}_g^{-1}\boldsymbol{B}_g(L)^{-1}x_{lgt} = \left[\frac{h_k(L)}{(1-\rho L)(L-\lambda_g)}\right]_+\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}
$$

$$= \left[\frac{\phi_2(L)}{L-\lambda_g}\right]_+ \frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt} = \frac{\phi_2(L)-\phi_2(\lambda_g)}{L-\lambda_g}\frac{\lambda_g\tau_{gu}}{\rho\tau_\varepsilon}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}, \quad \phi_2(z) = \frac{h_k(z)}{1-\rho z}$$

(A.9)

$$= \frac{\lambda_g\tau_g}{\rho}\left[h_k(L) - h_k(\lambda_g)\frac{1-\rho L}{1-\rho\lambda_g}\right]\frac{1}{(1-\lambda_g L)(L-\lambda_g)}x_{lgt} = G_{2gk}(L)x_{lgt}$$

$$\mathbb{E}_{lgt}a_{kt} = \left[A_k(L)\boldsymbol{M}_g^\top(L^{-1})\boldsymbol{B}_g(L^{-1})^{-1}\right]_+ \boldsymbol{V}_g^{-1}\boldsymbol{B}_g(L)^{-1}x_{lgt} = \left[\frac{h_k(L)L}{(1-\rho L)(L-\lambda_g)}\right]_+ \frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}$$

$$= \left[\frac{\phi_3(L)}{L-\lambda_g}\right]_+ \frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt} = \frac{\phi_3(L)-\phi_3(\lambda_g)}{L-\lambda_g}\frac{\lambda_g\tau_{gu}}{\rho\tau_\varepsilon}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}, \quad \phi_3(z) = \frac{h_k(z)z}{1-\rho z}$$

(A.10)

$$= \frac{\lambda_g\tau_g}{\rho}\left[h_k(L)L - h_k(\lambda_g)\lambda_g\frac{1-\rho L}{1-\rho\lambda_g}\right]\frac{1}{(1-\lambda_g L)(L-\lambda_g)}x_{lgt} = G_{3gk}(L)x_{lgt}$$

$$\mathbb{E}_{lgt}a_{lg,t+1} = \left[A_{ig}(L)\boldsymbol{M}_g^\top(L^{-1})\boldsymbol{B}_g(L^{-1})^{-1}\right]_+ \boldsymbol{V}_g^{-1}\boldsymbol{B}_g(L)^{-1}x_{lgt}$$

$$= \left[\frac{h_g(L)}{\tau_\varepsilon(1-\rho L)(L-\lambda_g)} + \frac{h_g(L)(L-\rho)}{\tau_{gu}L(L-\lambda_g)}\right]_+ \frac{\lambda_g\tau_{gu}}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}$$

$$= \left\{\left[\frac{h_g(L)}{\tau_\varepsilon(1-\rho L)(L-\lambda_g)}\right]_+ + \left[\frac{h_g(L)(L-\rho)}{\tau_{gu}L(L-\lambda_g)}\right]_+\right\}\frac{\lambda_g\tau_{gu}}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}$$

$$= \left\{\left[\frac{\phi_4(L)}{L-\lambda_g}\right]_+ + \left[\frac{\phi_5(L)}{L(L-\lambda_g)}\right]_+\right\}\frac{\lambda_g\tau_{gu}}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}$$

$$= \left\{\frac{\phi_4(L)-\phi_4(\lambda_g)}{L-\lambda_g} + \frac{\phi_5(L)-\phi_5(\lambda_g)}{\lambda_g(L-\lambda_g)} - \frac{\phi_5(L)-\phi_5(0)}{\lambda_g L}\right\}\frac{\lambda_g\tau_{gu}}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}$$

$$= \frac{\lambda_g}{\rho}\left\{\frac{h_g(L)}{L-\lambda_g}\left[\frac{\tau_{gu}}{\tau_\varepsilon(1-\rho L)} + \frac{L-\rho}{L}\right] - \frac{h_g(\lambda_g)}{L-\lambda_g}\left[\frac{\tau_{gu}}{\tau_\varepsilon(1-\rho\lambda_g)} + \frac{\lambda_g-\rho}{\lambda_g}\right] - \frac{\rho h_g(0)}{\lambda_g L}\right\}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}$$

$$= \left\{\frac{h_g(L)}{L-\lambda_g}\left[\left(1-\frac{\lambda_g}{\rho}\right)\frac{1-\rho\lambda_g}{1-\rho L} + \frac{\lambda_g(L-\rho)}{\rho L}\right] - \frac{h_g(0)}{L}\right\}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}$$

(A.11)

$$= G_{4g}(L)x_{lgt}, \quad \phi_4(z) = \frac{h_g(z)}{\tau_\varepsilon(1-\rho z)}, \quad \phi_5(z) = \frac{h_g(z)(z-\rho)}{\tau_{gu}}$$

Inserting the obtained expressions into (A.1),

$$h_g(L)x_{lgt} = \varphi_g G_{1g}(L)x_{lgt} + \beta_g G_{4g}(L)x_{lgt} + \sum_{k=1}^{n}\gamma_{gk}G_{3gk}(L)x_{lgt} + \sum_{k=1}^{n}\alpha_{gk}G_{2gk}(L)x_{lgt}$$

$$h_g(L)x_{lgt} = \varphi_g\left(1-\frac{\lambda_g}{\rho}\right)\frac{1}{1-\lambda_g L}x_{lgt} + \beta_g\left\{\frac{h_g(L)}{L-\lambda_g}\left[\left(1-\frac{\lambda_g}{\rho}\right)\frac{1-\rho\lambda_g}{1-\rho L} + \frac{\lambda_g(L-\rho)}{\rho L}\right] - \frac{h_g(0)}{L}\right\}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}$$

$$+\sum_{k=1}^{2}\gamma_{gk}\frac{\lambda_g\tau_g}{\rho}\left[h_k(L)L-h_k(\lambda_g)\lambda_g\frac{1-\rho L}{1-\rho\lambda_g}\right]\frac{1}{(1-\lambda_g L)(L-\lambda_g)}x_{l\,gt}$$
$$+\sum_{k=1}^{2}\alpha_{gk}\frac{\lambda_g\tau_g}{\rho}\left[h_k(L)-h_k(\lambda_g)\frac{1-\rho L}{1-\rho\lambda_g}\right]\frac{1}{(1-\lambda_g L)(L-\lambda_g)}x_{l\,gt}$$

Removing the $x_{l\,gt}$ terms, and rearranging terms on the LHS and RHS

$$h_g(z)\left\{1-\beta_g\left[\left(1-\frac{\lambda_g}{\rho}\right)\frac{1-\rho\lambda_g}{1-\rho z}+\frac{\lambda_g(z-\rho)}{\rho z}\right]\frac{1-\rho z}{(1-\lambda_g z)(L-\lambda_g)}\right\}$$
$$-\sum_{k=1}^{2}h_k(z)\frac{(\rho-\lambda_g)(1-\rho\lambda_g)}{\rho}\frac{\gamma_{gk}z+\alpha_{gk}}{(1-\lambda_g z)(z-\lambda_g)}$$
$$=\varphi_g\left(1-\frac{\lambda_g}{\rho}\right)\frac{1}{1-\lambda_g z}-\beta_g\frac{1-\rho z}{z(1-\lambda_g z)}h_g(0)-\sum_{k=1}^{2}h_k(\lambda_g)\left(1-\frac{\lambda_g}{\rho}\right)\frac{\gamma_{gk}\lambda_g+\alpha_{gk}}{(1-\lambda_g z)(z-\lambda_g)}(1-\rho z)$$

Multiplying both sides by $z(z-\lambda_g)(1-\lambda_g z)$,

$$h_g(z)\left[z(z-\lambda_g)(1-\lambda_g z)-\beta_g(z-\lambda_g)(1-\lambda_g z)\right]-\sum_{k=1}^{2}h_k(z)\frac{(\rho-\lambda_g)(1-\rho\lambda_g)}{\rho}z(\gamma_{gk}z+\alpha_{gk})$$
$$=\varphi_g\left(1-\frac{\lambda_g}{\rho}\right)z(z-\lambda_g)-\beta_g(1-\rho z)(z-\lambda_g)h_g(0)-\sum_{k=1}^{2}h_k(\lambda_g)\left(1-\frac{\lambda_g}{\rho}\right)(\gamma_{gk}\lambda_g+\alpha_{gk})z(1-\rho z)$$

I can write the above system of equations in terms of $\boldsymbol{h}(L)$ in matrix form

$$\boldsymbol{C}(z)\boldsymbol{h}(z)=\boldsymbol{d}(z) \tag{A.12}$$

where $\boldsymbol{C}(z)\equiv\operatorname{diag}\left\{\lambda_g\right\}\left[(\boldsymbol{\beta}-Iz)\operatorname{diag}\left\{\left(z-\frac{1}{\rho}\right)(z-\rho)\right\}-(\boldsymbol{\beta}-Iz)z\operatorname{diag}\left\{\frac{\tau_g}{\rho}\right\}-z\operatorname{diag}\left\{\frac{\tau_g}{\rho}\right\}(z\boldsymbol{\gamma}+\boldsymbol{\alpha})\right]$.
That is, I can write $\boldsymbol{C}(z)=\begin{bmatrix}C_{11}(z) & C_{12}(z)\\ C_{21}(z) & C_{22}(z)\end{bmatrix}$, where

$$C_{11}(z)=\lambda_1\left[(\beta_1-z)\left(z-\frac{1}{\rho}\right)(z-\rho)+\frac{\tau_1}{\rho}z\left[z(1-\gamma_{11})-\delta_{11}\right]\right],\quad C_{12}(z)=-\lambda_1 z\frac{\tau_1}{\rho}(z\gamma_{12}+\delta_{12})$$
$$C_{22}(z)=\lambda_2\left[(\beta_2-z)\left(z-\frac{1}{\rho}\right)(z-\rho)+\frac{\tau_2}{\rho}z\left[z(1-\gamma_{22})-\delta_{22}\right]\right],\quad C_{21}(z)=-\lambda_2 z\frac{\tau_2}{\rho}(z\gamma_{21}+\delta_{21})$$

I can also write $\boldsymbol{d}(z)=\left[d_1[z;h_1(\cdot)]\quad d_2[z;h_2(\cdot)]\right]^{\top}$, where $d_g(z)=\varphi_g\left(1-\frac{\lambda_g}{\rho}\right)z(z-\lambda_g)-$

$\beta_g(1-\rho z)(z-\lambda_g)h_g(0) - \sum_{k=1}^{2} h_k(\lambda_g)\left(1-\frac{\lambda_g}{\rho}\right)(\gamma_{gk}\lambda_g + \alpha_{gk})z(1-\rho z)$.

From (A.12), the solution to the policy function is given by $\boldsymbol{h}(z) = \boldsymbol{C}(z)^{-1}\boldsymbol{d}(z) = \frac{\operatorname{adj} \boldsymbol{C}(z)}{\det \boldsymbol{C}(z)}\boldsymbol{d}(z)$. Hence, I need to obtain $\det \boldsymbol{C}(z)$. Note that the degree of $\det \boldsymbol{C}(z)$ is a polynomial of degree 6 on z. Denote the inside roots of $\det \boldsymbol{C}(z)$ as $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, and the outside roots as $\left\{\vartheta_1^{-1}, \vartheta_2^{-1}\right\}$. Because agents cannot use future signals, the inside roots have to be removed. Note that the number of free constants in $\boldsymbol{d}(z)$ is 4: $\left\{h_g(0)\right\}$ and $\left\{\widetilde{h}(\lambda_g) = \sum_{k=1}^{2} h_k(\lambda_g)\left(1-\frac{\lambda_g}{\rho}\right)(\gamma_{gk}\lambda_g + \alpha_{gk})\right\}$ for each $g \in \{1,2\}$. With a unique solution, it has to be the case that the number of outside roots is 2.[31] By Cramer's rule, $h_g(L)$ is given by

$$h_1(z) = \frac{\det\begin{bmatrix} d_1(z) & C_{12}(z) \\ d_2(z) & C_{22}(z)\end{bmatrix}}{\det \boldsymbol{C}(z)}, \qquad h_2(z) = \frac{\det\begin{bmatrix} C_{11}(z) & d_1(z) \\ C_{21}(z) & d_2(z)\end{bmatrix}}{\det \boldsymbol{C}(z)}$$

which are the policy function for groups 1 (consumers) and 2 (firms). The degree of the numerator is 5, as the highest degree of $d_g(z)$ is 1 degree less than that of $\boldsymbol{C}(z)$. By choosing the appropriate constants $\left\{h_1(0), \widetilde{h}(\lambda_1), h_2(0), \widetilde{h}(\lambda_2)\right\}$, the 4 inside roots will be removed. Therefore, the 4 constants are solutions to the following system of linear equations $\det\begin{bmatrix} d_1(\zeta_n) & C_{12}(\zeta_n) \\ d_2(\zeta_n) & C_{22}(\zeta_n)\end{bmatrix} = 0$, for $\{\zeta_n\}_{n=1}^{4}$. After removing the inside roots in the denominator, the degree of the numerator is 1 and the degree of the denominator is 2. The above determinants can be written as a system of 4 equations and 4 unknowns (the free constants). Once I have set the appropriate free constants the policy functions will be $h_g(z) = \frac{\widetilde{\psi}_{g1}+\widetilde{\psi}_{g2}z}{(1-\vartheta_1 z)(1-\vartheta_2 z)}$, and hence I have $a_{gt} = h_g(L)\xi_t = \frac{\widetilde{\psi}_{g1}+\widetilde{\psi}_{g2}z}{(1-\vartheta_1 z)(1-\vartheta_2 z)}\xi_t = \sum_{j=1}^{2} \psi_{gj}\left(1-\frac{\vartheta_j}{\rho}\right)\frac{1}{1-\vartheta_j L}\xi_t = \sum_{j=1}^{2}\psi_{gj}\widetilde{\vartheta}_{jt}$. I can write $\boldsymbol{a}_t = \begin{bmatrix} a_{1t} \\ a_{2t}\end{bmatrix} = Q\widetilde{\vartheta}_t = \begin{bmatrix}\psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22}\end{bmatrix}\begin{bmatrix}\widetilde{\vartheta}_{1t} \\ \widetilde{\vartheta}_{2t}\end{bmatrix} = \begin{bmatrix}\psi_{11}\widetilde{\vartheta}_{1t} + \psi_{12}\widetilde{\vartheta}_{2t} \\ \psi_{21}\widetilde{\vartheta}_{1t} + \psi_{22}\widetilde{\vartheta}_{2t}\end{bmatrix}$. Notice that I can write $\widetilde{\vartheta}_{gt}(1-\vartheta_g L) = \left(1-\frac{\vartheta_g}{\rho}\right)\xi_t \implies \widetilde{\vartheta}_{gt} = \vartheta_g\widetilde{\vartheta}_{g,t-1} + \left(1-\frac{\vartheta_g}{\rho}\right)\xi_t$, which I can write as a system as $\widetilde{\vartheta}_t = \Lambda\widetilde{\vartheta}_{t-1} + \Gamma\xi_t$, where $\Lambda = \begin{bmatrix}\vartheta_1 & 0 \\ 0 & \vartheta_2\end{bmatrix}$, $\Gamma = \begin{bmatrix} 1-\frac{\vartheta_1}{\rho} \\ 1-\frac{\vartheta_2}{\rho}\end{bmatrix}$. Hence, I can write $\boldsymbol{a}_t = Q\widetilde{\theta}_t =$

[31] This is the proof of part (*ii*) of proposition 4.

$Q(\Lambda\widetilde{\theta}_{t-1} + \Gamma\xi_t) = Q\Lambda\widetilde{\theta}_{t-1} + Q\Gamma\xi_t = Q\Lambda Q^{-1}\boldsymbol{a}_{t-1} + Q\Gamma\xi_t = A\boldsymbol{a}_{t-1} + B\xi_t$, where

$$A = \begin{bmatrix} \frac{\psi_{11}\psi_{22}\vartheta_1 - \psi_{12}\psi_{21}\vartheta_2}{\psi_{11}\psi_{22}-\psi_{12}\psi_{21}} & -\frac{\psi_{11}\psi_{12}(\vartheta_1-\vartheta_2)}{\psi_{11}\psi_{22}-\psi_{12}\psi_{21}} \\ \frac{\psi_{21}\psi_{22}(\vartheta_1-\vartheta_2)}{\psi_{11}\psi_{22}-\psi_{12}\psi_{21}} & -\frac{(\psi_{12}\psi_{21}\vartheta_1-\psi_{11}\psi_{22}\vartheta_2)}{\psi_{11}\psi_{22}-\psi_{12}\psi_{21}} \end{bmatrix}, \quad B = \begin{bmatrix} \psi_{11}\left(1-\frac{\vartheta_1}{\rho}\right) + \psi_{12}\left(1-\frac{\vartheta_2}{\rho}\right) \\ \psi_{21}\left(1-\frac{\vartheta_1}{\rho}\right) + \psi_{22}\left(1-\frac{\vartheta_2}{\rho}\right) \end{bmatrix}$$

where $\{\psi_{gk}\}_{g=1,k=1}^{2}$ are fixed scalars that depend on deep parameters of the model, satisfying

$$\text{(A.13)} \quad \sum_{j=1}^{2}\psi_{1j} = -\frac{1-\rho\beta}{(1-\beta\rho)[\nu(1-\delta\rho)+\phi_y]+\kappa(\phi_\pi-\rho)}, \qquad \sum_{j=1}^{2}\psi_{2j} = -\frac{\kappa}{(1-\beta\rho)[\nu(1-\delta\rho)+\phi_y]+\kappa(\phi_\pi-\rho)}$$

Finally, I need to show that (A.13) hold. First, notice that in the standard FIRE framework, there is no information friction, $\vartheta_1 = \vartheta_2 = 0$. Therefore, the dynamics follow $\boldsymbol{a}_t = A_{\text{FIRE}}\boldsymbol{a}_{t-1} + B_{\text{FIRE}}\xi_t$ where $A_{\text{FIRE}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B_{\text{FIRE}} = \begin{bmatrix} \psi_{11}+\psi_{12} \\ \psi_{21}+\psi_{22} \end{bmatrix}$ Under the standard FIRE case, dynamics are given by (24). To find the solution dynamics under FIRE, I proceed with a guess and verify approach. Assume that $\boldsymbol{a}_t = D\xi_t$. Using the method of undetermined coefficients $D\xi_t = \overline{\boldsymbol{\varphi}}\xi_t + \overline{\boldsymbol{\delta}}\mathbb{E}_t D\xi_{t+1} = \overline{\boldsymbol{\varphi}}\xi_t + \overline{\boldsymbol{\delta}}D\rho\xi_t \implies D = \overline{\boldsymbol{\varphi}} + \overline{\boldsymbol{\delta}}D\rho$. Hence, it must be that $D = (\boldsymbol{I} - \overline{\boldsymbol{\delta}}\rho)^{-1}\overline{\boldsymbol{\varphi}}$. Notice that, for consistency, $B_{\text{FIRE}} = D$. As a result, even if I cannot find the analytical form of the individual $(\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})$, I know that conditions (A.13) hold. □

Proof of Proposition 2. Consider the best response of an agent i is $a_{it} = \varphi_u\mathbb{E}_{it}\xi_t + \beta_u\mathbb{E}_{it}a_{i,t+1} + \gamma_u\mathbb{E}_{it}a_t + \alpha_u\mathbb{E}_{it}a_{t+1}$. Notice that the best response is equivalent to (10) if $a_t = y_t$, $\xi_t = r_t$, $\overline{\mathbb{E}}_t(\cdot) = \overline{\mathbb{E}}^c_t(\cdot)$, and the following parametric restrictions are satisfied: $\varphi_u = -\beta\frac{1-\lambda}{\sigma(1-\lambda\chi)}$, $\beta_u = \beta s$, $\gamma_u = 1-\beta$, and $\alpha_u = \beta(\delta - s)$.

I now turn to solve the expectation terms. I can write the fundamental representation of the signal process as a system containing (15) and (16), which admits the following state-space representation:

$$\text{(A.14)} \qquad \boldsymbol{Z}_t = \boldsymbol{F}\boldsymbol{Z}_{t-1} + \boldsymbol{\Phi}\boldsymbol{s}_{it}, \qquad x_{it} = \boldsymbol{H}\boldsymbol{Z}_t + \boldsymbol{\Psi}\boldsymbol{s}_{it}$$

with $\boldsymbol{F} = \rho$, $\boldsymbol{\Phi} = \begin{bmatrix} \sigma_\varepsilon & 0 \end{bmatrix}$, $\boldsymbol{Z}_t = r_t$, $\boldsymbol{s}_{it} = \begin{bmatrix} \varepsilon^r_t & u_{it} \end{bmatrix}^\top$, $\boldsymbol{H} = 1$, and $\boldsymbol{\Psi} = \begin{bmatrix} 0 & \sigma_u \end{bmatrix}$. It is convenient to rewrite the uncertainty parameters in terms of precision: define $\tau_\varepsilon \equiv \frac{1}{\sigma^2_\varepsilon}$, $\tau_u \equiv \frac{1}{\sigma^2_u}$, and $\tau_{\varepsilon u} = \frac{\tau_u}{\tau_\varepsilon}$. The signal system can be written as

$$x_{it} = \frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t^r + \sigma_u u_{it} = \begin{bmatrix} \frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & \tau_u^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \varepsilon_t^r \\ u_{it} \end{bmatrix} = \boldsymbol{M}(L)\boldsymbol{s}_{it}, \qquad \boldsymbol{s}_{it} \sim \mathcal{N}(0, I) \tag{A.15}$$

The Wold theorem states that there exists another representation of the signal process (A.15), $x_{it} = \boldsymbol{B}(L)\boldsymbol{w}_{it}$ such that $\boldsymbol{B}(z)$ is invertible and $\boldsymbol{w}_{it} \sim (0, \boldsymbol{V})$ is white noise. Hence, I can write $x_{it} = \boldsymbol{M}(L)\boldsymbol{s}_{it} = \boldsymbol{B}(L)\boldsymbol{w}_{it}$. In the Wold representation of x_{it}, observing $\{x_{it}\}$ is equivalent to observing $\{\boldsymbol{w}_{it}\}$, and $\{x_i^t\}$ and $\{\boldsymbol{w}_i^t\}$ contain the same information. Furthermore, note that the Wold representation has the property that both processes share the autocovariance generating function, $\rho_{xx}(z) = \boldsymbol{M}(z)\boldsymbol{M}^\top(z^{-1}) = \boldsymbol{B}(z)\boldsymbol{V}\boldsymbol{B}^\top(z^{-1})$. Given the state-space representation of the signal process (A.14), optimal expectations of the exogenous fundamental take the form of a Kalman filter $\mathbb{E}_{it}\xi_t = \lambda_u\mathbb{E}_{it-1}\xi_{t-1} + \boldsymbol{K}x_{it}$, where $\lambda_u = (I - \boldsymbol{K}\boldsymbol{H})\boldsymbol{F}$, and $\boldsymbol{K}$ is given by

$$\begin{aligned} \boldsymbol{K} &= \boldsymbol{P}\boldsymbol{H}^\top\boldsymbol{V}^{-1} \\ \boldsymbol{P} &= \boldsymbol{F}[\boldsymbol{P} - \boldsymbol{P}\boldsymbol{H}^\top\boldsymbol{V}^{-1}\boldsymbol{H}\boldsymbol{P}]\boldsymbol{F} + \boldsymbol{\Phi}\boldsymbol{\Phi}^\top \end{aligned} \tag{A.16}$$

I still need to find the unknowns $\boldsymbol{B}(z)$ and $\boldsymbol{V}$. Propositions 13.1-13.4 in Hamilton (1994) provide us with these objects. Unknowns $\boldsymbol{B}(z)$ and $\boldsymbol{V}$ satisfy $\boldsymbol{B}(z) = I + \boldsymbol{H}(I - \boldsymbol{F}z)^{-1}\boldsymbol{F}\boldsymbol{K}$ and $\boldsymbol{V} = \boldsymbol{H}\boldsymbol{P}\boldsymbol{H}^\top + \boldsymbol{\Psi}\boldsymbol{\Psi}^\top$. I can write (A.6) as

$$\boldsymbol{P}^2 + \boldsymbol{P}[(1-\rho^2)\sigma_u^2 - \sigma_\varepsilon^2] - \sigma_\varepsilon^2\sigma_u^2 = 0 \tag{A.17}$$

from which I can infer that $\boldsymbol{P}$ is a scalar. Denote $k = \boldsymbol{P}^{-1}$ and rewrite (A.17) as $k = \frac{\tau_\varepsilon}{2}\left\{1 - \rho^2 - \tau_{\varepsilon u} \pm \sqrt{\left[\tau_{\varepsilon u} - (1-\rho^2)\right]^2 + 4\tau_{\varepsilon u}}\right\}$. I also need to find $\boldsymbol{K}$. Now that I have found $\boldsymbol{P}$ in terms of model primitives, I can obtain $\boldsymbol{K}$ using condition (A.16), $\boldsymbol{K} = \frac{1}{1+k\sigma_u^2}$. I can finally write $\lambda_u = \frac{k\sigma_u^2\rho}{1+k\sigma_u^2} = \frac{1}{2}\left[\frac{1}{\rho} + \rho + \frac{\tau_{\varepsilon u}}{\rho} \pm \sqrt{\left(\frac{1}{\rho} + \rho + \frac{\tau_{\varepsilon u}}{\rho}\right)^2 - 4}\right]$. One can show that one of the roots λ_u lies inside the unit circle and the other lies outside as long as $\rho \in (0, 1)$, which guarantees that the Kalman expectation process is stationary and unique. I set λ_u to the root that lies inside the unit circle (the one with the '–' sign). Notice that I can also write $\boldsymbol{V}$ in terms of λ, $\boldsymbol{V} = k^{-1} + \sigma_u^2 = \frac{\rho}{\lambda_u\tau_u}$, where I have used the identity $k = \frac{\lambda_u\tau_u}{\rho-\lambda_u}$. Finally, I can obtain $\boldsymbol{B}(z) = 1 + \frac{\rho z}{(1-\rho z)(1+k\sigma_u^2)} = \frac{1-\lambda_u z}{1-\rho z}$ and therefore one can verify that $\boldsymbol{B}(z)\boldsymbol{V}\boldsymbol{B}^\top(z^{-1}) = \boldsymbol{M}(z)\boldsymbol{M}^\top(z^{-1}) \implies (\rho - \lambda_u)(1 - \rho\lambda_u) = \lambda_u\tau_{\varepsilon u}$.

Let us now move to the forecast of endogenous variables. Consider a variable $f_t = A(L)\boldsymbol{s}_{it}$. Applying the Wiener-Hopf prediction filter, I can obtain the forecast as $\mathbb{E}_{it} f_t = \left[A(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it}$, where $[\cdot]_+$ denotes the annihilator operator.

I need to find the $A(z)$ polynomial for each of the forecasted variables. Let us start from the exogenous fundamental ξ_t to verify that the Kalman and Wiener-Hopf filters result in the same forecast. I can write the fundamental as $\xi_t = \begin{bmatrix} \frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & 0 \end{bmatrix} \boldsymbol{s}_{it} = A_\xi(L)\boldsymbol{s}_{it}$. Let me now move to the endogenous variables. Guess that household i's policy function satisfies $a_{it} = h(L)x_{it}$. The aggregate outcome can then be expressed as $a_t = \int a_{it}\, di = \int h(L)x_{it}\, di = h(L)\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t = \begin{bmatrix} h(L)\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & 0 \end{bmatrix} \boldsymbol{s}_{it} = A(L)\boldsymbol{s}_{it}$. Similarly, the own and average future actions can be written as $a_{t+1} = \frac{A(L)}{L}\boldsymbol{s}_{it}$ and $a_{it+1} = h(L)x_{i,t+1} = \begin{bmatrix} \tau_\varepsilon^{-\frac{1}{2}} \frac{h(L)}{L(1-\rho L)} & \tau_u^{-\frac{1}{2}} \frac{h(L)}{L} \end{bmatrix} \boldsymbol{s}_{it} = A_i(L)\boldsymbol{s}_{it}$. I now obtain the forecasts,

$$\begin{aligned}
\mathbb{E}_{it}\xi_t &= \left[A_\xi(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it} = \left[\frac{L}{(1-\rho L)(L-\lambda_u)}\right]_+ \frac{\lambda_u \tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left[\frac{\phi_1(L)}{L-\lambda_u}\right]_+ \frac{\lambda_u \tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} = \frac{\phi_1(L)-\phi_1(\lambda_u)}{L-\lambda_u}\frac{\lambda_u \tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it}, \quad \phi_1(z) = \frac{z}{1-\rho z} \\
&= \frac{\lambda_u \tau_{\varepsilon u}}{\rho(1-\rho\lambda_u)}\frac{1}{1-\lambda_u L}x_{it} = \left(1-\frac{\lambda_u}{\rho}\right)\frac{1}{1-\lambda_u L}x_{it} = G_1(L)x_{it}
\end{aligned} \tag{A.18}$$

$$\begin{aligned}
\mathbb{E}_{it}a_{t+1} &= \left[\frac{A(L)}{L}\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it} = \left[\frac{h(L)}{(1-\rho L)(L-\lambda_u)}\right]_+ \frac{\lambda_u \tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left[\frac{\phi_2(L)}{L-\lambda_u}\right]_+ \frac{\lambda_u \tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} = \frac{\phi_2(L)-\phi_2(\lambda_u)}{L-\lambda_u}\frac{\lambda_u \tau_u}{\rho\tau_\varepsilon}\frac{1-\rho L}{1-\lambda_u L}x_{it}, \quad \phi_2(z) = \frac{h(z)}{1-\rho z} \\
&= \frac{\lambda_u \tau_{\varepsilon u}}{\rho}\left[h(L)-h(\lambda_u)\frac{1-\rho L}{1-\rho\lambda_u}\right]\frac{1}{(1-\lambda_u L)(L-\lambda_u)}x_{it} = G_2(L)x_{it}
\end{aligned} \tag{A.19}$$

$$\begin{aligned}
\mathbb{E}_{it}a_t &= \left[A(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it} = \left[\frac{h(L)L}{(1-\rho L)(L-\lambda_u)}\right]_+ \frac{\lambda_u \tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left[\frac{\phi_3(L)}{L-\lambda_u}\right]_+ \frac{\lambda_u \tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} = \frac{\phi_3(L)-\phi_3(\lambda_u)}{L-\lambda_u}\frac{\lambda_u \tau_u}{\rho\tau_\varepsilon}\frac{1-\rho L}{1-\lambda_u L}x_{it}, \quad \phi_3(z) = \frac{h(z)z}{1-\rho z} \\
&= \frac{\lambda_u \tau_{\varepsilon u}}{\rho}\left[h(L)L-h(\lambda_u)\lambda_u\frac{1-\rho L}{1-\rho\lambda_u}\right]\frac{1}{(1-\lambda_u L)(L-\lambda_u)}x_{it} = G_3(L)x_{it}
\end{aligned} \tag{A.20}$$

$$\mathbb{E}_{it}a_{i,t+1} = \left[A_{ig}(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it}$$

$$
\begin{aligned}
&= \left[\frac{h(L)}{\tau_\varepsilon(1-\rho L)(L-\lambda_u)} + \frac{h(L)(L-\rho)}{\tau_u L(L-\lambda_u)} \right]_+ \frac{\lambda_u \tau_u}{\rho} \frac{1-\rho L}{1-\lambda_u L} x_{it} \\
&= \left\{ \left[\frac{h(L)}{\tau_\varepsilon(1-\rho L)(L-\lambda_u)} \right]_+ + \left[\frac{h(L)(L-\rho)}{\tau_u L(L-\lambda_u)} \right]_+ \right\} \frac{\lambda_u \tau_u}{\rho} \frac{1-\rho L}{1-\lambda_u L} x_{it} \\
&= \left\{ \left[\frac{\phi_4(L)}{L-\lambda_u} \right]_+ + \left[\frac{\phi_5(L)}{L(L-\lambda_u)} \right]_+ \right\} \frac{\lambda_u \tau_u}{\rho} \frac{1-\rho L}{1-\lambda_u L} x_{it} \\
&= \left\{ \frac{\phi_4(L)-\phi_4(\lambda_u)}{L-\lambda_u} + \frac{\phi_5(L)-\phi_5(\lambda_u)}{\lambda_u(L-\lambda_u)} - \frac{\phi_5(L)-\phi_5(0)}{\lambda_u L} \right\} \frac{\lambda_u \tau_u}{\rho} \frac{1-\rho L}{1-\lambda_u L} x_{it} \\
&= \frac{\lambda_u}{\rho} \left\{ \frac{h(L)}{L-\lambda_u} \left[\frac{\tau_u}{\tau_\varepsilon(1-\rho L)} + \frac{L-\rho}{L} \right] - \frac{h(\lambda_u)}{L-\lambda_u} \left[\frac{\tau_u}{\tau_\varepsilon(1-\rho\lambda_u)} + \frac{\lambda_u-\rho}{\lambda_u} \right] - \frac{\rho h(0)}{\lambda_u L} \right\} \frac{1-\rho L}{1-\lambda_u L} x_{it} \\
&= \left\{ \frac{h(L)}{L-\lambda_u} \left[\left(1-\frac{\lambda_u}{\rho}\right) \frac{1-\rho\lambda_u}{1-\rho L} + \frac{\lambda_u(L-\rho)}{\rho L} \right] - \frac{h(0)}{L} \right\} \frac{1-\rho L}{1-\lambda_u L} x_{it}
\end{aligned}
$$

(A.21)

$$
= G_4(L) x_{it}, \quad \phi_4(z) = \frac{h(z)}{\tau_\varepsilon(1-\rho z)}, \quad \phi_5(z) = \frac{h(z)(z-\rho)}{\tau_u}
$$

Inserting the obtained expressions into (A.1),

$$
\begin{aligned}
h(L)x_{it} &= \varphi_u G_1(L)x_{it} + \beta_u G_4(L)x_{it} + \gamma_u G_3(L)x_{it} + \alpha_u G_2(L)x_{it} \\
&= \varphi_u \left(1-\frac{\lambda_u}{\rho}\right) \frac{1}{1-\lambda_u L} x_{it} + \beta_u \left\{ \frac{h(L)}{L-\lambda_u} \left[\left(1-\frac{\lambda_u}{\rho}\right) \frac{1-\rho\lambda_u}{1-\rho L} + \frac{\lambda_u(L-\rho)}{\rho L} \right] - \frac{h(0)}{L} \right\} \frac{1-\rho L}{1-\lambda_u L} x_{it} \\
&+ \gamma_u \frac{\lambda_u \tau_{\varepsilon u}}{\rho} \left[h(L)L - h(\lambda_u)\lambda_u \frac{1-\rho L}{1-\rho\lambda_u} \right] \frac{1}{(1-\lambda_u L)(L-\lambda_u)} x_{it} \\
&+ \alpha_u \frac{\lambda_u \tau_{\varepsilon u}}{\rho} \left[h(L) - h(\lambda_u) \frac{1-\rho L}{1-\rho\lambda_u} \right] \frac{1}{(1-\lambda_u L)(L-\lambda_u)} x_{it}
\end{aligned}
$$

Removing the x_{it} terms, and rearranging terms on the LHS and RHS

$$
\begin{aligned}
& h(z) \left\{ 1 - \beta_u \left[\left(1-\frac{\lambda_u}{\rho}\right) \frac{1-\rho\lambda_u}{1-\rho z} + \frac{\lambda_u(z-\rho)}{\rho z} \right] \frac{1-\rho z}{(1-\lambda_u z)(L-\lambda_u)} \right\} \\
& - h(z) \frac{(\rho-\lambda_u)(1-\rho\lambda_u)}{\rho} \frac{\gamma_u z + \alpha_u}{(1-\lambda_u z)(z-\lambda_u)} \\
& = \varphi_u \left(1-\frac{\lambda_u}{\rho}\right) \frac{1}{1-\lambda_u z} - \beta_u \frac{1-\rho z}{z(1-\lambda_u z)} h(0) - h(\lambda_u) \left(1-\frac{\lambda_u}{\rho}\right) \frac{\gamma_u \lambda_u + \alpha_u}{(1-\lambda_u z)(z-\lambda_u)} (1-\rho z)
\end{aligned}
$$

Multiplying both sides by $z(z-\lambda_u)(1-\lambda_u z)$,

$$
h(z) \left[z(z-\lambda_u)(1-\lambda_u z) - \beta_u (z-\lambda_u)(1-\lambda_u z) \right] - h(z) \frac{(\rho-\lambda_u)(1-\rho\lambda_u)}{\rho} z(\gamma_u z + \alpha_u)
$$

$$= \varphi_u \left(1 - \frac{\lambda_u}{\rho}\right) z(z - \lambda_u) - \beta_u(1 - \rho z)(z - \lambda_u)h(0) - h(\lambda_u)\left(1 - \frac{\lambda_u}{\rho}\right)(\gamma_u \lambda_u + \alpha_u) z(1 - \rho z)$$

I can write the above system of equations in terms of $\boldsymbol{h}(L)$ in matrix form $\boldsymbol{C}(z)\boldsymbol{h}(z) = \boldsymbol{d}(z)$ where $\boldsymbol{C}(z) = (z - \beta_u)(z - \lambda_u)(1 - \lambda_u z) - \frac{(\rho - \lambda_u)(1 - \rho\lambda_u)}{\rho} z(\gamma_u z + \alpha_u) = \lambda_u \left\{(\beta_u - z)(z - \rho)\left(z - \frac{1}{\rho}\right) - \frac{\tau_{\varepsilon u}}{\rho} z[\alpha_u + \beta_u - (1 - \gamma_u)z]\right\} = \lambda_u \mathcal{P}(z)$. I can also write $\boldsymbol{d}(z) = \varphi_u \left(1 - \frac{\lambda_u}{\rho}\right) z(z - \lambda_u) - \beta_u(1 - \rho z)(z - \lambda_u)h(0) - h(\lambda_u)\left(1 - \frac{\lambda_u}{\rho}\right)(\gamma_u\lambda_u + \alpha_u)z(1 - \rho z)$. Note that $\boldsymbol{C}(z)$ is a polynomial of degree 3 on z. Denote the inside roots of det $\boldsymbol{C}(z)$ as $\{\zeta_1, \zeta_2\}$, and the outside root as $\{\vartheta^{-1}\}$. Because agents cannot use future signals, the inside roots have to be removed. Note that the number of free constants in $\boldsymbol{d}(z)$ is 2: $\{h(0)\}$ and $\left\{\widetilde{h}(\lambda_u) = h(\lambda_u)\left(1 - \frac{\lambda_u}{\rho}\right)(\gamma_u\lambda_u + \alpha_u)\right\}$. With a unique solution, it has to be the case that the number of outside roots is 2. By choosing the appropriate constants, the 2 inside roots will be removed. Therefore, the 2 constants are solutions to $d_1(\zeta_n) = 0$ for $\{\zeta_n\}_{n=1}^2$. Using $C(z) = -\lambda_u(z - \zeta_1)(z - \zeta_2)(z - \vartheta^{-1})$, the Vieta properties to eliminate the inside roots, and $C(\vartheta^{-1}) = 0$, I obtain $a_t = h(L)\xi_t = \left(1 - \frac{\vartheta}{\rho}\right) \frac{\varphi_u}{1 - \gamma_u - \rho(\alpha_u + \beta_u)} \frac{1}{1 - \vartheta L} \xi_t$. □

Proof of Corollary 1. I begin by showing that $\mathcal{P}(z)$ has two inside roots and one outside root: $\mathcal{P}(0) = \beta s > 0$, $\mathcal{P}(\beta s) = -\frac{\sigma_\varepsilon^2}{\sigma_u^2 \rho}\beta^2 s(\delta - \beta s) < 0$, and $\mathcal{P}(1) = (1 - \beta s)\frac{(1-\rho)^2}{\rho} - \frac{\sigma_\varepsilon^2}{\sigma_u^2\rho}\beta(\delta - 1) > 0$, where the second condition is satisfied when $\lambda < (1 + \varphi\tau_D)/(1 + \varphi)$ and $\delta > \beta s$, and the third condition is satisfied when there is a sufficient level of information frictions. □

Proof of Corollary 2. Note that $\mathcal{P}(\rho^{-1}) = \frac{\sigma_\varepsilon^2}{\rho^3\sigma_u^2}\beta(1 - \rho\delta) > 0$ and $\mathcal{P}(\lambda_u^{-1}) = -\frac{(\rho - \lambda_u)(1 - \rho\lambda_u)}{\rho\lambda_u^2}\left[1 - \beta + \beta(\delta - s)\lambda_u\right] < 0$, where both expressions are satisfied under $\lambda < (1 + \varphi\tau_D)/(1 + \varphi)$, $\delta < 1/\rho$ and $\beta < \delta$. By continuity of $\mathcal{P}(z)$, there exists a root between λ_u and ρ, such that $\lambda_u < \vartheta < \rho$. It also implies that $\mathcal{P}(z)$ is decreasing in the neighborhood of $z = \vartheta^{-1}$, a property that I use in the sequel to characterize comparative statics of ϑ. Taking the derivative, $\frac{\partial\mathcal{P}(z)}{\partial\lambda} = -\frac{\lambda_u\tau_{\varepsilon u}}{\rho} z \frac{\partial\alpha_u}{\partial\lambda}$. Evaluated at $z = \vartheta^{-1}$, $\frac{\partial\mathcal{P}(\vartheta^{-1})}{\partial\lambda} = -\frac{\lambda_u\tau_{\varepsilon u}}{\rho\vartheta}\frac{\partial\alpha_u}{\partial\lambda} = -\frac{\lambda_u\tau_{\varepsilon u}}{\rho\vartheta}\frac{\beta}{1 - \lambda\chi}\left[\frac{\varphi(1-s)\tau^D}{\lambda^2} + (\delta - 1)(1 + \varphi)\right] < 0$, where $\frac{\partial\alpha_u}{\partial\lambda} = \frac{\beta}{1 - \lambda\chi}\left[\frac{\varphi(1-s)\tau^D}{\lambda^2} + (\delta - 1)(1 + \varphi)\right] > 0$. Knowing $\mathcal{P}(\vartheta^{-1}) = 0$, making use of the Implicit Function Theorem, $\frac{\partial\mathcal{P}(\vartheta^{-1})}{\partial\vartheta}\frac{\partial\vartheta}{\partial\lambda} + \frac{\partial\mathcal{P}(\vartheta^{-1})}{\partial\lambda} = 0$, which implies $\frac{\partial\vartheta}{\partial\lambda} = -\frac{\frac{\partial\mathcal{P}(\vartheta^{-1})}{\partial\lambda}}{\frac{\partial\mathcal{P}(\vartheta^{-1})}{\partial\vartheta}}$. Combining this last result with the earlier observation that $\frac{\partial\mathcal{P}(\vartheta^{-1})}{\partial z} < 0$ in the neighborhood of $z = \vartheta^{-1}$, I infer that ϑ is an increasing function of λ.

Similarly, I can write $\frac{\partial\mathcal{P}(z)}{\partial s} = (z - \rho)(z - \rho^{-1})\frac{\partial\beta_u}{\partial s} - \frac{\lambda_u\tau_{\varepsilon u}}{\rho} z\frac{\partial\alpha_u}{\partial s}$. Evaluated at $z = \vartheta^{-1}$, $\frac{\partial\mathcal{P}(\vartheta^{-1})}{\partial s} = \frac{(1 - \rho\vartheta)(\rho - \vartheta)}{\rho\vartheta^2}\frac{\partial\beta_u}{\partial s} - \frac{\lambda_u\tau_{\varepsilon u}}{\rho\vartheta}\frac{\partial\alpha_u}{\partial s} = \frac{(1 - \rho\vartheta)(\rho - \vartheta)}{\rho\vartheta^2}\beta + \frac{\lambda_u\tau_{\varepsilon u}}{\rho\vartheta}\frac{\beta\chi(1 - \lambda)}{1 - \lambda\chi} > 0$, where $\frac{\partial\beta_u}{\partial s} = \beta$ and $\frac{\partial\alpha_u}{\partial s} = -\frac{\beta\chi(1-\lambda)}{1 - \lambda\chi}$. Combining the Implicit Function Theorem with the earlier observation that $\frac{\partial \mathcal{P}(\vartheta^{-1})}{\partial z} < 0$ in the neighborhood of $z = \vartheta^{-1}$, I infer that ϑ is a decreasing function of s. $\square$

Proof of Corollary 3. Output dynamics (17) can be written as $y_t = \psi\left(1 - \frac{\vartheta}{\rho}\right)\frac{\sigma_\varepsilon}{(1-\vartheta L)(1-\rho L)}\varepsilon_t = \frac{\psi\sigma_\varepsilon}{\rho}\sum_{k=0}^{\infty}(\rho^{k+1} - \vartheta^{k+1})\varepsilon_{t-k}$, with $\psi = -[\nu(1-\rho\delta)]^{-1}$. Denote the variance of output in the FIRE economy as $\mathbb{V}(y_{t,\text{FIRE}}) = \frac{(\psi\sigma_\varepsilon)^2}{1-\rho^2}$. As long as $\chi > 1$, I know that the share of HtM agents amplifies the effect of real interest rates ($\partial\psi^2/\partial\lambda > 0$). The variance of output in the beyond FIRE economy is given by $\mathbb{V}(y_t) = \mathbb{V}(y_{t,\text{FIRE}}) - \left(\frac{\psi\sigma_\varepsilon}{\rho}\right)^2\frac{\vartheta[\rho-\vartheta+\rho(1-\vartheta^2)]}{(1-\vartheta^2)(1-\rho\vartheta)}$. Taking the derivative with respect to the share of HtM agents, $\partial\mathbb{V}(y_t)/\partial\lambda = \partial\mathbb{V}(y_{t,\text{FIRE}})/\partial\lambda - \partial\mathbb{V}(y_{t,\text{FIRE}})/\partial\lambda\frac{(1-\rho^2)\vartheta[\rho+\vartheta+\rho(1-\vartheta^2)]}{\rho^2(1-\vartheta^2)(1-\rho\vartheta)} - \mathbb{V}(y_{t,\text{FIRE}})\frac{(1-\rho^2)2(\rho-\vartheta)(1-\rho\vartheta^3)}{\rho^2(1-\vartheta^2)^2(1-\rho\vartheta)^2}\frac{\partial\vartheta}{\partial\lambda} < \partial\mathbb{V}(y_{t,\text{FIRE}})/\partial\lambda$.

As long as $1-\rho\delta > 1$, I know that the persistence of the income process agents reduces the effect of real interest rates ($\partial\psi^2/\partial s < 0$). Taking the derivative of the variance with respect to the persistence of the income process, $\partial\mathbb{V}(y_t)/\partial s = \partial\mathbb{V}(y_{t,\text{FIRE}})/\partial s - \partial\mathbb{V}(y_{t,\text{FIRE}})/\partial s\frac{(1-\rho^2)\vartheta[\rho+\vartheta+\rho(1-\vartheta^2)]}{\rho^2(1-\vartheta^2)(1-\rho\vartheta)} - \mathbb{V}(y_{t,\text{FIRE}})\frac{(1-\rho^2)2(\rho-\vartheta)(1-\rho\vartheta^3)}{\rho^2(1-\vartheta^2)^2(1-\rho\vartheta)^2}\frac{\partial\vartheta}{\partial s} > \partial\mathbb{V}(y_{t,\text{FIRE}})/\partial s$

The first-order autocorrelation of output is given by $\mathbb{C}\text{orr}(y_t, y_{t-1}) = \frac{\frac{\rho^3}{1-\rho^2} - \frac{\rho\vartheta(\rho+\vartheta)}{1-\rho\vartheta} + \frac{\vartheta^3}{1-\vartheta^2}}{\frac{\rho^2}{1-\rho^2} - \frac{2\rho\vartheta}{1-\rho\vartheta} + \frac{\vartheta^2}{1-\vartheta^2}}$. Taking the derivative of the first-order autocorrelation with respect to the share of HtM agents, $\partial\mathbb{C}\text{orr}(y_t, y_{t-1})/\partial\lambda = \frac{1-\rho^2}{(1+\rho\vartheta)^2}\frac{\partial\vartheta}{\partial\lambda} > 0$. Taking the derivative of the first-order autocorrelation with respect to the share of HtM agents, $\partial\mathbb{C}\text{orr}(y_t, y_{t-1})/\partial\lambda = \frac{1-\rho^2}{(1+\rho\vartheta)^2}\frac{\partial\vartheta}{\partial s} < 0$ $\square$

Proof of Corollary 4. Here I prove that second-order beliefs are more persistent that first-order beliefs. The extension to higher-order beliefs is a direct application of the proof. The first-order forecast of the exogenous variable is given by

$$\overline{\mathbb{E}}_t^c r_{t+k+1} = \left[\rho^{k+1}\frac{1}{1-\rho L} - \rho^k\lambda_u\frac{1}{1-\lambda_u L}\right]\varepsilon_t = \rho^k(\rho-\lambda)\frac{1}{1-\lambda L}r_t$$

The second-order forecast of the exogenous variable is given by

$$\begin{aligned}
\overline{\mathbb{E}}_{t+1}^c r_{t+k+1} &= \rho^{k-1}(\rho-\lambda_u)\frac{1}{1-\lambda_u L}r_{t+1} \\
\overline{\mathbb{E}}_t^c\left[\overline{\mathbb{E}}_{t+1}^c r_{t+k+1}\right] &= \rho^{k-1}(\rho-\lambda_u)\overline{\mathbb{E}}_t\left[\frac{1}{1-\lambda_u L}r_{t+1}\right] \\
&= \rho^{k-1}(\rho-\lambda_u)\left[\begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{(1-\lambda_u L)(1-\rho L)} & 0\end{bmatrix}\begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L^{-1}} \\ \tau_u^{-\frac{1}{2}}\end{bmatrix}\frac{1-\rho L^{-1}}{1-\lambda_u L^{-1}}\right]_+\frac{\lambda_u\tau_u}{\rho}\frac{1}{1-\lambda_u L}\varepsilon_t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-\lambda_u L}\left[\frac{\rho^{k-1}(\rho-\lambda_u)[\lambda_u(1-\rho\lambda_u)+\rho]}{1-\lambda_u^2} - \frac{\rho^{k-2}(\rho-\lambda_u)\lambda_u^2(1-\rho\lambda_u)}{1-\lambda_u^2}\frac{1-\rho L}{1-\lambda_u L}\right] r_t \\
&= \frac{\rho^k(\rho-\lambda_u)}{(1-\lambda_u^2)}\frac{1}{1-\lambda_u L} r_t + \frac{\rho^{k-2}(\rho-\lambda_u)^2\lambda_u(1-\rho\lambda_u)}{(1-\lambda_u^2)}\frac{1}{(1-\lambda_u L)^2} r_t \\
&= \frac{1}{1-\lambda_u^2}\overline{\mathbb{E}}_t^c r_{t+k+1} + \frac{(\rho-\lambda_u)\lambda_u(1-\rho\lambda_u)}{\rho^2(1-\lambda_u^2)}\frac{1}{1-\lambda_u L}\overline{\mathbb{E}}_t^c r_{t+k+1}
\end{aligned}
$$

That is, second-order beliefs are more persistent than first-order beliefs (second term). The first-order forecast of the endogenous variable is given by

$$
\begin{aligned}
\overline{\mathbb{E}}_t^c y_{t+k+2} &= \vartheta^{k+2}\overline{\mathbb{E}}_t^c y_t + \psi\left[\vartheta^{k+1}\rho + \vartheta^k\rho^2 + \ldots + \vartheta\rho^{k+1} + \rho^{k+2}\right]\overline{\mathbb{E}}_t^c r_t = \vartheta^{k+2}\overline{\mathbb{E}}_t^c y_t + \rho\psi\frac{\rho^{k+2}-\vartheta^{k+2}}{\rho-\vartheta}\overline{\mathbb{E}}_t^c r_t \\
&= \vartheta^{k+2}\frac{(\rho-\lambda)\psi}{\rho(1-\lambda\vartheta)}\frac{1-\rho\lambda\vartheta L}{(1-\lambda L)(1-\vartheta L)} r_t + \frac{\psi(\rho-\lambda)(\rho^{k+2}-\vartheta^{k+2})}{\rho-\vartheta}\frac{1}{1-\lambda L} r_t \\
&= \frac{\rho^{k+3}\psi}{\rho-\vartheta} r_t - \psi\lambda\left[\frac{\vartheta^{k+2}\lambda(1-\rho\vartheta)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)} + \frac{(\rho^{k+2}-\vartheta^{k+2})}{\rho-\vartheta}\right]\frac{1-\rho L}{1-\lambda L} r_t \\
&+ \vartheta^{k+3}\psi\left[\frac{\lambda(1-\rho\vartheta)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)} - \frac{1}{\rho-\vartheta}\right]\frac{1-\rho L}{1-\vartheta L} r_t \\
&= \psi\,\frac{\rho-\lambda}{\rho-\vartheta}\left[\rho^{k+2} + \frac{(\rho-\lambda)(1-\rho\lambda)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)}\vartheta^{k+3}\right]\frac{1}{1-\lambda L} r_t - \psi\,\frac{\rho-\lambda}{\rho}\,\frac{\vartheta^{k+3}(1-\rho\lambda)}{(1-\lambda\vartheta)(\lambda-\vartheta)}\frac{1}{1-\vartheta L} r_t \\
&= \psi\,\frac{\rho-\lambda}{\rho-\vartheta}\left[\rho^{k+2} + \frac{(\rho-\lambda)(1-\rho\lambda)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)}\vartheta^{k+3}\right]\frac{1}{1-\lambda L} r_t - \psi\,\frac{\rho-\lambda}{\rho}\,\frac{\vartheta^{k+3}(1-\rho\lambda)}{(1-\lambda\vartheta)(\lambda-\vartheta)}\frac{1-\lambda L}{1-\vartheta L}\frac{1}{1-\lambda L} r_t \\
&= \psi\left[\frac{\rho-\lambda}{\rho-\vartheta}\,\rho^{k+2} - \frac{\vartheta^{k+2}(1-\rho\lambda)}{\rho\,(\rho-\vartheta)(1-\lambda\vartheta)}\right]\frac{1}{1-\lambda L} r_t + \psi\,\frac{\rho-\lambda}{\rho}\,\frac{\vartheta^{k+2}(1-\rho\lambda)}{1-\lambda\vartheta}\,\frac{1}{1-\vartheta L}\frac{1}{1-\lambda L} r_t \\
&= \frac{\psi}{\rho-\vartheta}\left[\rho^2 - \frac{\vartheta^{k+2}(1-\rho\lambda)}{\rho^{k+1}(\rho-\lambda)(1-\lambda\vartheta)}\right]\overline{\mathbb{E}}_t^c r_{t+k+1} + \psi\,\frac{\vartheta^{k+2}(1-\rho\lambda)}{\rho^{k+1}(1-\lambda\vartheta)}\frac{1}{1-\vartheta L}\overline{\mathbb{E}}_t^c r_{t+k+1}
\end{aligned}
$$

The second-order forecast of the endogenous variable is given by

$$
\begin{aligned}
\overline{\mathbb{E}}_{t+1}^c y_{t+k+2} &= \vartheta^{k+1}\overline{\mathbb{E}}_{t+1}^c y_{t+1} + \rho\psi\frac{\rho^{k+1}-\vartheta^{k+1}}{\rho-\vartheta}\overline{\mathbb{E}}_{t+1}^c r_{t+1} \\
&= \vartheta^{k+1}\frac{(\rho-\lambda)\psi}{\rho(1-\lambda\vartheta)}\frac{1-\rho\lambda\vartheta L}{(1-\lambda L)(1-\vartheta L)} r_{t+1} + \frac{\psi(\rho-\lambda)(\rho^{k+1}-\vartheta^{k+1})}{\rho-\vartheta}\frac{1}{1-\lambda L} r_{t+1} \\
&= \frac{\rho^{k+2}\psi}{\rho-\vartheta}\frac{1}{1-\rho L}\varepsilon_{t+1} - \psi\lambda\left[\frac{\vartheta^{k+1}\lambda(1-\rho\vartheta)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)} + \frac{(\rho^{k+1}-\vartheta^{k+1})}{\rho-\vartheta}\right]\frac{1}{1-\lambda L}\varepsilon_{t+1} \\
&+ \vartheta^{k+2}\psi\left[\frac{\lambda(1-\rho\vartheta)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)} - \frac{1}{\rho-\vartheta}\right]\frac{1}{1-\vartheta L}\varepsilon_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
\overline{\mathbb{E}}_t^c\left[\overline{\mathbb{E}}_{t+1}^c y_{t+k+2}\right] &= \frac{\rho^{k+2}\psi}{\rho-\vartheta}\overline{\mathbb{E}}_t^c\left[\frac{1}{1-\rho L}\varepsilon_{t+1}\right] - \psi\lambda\left[\frac{\vartheta^{k+1}\lambda(1-\rho\vartheta)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)}+\frac{(\rho^{k+1}-\vartheta^{k+1})}{\rho-\vartheta}\right]\overline{\mathbb{E}}_t^c\left[\frac{1}{1-\lambda L}\varepsilon_{t+1}\right] \\
&+\vartheta^{k+2}\psi\left[\frac{\lambda(1-\rho\vartheta)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)}-\frac{1}{\rho-\vartheta}\right]\overline{\mathbb{E}}_t^c\left[\frac{1}{1-\vartheta L}\varepsilon_{t+1}\right] \\
&= \left\{\frac{\rho^{k+2}\psi(\rho-\lambda)}{\rho-\vartheta} - \frac{\psi\lambda^2(\rho-\lambda)(1-\rho\lambda)}{\rho(1-\lambda^2)}\left[\frac{\vartheta^{k+1}\lambda(1-\rho\vartheta)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)}+\frac{\rho^{k+1}-\vartheta^{k+1}}{\rho-\vartheta}\right]\frac{1-\rho L}{1-\lambda L}\right. \\
&\left.+\frac{\vartheta^{k+3}\psi(\rho-\lambda)(1-\rho\lambda)}{\rho(1-\lambda\vartheta)}\left[\frac{\lambda(1-\rho\vartheta)}{\rho(1-\lambda\vartheta)(\lambda-\vartheta)}-\frac{1}{\rho-\vartheta}\right]\frac{1-\rho L}{1-\vartheta L}\right\}\frac{1}{1-\lambda L}r_t \\
&= \frac{1}{1-\lambda L}\left\{\left[\psi\,\frac{\rho-\lambda}{\rho-\vartheta}\left(\rho-\frac{\lambda(1-\rho\lambda)}{1-\lambda^2}\right)\rho^{k+1}\right.\right. \\
&\left.+\psi\,(\rho-\lambda)(1-\rho\lambda)\frac{(\lambda\rho-\vartheta)(1-\lambda\vartheta)-\lambda(1-\rho\vartheta)(\rho-\vartheta)}{\rho(1-\lambda^2)(1-\lambda\vartheta)^2(\rho-\vartheta)}\,\vartheta^{k+1}\right]r_t \\
&+\left[\frac{\psi\,\lambda(\rho-\lambda)^2(1-\rho\lambda)}{\rho(1-\lambda^2)(\rho-\vartheta)}\,\rho^{k+1}+\frac{\psi\,\lambda\,\vartheta\,(\rho-\lambda)^3(1-\rho\lambda)^2}{\rho^2(1-\lambda^2)(1-\lambda\vartheta)(\lambda-\vartheta)(\rho-\vartheta)}\,\vartheta^{k+1}\right]\frac{1}{1-\lambda L}r_t \\
&-\frac{\psi\,\vartheta^{k+3}(\rho-\lambda)^2(1-\rho\lambda)^2}{\rho^2(1-\lambda\vartheta)^2(\lambda-\vartheta)}\frac{1}{1-\vartheta L}r_t \\
&= \frac{1}{1-\lambda L}\left\{\left[\psi\,\frac{\rho-\lambda}{\rho-\vartheta}\left(\rho-\frac{\lambda(1-\rho\lambda)}{1-\lambda^2}\right)\rho^{k+1}\right.\right. \\
&\left.+\psi\,(\rho-\lambda)(1-\rho\lambda)\frac{(\lambda\rho-\vartheta)(1-\lambda\vartheta)-\lambda(1-\rho\vartheta)(\rho-\vartheta)}{\rho(1-\lambda^2)(1-\lambda\vartheta)^2(\rho-\vartheta)}\,\vartheta^{k+1}\right]r_t \\
&+\left[\frac{\psi\,\lambda(\rho-\lambda)^2(1-\rho\lambda)}{\rho(1-\lambda^2)(\rho-\vartheta)}\,\rho^{k+1}+\frac{\psi\,\lambda\,\vartheta\,(\rho-\lambda)^3(1-\rho\lambda)^2}{\rho^2(1-\lambda^2)(1-\lambda\vartheta)(\lambda-\vartheta)(\rho-\vartheta)}\,\vartheta^{k+1}\right]\frac{1}{1-\lambda L}r_t \\
&\left.-\frac{\psi\,\vartheta^{k+3}(\rho-\lambda)^2(1-\rho\lambda)^2}{\rho^2(1-\lambda\vartheta)^2(\lambda-\vartheta)}\frac{1-\lambda L}{1-\vartheta L}\frac{1}{1-\lambda L}r_t\right\} \\
&= \frac{1}{1-\lambda L}\left\{\left[\frac{\psi(\rho-\lambda)^2}{(1-\lambda^2)(\rho-\vartheta)}\,\rho^{k+1}\right.\right. \\
&\left.+\frac{\psi(\rho-\lambda)(1-\rho\lambda)\left[(\lambda\rho-\vartheta)(1-\lambda\vartheta)-\lambda(1-\rho\vartheta)(\rho-\vartheta)\right]}{\rho(1-\lambda^2)(1-\lambda\vartheta)^2(\rho-\vartheta)}\vartheta^{k+1}\right]r_t \\
&+\left[\frac{\psi\,\lambda(\rho-\lambda)^2(1-\rho\lambda)}{\rho(1-\lambda^2)(\rho-\vartheta)}\,\rho^{k+1}\right. \\
&\left.+\frac{\psi\,\lambda(\rho-\lambda)^2(1-\rho\lambda)^2\left[(\rho-\lambda)(1-\lambda\vartheta)-(1-\lambda^2)(\rho-\vartheta)\right]}{\rho^2(\lambda-\vartheta)(1-\lambda^2)(1-\lambda\vartheta)^2(\rho-\vartheta)}\vartheta^{k+2}\right]\frac{1}{1-\lambda L}r_t
\end{aligned}
$$

$$+\frac{\psi\,\vartheta^{k+2}(\rho-\lambda)^2(1-\rho\lambda)^2}{\rho^2(1-\lambda\vartheta)^2}\frac{1}{1-\vartheta L}\frac{1}{1-\lambda L}r_t\Bigg\}$$

$$=\left[\frac{\psi(\rho-\lambda)\rho}{(1-\lambda^2)(\rho-\vartheta)}+\frac{\psi(1-\rho\lambda)\left[(\lambda\rho-\vartheta)(1-\lambda\vartheta)-\lambda(1-\rho\vartheta)(\rho-\vartheta)\right]}{\rho^{k+1}(1-\lambda^2)(1-\lambda\vartheta)^2(\rho-\vartheta)}\vartheta^{k+1}\right]\overline{\mathbb{E}}^c_t r_{t+k+1}$$

$$+\frac{\psi\,\lambda(\rho-\lambda)(1-\rho\lambda)}{(1-\lambda^2)(\rho-\vartheta)}\left[1+\frac{(1-\rho\lambda)\left[(\rho-\lambda)(1-\lambda\vartheta)-(1-\lambda^2)(\rho-\vartheta)\right]}{\rho^{k+2}(\lambda-\vartheta)(1-\lambda\vartheta)^2}\vartheta^{k+2}\right]\frac{1}{1-\lambda L}\overline{\mathbb{E}}^c_t r_{t+k+1}$$

$$+\frac{\psi\,\vartheta^{k+2}(\rho-\lambda)(1-\rho\lambda)^2}{\rho^{k+2}(1-\lambda\vartheta)^2}\frac{1}{1-\vartheta L}\overline{\mathbb{E}}^c_t r_{t+k+1}$$

For a given variable, second-order beliefs are more persistent than first-order beliefs. Second, for a given hierarchy in the order of beliefs, beliefs on endogenous variables are more persistent than beliefs on exogenous variables. □

Proof of Proposition 3. The aggregate outcome is

$$y_t=-\frac{1}{\nu(1-\rho\delta)}\left(1-\frac{\vartheta}{\rho}\right)\frac{1}{(1-\vartheta L)(1-\rho L)}\varepsilon_t=-\frac{1}{\rho\nu(1-\rho\delta)}\sum_{k=0}^{\infty}(\rho^{k+1}-\vartheta^{k+1})\varepsilon_{t-k}$$

The direct component is given by

$$\text{direct}_t=-\frac{\beta(1-\lambda)}{\sigma}\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t r_{t+k}=-\frac{\beta(1-\lambda)}{\sigma}\sum_{k=0}^{\infty}(\beta s\rho)^k\overline{\mathbb{E}}_t r_t=-\frac{\beta(1-\lambda)}{\sigma(1-\rho s\beta)}\overline{\mathbb{E}}_t r_t$$

$$=-\frac{\beta(1-\lambda)}{\sigma(1-\rho s\beta)}\left(1-\frac{\lambda_u}{\rho}\right)\frac{1}{(1-\lambda_u L)(1-\rho L)}\varepsilon_t=-\frac{\beta(1-\lambda)}{\rho\sigma(1-\rho s\beta)}\sum_{k=0}^{\infty}(\rho^{k+1}-\lambda_u^{k+1})\varepsilon_{t-k}$$

Therefore, the direct share μ_τ is given by $\alpha_\tau=\frac{\partial\text{direct}_\tau/\partial\varepsilon_t}{\partial\text{TE}_\tau/\partial\varepsilon_t}=\frac{\beta(1-\lambda\chi)(1-\rho\delta)}{1-\rho s\beta}\frac{\rho^{\tau+1}-\lambda_u^{\tau+1}}{\rho^{\tau+1}-\vartheta^{\tau+1}}$ □

Proof of Corollary 5. The derivative of the long-term direct share with respect to λ is given by $\partial\mu/\partial\lambda=-\beta\chi(1-\rho)/(1-\rho\beta s)<0$. Using this result, I can compute

$$\frac{\partial\mu_t}{\partial\lambda}=\frac{\partial\mu}{\partial\lambda}\left(1-\frac{\lambda_u^{t+1}-\vartheta^{t+1}}{\rho^{t+1}-\vartheta^{t+1}}\right)+\frac{(1+t)\vartheta^t(\rho^{t+1}-\lambda_u^{t+1})}{(\rho^{t+1}-\vartheta^{t+1})^2}\frac{\partial\vartheta}{\partial\lambda}>\frac{\partial\mu}{\partial\lambda}$$

where the first term in parenthesis on the right-hand side is inside the unit interval since $\rho>\lambda_u>\vartheta$. The derivative of the long-term direct share with respect to s is given by

$\partial\mu/\partial s = \rho\beta[(1-\beta)(\chi-1)+\beta\chi(1-\lambda)(1-\rho)]/(1-\rho\beta s)^2 > 0$. Using this result, I can compute

$$\frac{\partial\mu_t}{\partial s} = \frac{\partial\mu}{\partial s}\left(1-\frac{\lambda_u^{t+1}-\vartheta^{t+1}}{\rho^{t+1}-\vartheta^{t+1}}\right) + \frac{(1+t)\vartheta^t(\rho^{t+1}-\lambda_u^{t+1})}{(\rho^{t+1}-\vartheta^{t+1})^2}\frac{\partial\vartheta}{\partial s} < \frac{\partial\mu}{\partial s}$$

□

Proof of Proposition 4. (*i*) Equilibrium dynamics satisfy (24). Equilibrium determinacy boils down to having all the eigenvalues in the matrix $\overline{\delta}^{-1}$ outside the unit circle. This restriction is satisfied if

$$\det\overline{\delta}^{-1} > 1 \tag{A.22}$$

$$\det\overline{\delta}^{-1} - \operatorname{tr}\overline{\delta}^{-1} > -1 \tag{A.23}$$

$$\det\overline{\delta}^{-1} + \operatorname{tr}\overline{\delta}^{-1} > -1 \tag{A.24}$$

Introducing the respective values in (A.22)-(A.24), condition (A.24) is irrelevant for positive values of the coefficients of the monetary rule. Part (*ii*) is proved in proof of proposition 1. □

Proof of Proposition 5. I first prove (*i*). Guess an ad hoc system of dynamics, such that

$$\boldsymbol{x}_t = \omega_b\boldsymbol{x}_{t-1} + \overline{\delta}\omega_f\mathbb{E}_t\boldsymbol{x}_{t+1} + \overline{\varphi}v_t \tag{A.25}$$

for some arbitrary 2×2 matrices (ω_b, ω_f). To show that the ad hoc model presented above captures the HANK beyond FIRE under certain (ω_f, ω_b), I rely on the Method for Undetermined Coefficients. Both dynamics are observationally equivalent if

$$\begin{aligned}
A\boldsymbol{x}_{t-1} + Bv_t &= \overline{\varphi}v_t + \overline{\delta}\omega_f\mathbb{E}_t\boldsymbol{x}_{t+1} + \omega_b\boldsymbol{x}_{t-1} = \overline{\varphi}v_t + \overline{\delta}\omega_f\mathbb{E}_t(A\boldsymbol{x}_t + Bv_{t+1}) + \omega_b\boldsymbol{x}_{t-1}\\
&= \overline{\varphi}v_t + \overline{\delta}\omega_f(A\boldsymbol{x}_t + B\mathbb{E}_tv_{t+1}) + \omega_b\boldsymbol{x}_{t-1} = \overline{\varphi}v_t + \overline{\delta}\omega_f(A\boldsymbol{x}_t + B\rho v_t) + \omega_b\boldsymbol{x}_{t-1}\\
&= \overline{\varphi}v_t + \overline{\delta}\omega_f[A(A\boldsymbol{x}_{t-1} + Bv_t) + B\rho v_t] + \omega_b\boldsymbol{x}_{t-1}\\
&= \left[\overline{\delta}\omega_fAA + \omega_b\right]\boldsymbol{x}_{t-1} + \left[\overline{\varphi} + \overline{\delta}\omega_f(A+\rho)B\right]v_t
\end{aligned}$$

They are thus equivalent when (25) is satisfied. Now that I have the system dynamics (A.25), I just need to multiply the system by $\widetilde{A}$ to back out the DIS curve, which I can write as (26). I now move to (*ii*). Using the lag operator, I can factorize (26)

$$\begin{aligned}\mathbb{E}_t\left[\left(\frac{1}{\nu}+\omega_{f\pi}\right)r_t-\omega_{b\pi}\pi_{t-1}\right]&=\mathbb{E}_t\left[\left(\omega_{fy}L^{-2}-L^{-1}+\omega_{by}\right)y_{t-1}\right]\\&=\mathbb{E}_t\left[\omega_{fy}\left(L^{-1}-\gamma_1^{-1}\right)\left(L^{-1}-\gamma_2^{-1}\right)y_{t-1}\right]\end{aligned}$$

where γ_1^{-1} and γ_2^{-1} are the roots of the polynomial $\mathcal{P}(x)\equiv\omega_{fy}x^2-x+\omega_{by}$. Dividing both sides by $(L^{-1}-\gamma_2^{-1})$

$$\begin{aligned}\omega_{fy}\mathbb{E}_t[(L^{-1}-\gamma_1^{-1})y_{t-1}]&=\mathbb{E}_t\left[\left(\frac{1}{\nu}+\omega_{f\pi}\right)\frac{1}{L^{-1}-\gamma_2^{-1}}r_t-\omega_{b\pi}\frac{1}{L^{-1}-\gamma_2^{-1}}\pi_{t-1}\right]\\&=\mathbb{E}_t\left[-\left(\frac{1}{\nu}+\omega_{f\pi}\right)\frac{\gamma_2}{1-\gamma_2L^{-1}}r_t+\omega_{b\pi}\frac{\gamma_2}{1-\gamma_2L^{-1}}\pi_{t-1}\right]\end{aligned}$$

Hence, I can write the dynamics as

$$\begin{aligned}y_t&=\gamma_1^{-1}y_{t-1}+\frac{\gamma_2\omega_{b\pi}}{\omega_{fy}}(\pi_{t-1}+\gamma_2\pi_t)-\frac{\gamma_2}{\omega_{fy}}\left(\frac{1}{\nu}+\omega_{f\pi}+\omega_{b\pi}\gamma_2^2\right)\sum_{k=0}^{\infty}\gamma_2^k\mathbb{E}_tr_{t+k}\\&=\gamma_1^{-1}y_{t-1}+\frac{\omega_{b\pi}}{\gamma_1\omega_{by}}\left(\pi_{t-1}+\frac{\omega_{fy}}{\gamma_1\omega_{by}}\pi_t\right)-\frac{1}{\gamma_1\omega_{by}}\left(\frac{1}{\nu}+\omega_{f\pi}+\omega_{b\pi}\frac{\omega_{fy}^2}{\gamma_1^2\omega_{by}^2}\right)\sum_{k=0}^{\infty}\left(\frac{\omega_{fy}}{\gamma_1\omega_{by}}\right)^k\mathbb{E}_tr_{t+k}\end{aligned}$$

where I have applied the Vieta properties. Therefore, the effect of a forward guidance shock promised at time t in period τ is $FG_{t,t+\tau}=\frac{\partial y_t}{\partial\mathbb{E}_tr_{t+\tau}}=-\frac{1}{\gamma_1\omega_{by}}\left(\frac{1}{\nu}+\omega_{f\pi}+\omega_{b\pi}\frac{\omega_{fy}^2}{\gamma_1^2\omega_{by}^2}\right)\left(\frac{\omega_{fy}}{\gamma_1\omega_{by}}\right)^{\tau}$, which is decreasing in τ provided that $\gamma_1\in(0,1)$ is the only inside root, $\lim_{\tau\to\infty}FG_{t+\tau}=0$, and the FGP is solved. □

Proof of Proposition 6. The proof is identical to the proof of Proposition 1, modulo the replacement of σ_g for σ_ϵ. In the public information case, the individual action is given by $a_{lgt}=h_g(L)z_t=h_g(L)(v_t+\epsilon_t)$. The policy function of an agent in group g is given by $h_g(z)=\frac{\tilde{\psi}_{g1}+\tilde{\psi}_{g2}z}{(1-\vartheta_1z)(1-\vartheta_2z)}$, and hence I have $a_{gt}=h_g(L)(v_t+\epsilon_t)=\frac{\tilde{\psi}_{g1}+\tilde{\psi}_{g2}z}{(1-\vartheta_1z)(1-\vartheta_2z)}(v_t+\epsilon_t)=\psi_{g1}\left(1-\frac{\vartheta_1}{\rho}\right)\frac{1}{1-\vartheta_1L}(v_t+\epsilon_t)+\psi_{g2}\left(1-\frac{\vartheta_2}{\rho}\right)\frac{1}{1-\vartheta_2L}(v_t+\epsilon_t)=\psi_{g1}\tilde{\vartheta}_{1t}+\psi_{g2}\tilde{\vartheta}_{2t}$. I can write $\boldsymbol{a}_t=\begin{bmatrix}a_{1t}\\a_{2t}\end{bmatrix}=Q\tilde{\vartheta}_t=\begin{bmatrix}\psi_{11}&\psi_{12}\\\psi_{21}&\psi_{22}\end{bmatrix}\begin{bmatrix}\tilde{\vartheta}_{1t}\\\tilde{\vartheta}_{2t}\end{bmatrix}=\begin{bmatrix}\psi_{11}\tilde{\vartheta}_{1t}+\psi_{12}\tilde{\vartheta}_{2t}\\\psi_{21}\tilde{\vartheta}_{1t}+\psi_{22}\tilde{\vartheta}_{2t}\end{bmatrix}$. Notice that I can write $\tilde{\vartheta}_{gt}(1-\vartheta_gL)=\left(1-\frac{\vartheta_g}{\rho}\right)(v_t+\epsilon_t)\implies\tilde{\vartheta}_{gt}=\vartheta_g\tilde{\vartheta}_{g,t-1}+\left(1-\frac{\vartheta_g}{\rho}\right)(v_t+\epsilon_t)$, which I can write as a system as $\widetilde{\vartheta}_t = \Lambda\widetilde{\vartheta}_{t-1} + \Gamma(\nu_t + \epsilon_t)$, where $\Lambda = \begin{bmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 - \frac{\vartheta_1}{\rho} \\ 1 - \frac{\vartheta_2}{\rho} \end{bmatrix}$. Hence, I can write $\boldsymbol{a}_t = Q\widetilde{\vartheta}_t = Q[\Lambda\widetilde{\vartheta}_{t-1} + \Gamma(\nu_t + \epsilon_t)] = Q\Lambda\widetilde{\vartheta}_{t-1} + Q\Gamma(\nu_t + \epsilon_t) = Q\Lambda Q^{-1}\boldsymbol{a}_{t-1} + Q\Gamma(\nu_t + \epsilon_t) = A\boldsymbol{a}_{t-1} + B\nu_t + B\epsilon_t$. □

Proof of Proposition 7. The proof is identical to the proof of Proposition 1, modulo the replacement of σ_g for $\widetilde{\sigma}_g = \sqrt{\gamma^2\sigma_\epsilon^2 + (1-\gamma)^2\sigma_g^2}$. In the correlated information case, the individual action is given by $a_{lgt} = h_g(L)(\nu_t + \gamma\epsilon_t)$. The policy function of an agent in group g is given by $h_g(z) = \frac{\widetilde{\psi}_{g1} + \widetilde{\psi}_{g2} z}{(1-\vartheta_1 z)(1-\vartheta_2 z)}$, and hence I have $a_{gt} = h_g(L)(\nu_t + \gamma\epsilon_t) = \frac{\widetilde{\psi}_{g1} + \widetilde{\psi}_{g2} z}{(1-\vartheta_1 z)(1-\vartheta_2 z)}(\nu_t + \gamma\epsilon_t) = \psi_{g1}\left(1 - \frac{\vartheta_1}{\rho}\right)\frac{1}{1-\vartheta_1 L}(\nu_t + \gamma\epsilon_t) + \psi_{g2}\left(1 - \frac{\vartheta_2}{\rho}\right)\frac{1}{1-\vartheta_2 L}(\nu_t + \gamma\epsilon_t) = \psi_{g1}\widetilde{\vartheta}_{1t} + \psi_{g2}\widetilde{\vartheta}_{2t}$. I can write $\boldsymbol{a}_t = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} = Q\widetilde{\vartheta}_t = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}\begin{bmatrix} \widetilde{\vartheta}_{1t} \\ \widetilde{\vartheta}_{2t} \end{bmatrix} = \begin{bmatrix} \psi_{11}\widetilde{\vartheta}_{1t} + \psi_{12}\widetilde{\vartheta}_{2t} \\ \psi_{21}\widetilde{\vartheta}_{1t} + \psi_{22}\widetilde{\vartheta}_{2t} \end{bmatrix}$. Notice that I can write $\widetilde{\vartheta}_{gt}(1 - \vartheta_g L) = \left(1 - \frac{\vartheta_g}{\rho}\right)(\nu_t + \gamma\epsilon_t) \implies \widetilde{\vartheta}_{gt} = \vartheta_g\widetilde{\vartheta}_{g,t-1} + \left(1 - \frac{\vartheta_g}{\rho}\right)(\nu_t + \gamma\epsilon_t)$, which I can write as a system as $\widetilde{\vartheta}_t = \Lambda\widetilde{\vartheta}_{t-1} + \Gamma(\nu_t + \gamma\epsilon_t)$, where $\Lambda = \begin{bmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 - \frac{\vartheta_1}{\rho} \\ 1 - \frac{\vartheta_2}{\rho} \end{bmatrix}$. Hence, I can write $\boldsymbol{a}_t = Q\widetilde{\vartheta}_t = Q[\Lambda\widetilde{\vartheta}_{t-1} + \Gamma(\nu_t + \gamma\epsilon_t)] = Q\Lambda\widetilde{\vartheta}_{t-1} + Q\Gamma(\nu_t + \gamma\epsilon_t) = Q\Lambda Q^{-1}\boldsymbol{a}_{t-1} + Q\Gamma(\nu_t + \gamma\epsilon_t) = A\boldsymbol{a}_{t-1} + B\nu_t + B\gamma\epsilon_t$. □

Proof of Proposition 8. This proof mimics the proof of Proposition 1 and extends it to allow for a public signal. In this case the fundamental representation of the signal process as a system containing (7), (8) and (27), which admits the state-space representation $\boldsymbol{Z}_t = \boldsymbol{F}\boldsymbol{Z}_{t-1} + \boldsymbol{\Phi}\boldsymbol{s}_{lgt}$ and $\boldsymbol{X}_{gt} = \boldsymbol{H}\boldsymbol{Z}_t + \boldsymbol{\Psi}_g\boldsymbol{s}_{lgt}$, with $\boldsymbol{F} = \rho$, $\boldsymbol{\Phi} = \begin{bmatrix} 0 & 0 & \sigma_\varepsilon \end{bmatrix}$, $\boldsymbol{Z}_t = \nu_t$, $\boldsymbol{s}_{lgt} = \begin{bmatrix} \epsilon_t & u_{lgt} & \varepsilon_t^\nu \end{bmatrix}^\intercal$, $\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\boldsymbol{\Psi}_g = \begin{bmatrix} \sigma_\epsilon & 0 & 0 \\ 0 & \sigma_g & 0 \end{bmatrix}$ and $\boldsymbol{X}_{gt} = \begin{bmatrix} z_t & x_{lgt} \end{bmatrix}^\intercal$. It is convenient to rewrite the uncertainty parameters in terms of precision: define $\tau_\varepsilon \equiv \frac{1}{\sigma_\varepsilon^2}$, $\tau_g \equiv \frac{1}{\sigma_g^2}$, and $\tau_\epsilon = \frac{1}{\sigma_\epsilon^2}$. The signal system can be written as $\boldsymbol{X}_{gt} = \begin{bmatrix} \tau_\epsilon^{-\frac{1}{2}} & 0 & \frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} \\ 0 & \tau_g^{-\frac{1}{2}} & \frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} \end{bmatrix}\begin{bmatrix} \widehat{\varepsilon}_t \\ \widehat{u}_{lgt} \end{bmatrix} = \boldsymbol{M}_g(L)\boldsymbol{s}_{lgt}$, $\boldsymbol{s}_{lgt} \sim \mathcal{N}(0, I)$. Denote λ_g as the inside root of $\det[\boldsymbol{M}_g(L)\boldsymbol{M}_g'(L)]$, which is given by $\lambda_g = \frac{1}{2}\left[\frac{1}{\rho} + \rho + \frac{\tau_g + \tau_\epsilon}{\rho\tau_\varepsilon} - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{\tau_g + \tau_\epsilon}{\rho\tau_\varepsilon}\right)^2 - 4}\right]$. I can also write $V_g^{-1} = \frac{\tau_g\tau_\epsilon}{\rho\tau_\varepsilon(\tau_g + \tau_\epsilon)}\begin{bmatrix} \frac{\rho\tau_g + \lambda_g\tau_\epsilon}{\tau_g} & \lambda_g - \rho \\ \lambda_g - \rho & \frac{\lambda_g\tau_g + \rho\tau_\epsilon}{\tau_\epsilon} \end{bmatrix}$ and $B_g(L)^{-1} = \frac{1}{1-\lambda_g L}\begin{bmatrix} 1 - \frac{\lambda_g\tau_g + \rho\tau_\epsilon}{\tau_g + \tau_\epsilon}L & \frac{\tau_g(\lambda_g - \rho)}{\tau_g + \tau_\epsilon}L \\ \frac{\tau_\epsilon(\lambda_g - \rho)}{\tau_g + \tau_\epsilon}L & 1 - \frac{\rho\tau_g + \lambda_g\tau_\epsilon}{\tau_g + \tau_\epsilon}L \end{bmatrix}$.

Let us now move to the forecasting part. Denote agent i in group g policy function $a_{lgt} = h_{g1}(L)z_t + h_{g2}(L)x_{lgt}$. The aggregate outcome in group g can then be expressed as follows

$$\begin{aligned}
a_{gt} &= \int a_{lgt}\, di = \int h_{g1}(L)z_t + h_{g2}(L)x_{lgt}\, di \\
&= \int h_{g1}(L)\left(\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t + \sigma_\epsilon \epsilon_t\right) + h_{g2}(L)\left(\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t + \sigma_g u_{lgt}\right)\, di \\
&= [h_{g1}(L) + h_{g2}(L)]\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t + h_{g1}(L)\sigma_\epsilon \epsilon_t
\end{aligned}$$

Hence, the forecasts are

$$\begin{aligned}
\mathbb{E}_{lgt}\nu_t &= \frac{\lambda_g}{\rho\tau_\varepsilon(1-\lambda_g\rho)}\frac{1}{1-\lambda_g L}\begin{bmatrix}\tau_\epsilon & \tau_g\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix} \\
\mathbb{E}_{lgt}a_{kt+1} &= \begin{bmatrix}\frac{h_{k1}(L)}{L\tau_\varepsilon} + h_{k2}(L)\frac{\lambda_g\tau_\epsilon}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2} & h_{k2}(L)\frac{\lambda_g\tau_g}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix} - \\
&\quad - \frac{\lambda_g(1-\rho L)h_{k2}(\lambda_g)}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}\tau_\epsilon & \tau_g\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix} - \\
&\quad - \frac{\lambda_g h_{k1}(0)}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}\frac{(1-\lambda_g L)\tau_g + (1-\rho L)\tau_\epsilon}{L(\rho-\lambda_g)} & -\tau_g\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix} \\
\mathbb{E}_{lgt}a_{kt} &= \begin{bmatrix}\frac{h_{k1}(L)}{\tau_\varepsilon} + h_{k2}(L)\frac{L\lambda_g\tau_\epsilon}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2} & h_{k2}(L)\frac{L\lambda_g\tau_g}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix} - \\
&\quad - \frac{\lambda_g^2(1-\rho L)h_{k2}(\lambda_g)}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}\tau_\epsilon & \tau_g\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix} \\
\mathbb{E}_{lgt}\left(a_{igt+1} - a_{gt+1}\right) &= \frac{\lambda_g h_{g2}(L)}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\begin{bmatrix}-\tau_\epsilon & \frac{(L-\rho)(1-\rho L)\lambda_g\tau_g + (L-\lambda_g)(1-\lambda_g L)\rho\tau_\epsilon}{L(\rho-\lambda_g)(1-\rho\lambda_g)}\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix} - \\
&\quad - \frac{\lambda_g(1-\rho L)h_{g2}(\lambda_g)}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}-\tau_\epsilon & \tau_g\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix} - \\
&\quad - \frac{\lambda_g h_{g2}(0)}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}-\tau_\epsilon & \frac{(1-\rho L)\tau_g + (1-\lambda_g L)\tau_\epsilon}{L(\rho-\lambda_g)}\end{bmatrix}\begin{bmatrix}z_t \\ x_{lgt}\end{bmatrix}
\end{aligned}$$

Introducing the expectations just calculated into the best response (A.1), and rearranging terms,

$$
\begin{aligned}
&\Bigg[h_{g1}(L)\left(1-\frac{\beta_g}{L\tau_\varepsilon}\right)-\sum_{k=1}^{2}\frac{h_{k1}(L)}{\tau_\varepsilon}\left(\gamma_{gk}+\frac{\alpha_{gk}}{L}\right)-\sum_{k=1}^{2}\frac{h_{k2}(L)\lambda_g\tau_\epsilon}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\left(\gamma_{gk}L+\alpha_{gk}\right),\\
&\qquad h_{g2}(L)\left(1-\frac{\beta_g}{L\tau_\varepsilon}\right)-\sum_{k=1}^{2}\frac{h_{k2}(L)\lambda_g\tau_g}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\left(\gamma_{gk}L+\alpha_{gk}\right)\Bigg]\begin{bmatrix} z_t\\ x_{lgt}\end{bmatrix}=\\
&=\Bigg[\frac{\varphi_g\lambda_g\tau_\epsilon}{\rho\tau_\varepsilon(1-\rho\lambda_g)(1-\lambda_g L)}-h_{g1}(0)\frac{\beta_g\lambda_g[(1-\lambda_g L)\tau_g+(1-\rho L)\tau_\epsilon]}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2 L(\rho-\lambda_g)}+h_{g2}(0)\frac{\beta_g\lambda_g\tau_\epsilon}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2}-\\
&\qquad-\sum_{k=1}^{2}h_{k1}(0)\frac{\alpha_{gk}\lambda_g[(1-\lambda_g L)\tau_g+(1-\rho L)\tau_\epsilon]}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2 L(\rho-\lambda_g)}-\sum_{k=1}^{2}h_{k2}(\lambda_g)\frac{\lambda_g(1-\rho L)\tau_\epsilon}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}(\alpha_{gk}+\lambda_g\gamma_{gk}),\\
&\qquad\frac{\varphi_g\lambda_g\tau_g}{\rho\tau_\varepsilon(1-\rho\lambda_g)(1-\lambda_g L)}+h_{g1}(0)\frac{\beta_g\lambda_g\tau_g}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2}-h_{g2}(0)\frac{\beta_g\lambda_g[(1-\rho L)\tau_g+(1-\lambda_g L)\tau_\epsilon]}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2 L(\rho-\lambda_g)}+\\
&\qquad+\sum_{k=1}^{2}h_{k1}(0)\frac{\alpha_{gk}\lambda_g\tau_g}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2}-\sum_{k=1}^{2}h_{k2}(\lambda_g)\frac{\lambda_g(1-\rho L)\tau_g}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}(\alpha_{gk}+\lambda_g\gamma_{gk})\Bigg]\begin{bmatrix} z_t\\ x_{lgt}\end{bmatrix}
\end{aligned}
$$

I can write the above system of equations in terms of $\boldsymbol{h}(L)$ in matrix form

$$\boldsymbol{C}(L)\boldsymbol{h}(L)=\boldsymbol{d}[L;h(\lambda),h(0)] \tag{A.26}$$

where

$$
\boldsymbol{C}(L)=\begin{bmatrix} C_{11}(L) & C_{12}(L) & C_{13}(L) & C_{14}(L)\\ C_{21}(L) & C_{22}(L) & C_{23}(L) & C_{24}(L)\\ C_{31}(L) & C_{32}(L) & C_{33}(L) & C_{34}(L)\\ C_{41}(L) & C_{42}(L) & C_{43}(L) & C_{44}(L)\end{bmatrix},\quad \boldsymbol{h}(L)=\begin{bmatrix} h_{11}(L)\\ h_{12}(L)\\ h_{21}(L)\\ h_{22}(L)\end{bmatrix},\quad \boldsymbol{d}[L;h(\lambda),h(0)]=\begin{bmatrix} d_1(L)\\ d_2(L)\\ d_3(L)\\ d_4(L)\end{bmatrix}
$$

where $C_{11}(L)=1-\frac{\beta_1+\alpha_{11}}{L\tau_\varepsilon}-\frac{\gamma_{11}}{\tau_\varepsilon}$, $C_{12}(L)=-\frac{\lambda_1\tau_\epsilon(\alpha_{11}+\gamma_{11}L)}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2}$, $C_{13}(L)=-\frac{\gamma_{12}}{\tau_\varepsilon}-\frac{\alpha_{12}}{L\tau_\varepsilon}$, $C_{14}(L)=-\frac{\lambda_1\tau_\epsilon(\alpha_{12}+\gamma_{12}L)}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2}$, $C_{21}(L)=0$, $C_{22}(L)=1-\frac{\beta_1}{L\tau_\varepsilon}-\frac{\lambda_1\tau_1(\alpha_{11}+\gamma_{11}L)}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2}$, $C_{23}(L)=0$, $C_{24}(L)=-\frac{\lambda_1\tau_1(\alpha_{12}+\gamma_{12}L)}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2}$, $C_{31}(L)=-\frac{\gamma_{21}}{\tau_\varepsilon}-\frac{\alpha_{21}}{L\tau_\varepsilon}$, $C_{32}(L)=-\frac{\lambda_2\tau_\epsilon(\alpha_{21}+\gamma_{21}L)}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2}$, $C_{33}(L)=1-\frac{\beta_2+\alpha_{22}}{L\tau_\varepsilon}-\frac{\gamma_{22}}{\tau_\varepsilon}$, $C_{34}(L)=-\frac{\lambda_2\tau_\epsilon(\alpha_{22}+\gamma_{22}L)}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2}$, $C_{41}(L)=0$, $C_{42}(L)=-\frac{\lambda_2\tau_2(\alpha_{21}+\gamma_{21}L)}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2}$, $C_{43}(L)=0$, $C_{44}(L)=1-\frac{\beta_2}{L\tau_\varepsilon}-\frac{\lambda_2\tau_2(\alpha_{22}+\gamma_{22}L)}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2}$, and

$$d_1(L)=\frac{\varphi_1\lambda_1\tau_\epsilon}{\rho\tau_\varepsilon(1-\rho\lambda_1)(1-\lambda_1 L)}-h_{11}(0)\frac{(\beta_1+\alpha_{11})\lambda_1[(1-\lambda_1 L)\tau_1+(1-\rho L)\tau_\epsilon]}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2 L(\rho-\lambda_1)}+$$

$$
\begin{aligned}
&+ h_{12}(0)\frac{\beta_1\lambda_1\tau_\epsilon}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2} - h_{21}(0)\frac{\alpha_{12}\lambda_1[(1-\lambda_1 L)\tau_1 + (1-\rho L)\tau_\epsilon]}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2 L(\rho-\lambda_1)} - \\
&- [h_{12}(\lambda_1)(\alpha_{11}+\lambda_1\gamma_{11}) + h_{22}(\lambda_1)(\alpha_{12}+\lambda_1\gamma_{12})]\frac{\lambda_1(1-\rho L)\tau_\epsilon}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2(1-\rho\lambda_1)}
\end{aligned}
$$

$$
\begin{aligned}
d_2(L) = &\frac{\varphi_1\lambda_1\tau_1}{\rho\tau_\varepsilon(1-\rho\lambda_1)(1-\lambda_1 L)} + h_{11}(0)\frac{(\beta_1+\alpha_{11})\lambda_1\tau_1}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2} - \\
&- h_{12}(0)\frac{\beta_1\lambda_1[(1-\rho L)\tau_1 + (1-\lambda_1 L)\tau_\epsilon]}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2 L(\rho-\lambda_1)} + h_{21}(0)\frac{\alpha_{12}\lambda_1\tau_1}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2} - \\
&- [h_{12}(\lambda_1)(\alpha_{11}+\lambda_1\gamma_{11}) + h_{22}(\lambda_1)(\alpha_{12}+\lambda_1\gamma_{12})]\frac{\lambda_1(1-\rho L)\tau_1}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2(1-\rho\lambda_1)}
\end{aligned}
$$

$$
\begin{aligned}
d_3(L) = &\frac{\varphi_2\lambda_2\tau_\epsilon}{\rho\tau_\varepsilon(1-\rho\lambda_2)(1-\lambda_2 L)} - h_{11}(0)\frac{\alpha_{21}\lambda_2[(1-\lambda_2 L)\tau_2 + (1-\rho L)\tau_\epsilon]}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2 L(\rho-\lambda_2)} - \\
&- h_{21}(0)\frac{(\beta_2+\alpha_{22})\lambda_2[(1-\lambda_2 L)\tau_2 + (1-\rho L)\tau_\epsilon]}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2 L(\rho-\lambda_2)} + h_{22}(0)\frac{\beta_2\lambda_2\tau_\epsilon}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2} - \\
&- [h_{12}(\lambda_2)(\alpha_{21}+\lambda_2\gamma_{21}) + h_{22}(\lambda_2)(\alpha_{22}+\lambda_1 2\gamma_{22})]\frac{\lambda_2(1-\rho L)\tau_\epsilon}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2(1-\rho\lambda_2)}
\end{aligned}
$$

$$
\begin{aligned}
d_4(L) = &\frac{\varphi_2\lambda_2\tau_2}{\rho\tau_\varepsilon(1-\rho\lambda_2)(1-\lambda_2 L)} + h_{11}(0)\frac{\alpha_{21}\lambda_2\tau_2}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2} + h_{21}(0)\frac{(\beta_2+\alpha_{22})\lambda_2\tau_2}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2} - \\
&- h_{22}(0)\frac{\beta_2\lambda_2[(1-\rho L)\tau_2 + (1-\lambda_2 L)\tau_\epsilon]}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2 L(\rho-\lambda_2)} - \\
&- [h_{12}(\lambda_2)(\alpha_{21}+\lambda_2\gamma_{21}) + h_{22}(\lambda_2)(\alpha_{22}+\lambda_1 2\gamma_{22})]\frac{\lambda_2(1-\rho L)\tau_2}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2(1-\rho\lambda_2)}
\end{aligned}
$$

From (A.26), the solution to the policy function is given by $\boldsymbol{h}(L) = \boldsymbol{C}(L)^{-1}\boldsymbol{d}(L) = \frac{\text{adj}\,\boldsymbol{C}(L)}{\det\boldsymbol{C}(L)}\boldsymbol{d}(L)$. Hence, I need to obtain $\det\boldsymbol{C}(L)$. Note that the degree of $\det\boldsymbol{C}(L)$ is a polynomial of degree 8 on L. Denote the inside roots of $\det\boldsymbol{C}(L)$ as $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$, and the outside roots as $\{\vartheta_1^{-1}, \vartheta_2^{-1}\}$. Because agents cannot use future signals, the inside roots have to be removed. Note that the number of free constants in $\boldsymbol{d}(L)$ is 6:

$$
\Big\{h_{11}(0), h_{12}(0), h_{21}(0), h_{22}(0), \underbrace{h_{12}(\lambda_1)(\alpha_{11}+\lambda_1\gamma_{11}) + h_{22}(\lambda_1)(\alpha_{12}+\lambda_1\gamma_{12})}_{h(\lambda_1)},
$$
$$
\underbrace{h_{12}(\lambda_2)(\alpha_{21}+\lambda_2\gamma_{21}) + h_{22}(\lambda_2)(\alpha_{22}+\lambda_2\gamma_{22})}_{h(\lambda_2)}\Big\}
$$

For a unique solution, it has to be the case that the number of outside roots is 2. By

Cramer's rule, $h_{11}(L)$ is given by

$$h_{11}(L) = \frac{\det \begin{bmatrix} d_1(L) & C_{12}(L) & C_{13}(L) & C_{14}(L) \\ d_2(L) & C_{22}(L) & C_{23}(L) & C_{24}(L) \\ d_3(L) & C_{32}(L) & C_{33}(L) & C_{34}(L) \\ d_4(L) & C_{42}(L) & C_{43}(L) & C_{44}(L) \end{bmatrix}}{\det \boldsymbol{C}(L)}$$

and similarly with the rest of policy functions. The degree of the numerator is 7, as the highest degree of $D_g(L)$ is 1 degree less than that of $\boldsymbol{C}(L)$. By choosing the appropriate constants $\left\{h_{11}(0), h_{12}(0), h_{21}(0), h_{22}(0), h(\lambda_1), h(\lambda_2)\right\}$, the 6 inside roots will be removed. Therefore, the 6 constants are solutions to the following system of linear equations

$$\det \begin{bmatrix} d_1(\zeta_i) & C_{12}(\zeta_i) & C_{13}(\zeta_i) & C_{14}(\zeta_i) \\ d_2(\zeta_i) & C_{22}(\zeta_i) & C_{23}(\zeta_i) & C_{24}(\zeta_i) \\ d_3(\zeta_i) & C_{32}(\zeta_i) & C_{33}(\zeta_i) & C_{34}(\zeta_i) \\ d_4(\zeta_i) & C_{42}(\zeta_i) & C_{43}(\zeta_i) & C_{44}(\zeta_i) \end{bmatrix} = 0$$

for $i = 1, 2, ..., 6$. After removing the inside roots in the denominator, the degree of the numerator is 1 and the degree of the denominator is 2. The policy functions will be $h_{g1}(L) = \frac{\widetilde{\psi}_{g1,1}+\widetilde{\psi}_{g2,1}L}{(1-\vartheta_1 L)(1-\vartheta_2 L)}$, and $h_{g2}(L) = \frac{\widetilde{\psi}_{g1,2}+\widetilde{\psi}_{g2,2}L}{(1-\vartheta_1 L)(1-\vartheta_2 L)}$, and hence I have

$$\begin{aligned} a_{gt} &= [h_{g1}(L) + h_{g2}(L)]\nu_t + h_{g1}(L)\epsilon_t = \frac{(\widetilde{\psi}_{g1,1} + \widetilde{\psi}_{g1,2}) + (\widetilde{\psi}_{g2,1} + \widetilde{\psi}_{g2,2})L}{(1-\vartheta_1 L)(1-\vartheta_2 L)}\nu_t + \frac{\widetilde{\psi}_{g1,2} + \widetilde{\psi}_{g2,2}L}{(1-\vartheta_1 L)(1-\vartheta_2 L)}\epsilon_t \\ &= \sum_{j=1} \psi_{gj}\left(1 - \frac{\vartheta_j}{\rho}\right)\frac{1}{1-\vartheta_j L}\nu_t + \sum_{j=1}^{2} \phi_{gj}\left(1 - \frac{\vartheta_j}{\rho}\right)\frac{1}{1-\vartheta_j L}\epsilon_t = \psi_{g1}\widetilde{\vartheta}^{\nu}_{1t} + \psi_{g2}\widetilde{\vartheta}^{\nu}_{2t} + \phi_{g1}\widetilde{\vartheta}^{\epsilon}_{1t} + \phi_{g2}\widetilde{\vartheta}^{\epsilon}_{2t} \end{aligned}$$

I can write $\boldsymbol{a}_t = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} = Q_\nu \widetilde{\vartheta}^{\nu}_t + Q_u \widetilde{\vartheta}^{\epsilon}_t = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{bmatrix} \widetilde{\vartheta}^{\nu}_{1t} \\ \widetilde{\vartheta}^{\nu}_{2t} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} \widetilde{\vartheta}^{\epsilon}_{1t} \\ \widetilde{\vartheta}^{\epsilon}_{2t} \end{bmatrix}$. Notice that I can write $\widetilde{\vartheta}^x_t = \Lambda \widetilde{\vartheta}^x_{t-1} + \Gamma x_t = (I - \Lambda L)^{-1}\Gamma x_t$ for $x \in \{\nu, \epsilon\}$, where $\Lambda = \begin{bmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 - \frac{\vartheta_1}{\rho} \\ 1 - \frac{\vartheta_2}{\rho} \end{bmatrix}$. Hence, I can write $\boldsymbol{a}_t = Q_\nu (I - \Lambda L)^{-1}\Gamma \nu_t + Q_u (I - \Lambda L)^{-1}\Gamma \epsilon_t = Q_\nu \sum_{k=0}^{\infty} \Lambda^k \Gamma \nu_{t-k} + Q_u \sum_{k=0}^{\infty} \Lambda^k \Gamma u_{t-k}$. $\{\psi_{gk}, \phi_{gk}\}^2_{g=1,k=1}$ are fixed scalars that depend on deep parameters of the model, and $(\vartheta_1, \vartheta_2)$ are two scalars that are given by the reciprocal of the two largest roots of the characteristic polynomial of $\boldsymbol{C}(z) = \begin{bmatrix} C_{11}(z) & C_{12}(z) & C_{13}(z) & C_{14}(z) \\ C_{21}(z) & C_{22}(z) & C_{23}(z) & C_{24}(z) \\ C_{31}(z) & C_{32}(z) & C_{33}(z) & C_{34}(z) \\ C_{41}(z) & C_{42}(z) & C_{43}(z) & C_{44}(z) \end{bmatrix}$, where $C_{11}(z) = 1 - \frac{\beta\delta\sigma_\varepsilon^2}{z} - [1 - \beta(1 + \phi_y/\nu)]\sigma_\varepsilon^2 + \frac{\phi_y(1-\lambda)}{\sigma\sigma_\varepsilon^2}$, $C_{12}(z) = -\frac{\lambda_1\sigma_\varepsilon^4\{\beta(\delta-s)+[1-\beta(1+\phi_y/\nu)]z\}}{(z-\lambda_1)(1-\lambda_1 z)\rho\sigma_\epsilon^2}$, $C_{13}(z) = \beta\phi_\pi/\nu\sigma_\varepsilon^2 - \frac{\beta/\nu\sigma_\varepsilon^2}{z}$, $C_{14}(z) = -\frac{\lambda_1\sigma_\varepsilon^4\beta/\nu(1-\phi_\pi z)}{(z-\lambda_1)(1-\lambda_1 z)\rho\sigma_\epsilon^2}$, $C_{21}(z) = 0$, $C_{22}(z) = 1 - \frac{\beta s\sigma_\varepsilon^2}{z} - \frac{\lambda_1\sigma_\varepsilon^4\{\beta(\delta-s)+[1-\beta(1+\phi_y/\nu)]z\}}{(z-\lambda_1)(1-\lambda_1 z)\rho\sigma_1^2}$, $C_{23}(z) = 0$, $C_{24}(z) = C_{14}\sigma_\epsilon^2/\sigma_1^2$, $C_{31}(z) = -\kappa\theta\sigma_\varepsilon^2$, $C_{32}(z) = -\frac{\lambda_2\sigma_\varepsilon^4\kappa\theta z}{(z-\lambda_2)(1-\lambda_2 z)\rho\sigma_\epsilon^2}$, $C_{33}(z) = 1 - \frac{\beta\theta\sigma_\varepsilon^2}{z} - (1-\theta)\sigma_\varepsilon^2$, $C_{34}(z) = -\frac{\lambda_2\sigma_\varepsilon^4(1-\theta)z}{(z-\lambda_2)(1-\lambda_2 z)\rho\sigma_\epsilon^2}$, $C_{41}(z) = 0$, $C_{42}(z) = -\frac{\lambda_2\sigma_\varepsilon^4\kappa\theta z}{(z-\lambda_2)(1-\lambda_2 z)\rho\sigma_2^2}$, $C_{43}(z) = 0$, $C_{44}(z) = 1 - \frac{\beta\theta\sigma_\varepsilon^2}{z} - \frac{\lambda_2\sigma_\varepsilon^4(1-\theta)z}{(z-\lambda_2)(1-\lambda_2 z)\rho\sigma_2^2}$, and λ_g, $g \in \{1, 2\}$ is the inside root of the polynomial $\boldsymbol{D}_g(z) \equiv (1-\rho z)(\rho - z) - \frac{(\sigma_g^2+\sigma_\epsilon^2)\sigma_\varepsilon^2}{\sigma_g^2\sigma_\epsilon^2} z$. □

PROPOSITION A1. *In the beyond FIRE framework the underrevision coefficient $\beta_{c\pi}^{\mathcal{M}}$ is given by*

$$\beta_{c\pi}^{\mathcal{M}} = \frac{\lambda_1^3}{(\rho-\lambda_1)(1+\lambda_1+\lambda_1^2+\lambda_1^3)\sum_{k=1}^{2}\psi_{2g}\frac{\rho-\vartheta_k}{1-\lambda_1\vartheta_k}} \times$$
$$\times \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}}{(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)} \left[\frac{\lambda_1\vartheta_g(1-\lambda_1^2)(1+\vartheta_g)(1+\vartheta_g^2)(1-\rho\vartheta_g)}{1-\lambda_1\vartheta_g}\right.$$
$$\left. + (1+\lambda_1^2)\{(\rho-\lambda_1)[\vartheta_g(1+\lambda_1) - \lambda_1(1-\lambda_1\vartheta_g)] - \rho\lambda_1^2(1+\lambda_1)(1-\lambda_1\vartheta_g)\}\right] \tag{A.27}$$

where λ_1 is the inside root of the polynomial $\boldsymbol{D}(z) \equiv (1-\rho z)(\rho - z) - \sigma_\varepsilon^2/\sigma_1^2 z$.

Proof of Proposition A1. From the proof of proposition 1, I have the following objects: $\pi_{t+k} = h_2(L)\nu_{t+k}$, $\overline{\mathbb{E}}_t^c\pi_{t+k} = \frac{(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho(L-\lambda_1)(1-\lambda_1 L)}\left[L^{1-k}h_2(L) - \frac{1-\rho L}{1-\rho\lambda_1}\lambda_1^{1-k}h_2(\lambda_1)\right]$, and $\pi_{t+k} - \overline{\mathbb{E}}_t^c\pi_{t+k} = \frac{\lambda_1}{\rho(L-\lambda_1)(1-\lambda_1 L)}\left[(L-\rho)L^{-k}h_2(L) + (\rho-\lambda_1)\lambda_1^{-k}h_2(\lambda_1)\right]\varepsilon_t$. The forecast error of annual inflation is

$$\begin{aligned}\pi_{t+3,3} - \overline{\mathbb{E}}_t^c\pi_{t+3,t} &= (\pi_t - \overline{\mathbb{E}}_t^c\pi_t) + (\pi_{t+1} - \overline{\mathbb{E}}_t^c\pi_{t+1}) + (\pi_{t+2} - \overline{\mathbb{E}}_t^c\pi_{t+2}) + (\pi_{t+3} - \overline{\mathbb{E}}_t^c\pi_{t+3}) \\ &= \frac{\lambda_1}{\rho(L-\lambda_1)(1-\lambda_1 L)}\left[(L-\rho)\left(\sum_{k=0}^{3}L^{-k}\right)h_2(L) + (\rho-\lambda_1)\left(\sum_{k=0}^{3}\lambda_1^{-k}\right)h_2(\lambda_1)\right]\varepsilon_t\end{aligned}$$

$$= \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)(1-\rho\vartheta_g)\lambda_1\psi_{2g}}{\rho^2(1-\lambda_1\vartheta_g)(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t + \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\left[\rho(1-\lambda_1\vartheta_g)-\vartheta_g(\rho-\lambda_1)L\right]\psi_{2g}}{\rho^2(1-\lambda_1\vartheta_g)L(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t$$

$$+ \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\left[\rho\lambda_1(1-\lambda_1\vartheta_g)+(\rho-\lambda_1)(1-\lambda_1\vartheta_g)L-\vartheta_g(\rho-\lambda_1)L^2\right]\psi_{2g}}{\rho^2\lambda_1(1-\lambda_1\vartheta_g)L^2(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t$$

$$+ \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\left[(L^2+\lambda_1 L+\lambda_1^2)(\rho+\lambda_1\vartheta_g L)-(L+\lambda_1)[\lambda_1 L+(L^2+\lambda_1^2)\rho\vartheta_g]\right]\psi_{2g}}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)L^3(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t$$

$$= \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)L^3(1-\lambda_1 L)(1-\vartheta_1 L)} \times \Big\{ \rho\lambda_1^2(1-\lambda_1\vartheta_g)+\lambda_1(1-\lambda_1\vartheta_g)[\rho-(1-\rho)\lambda_1]L$$

$$+(1-\lambda_1\vartheta_g)[\rho-(1-\rho)\lambda_1(1+\lambda_1)]L^2+[\lambda_1^3-\vartheta_g(\rho-(1-\rho)\lambda_1(1+\lambda_1+\lambda_1^2))]L^3 \Big\}\varepsilon_t$$

$$= \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)} \frac{(1-\xi_{1g}L)(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t$$

$$= \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)} \frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3} \left(\frac{\vartheta_g-\xi_{1g}}{\vartheta_g-\lambda_1}\frac{1}{1-\vartheta_g L} - \frac{\lambda_1-\xi_{1g}}{\vartheta_g-\lambda_1}\frac{1}{1-\lambda_1 L} \right)\varepsilon_t$$

$$= \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}(\vartheta_g-\xi_{1g})}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)} \frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\vartheta_g L)}\varepsilon_t$$

$$+ \sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}(\xi_{1g}-\lambda_1)}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)} \frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\lambda_1 L)}\varepsilon_t$$

$$= \sum_{g=1}^{2} \gamma_{1g}\frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\vartheta_g L)}\varepsilon_t + \sum_{g=1}^{2} \gamma_{2g}\frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\lambda_1 L)}\varepsilon_t$$

$$= \sum_{g=1}^{2} \gamma_{1g}\frac{1-(\xi_{2g}+\xi_{3g})L+\xi_{2g}\xi_{3g}L^2}{L^3(1-\vartheta_g L)}\varepsilon_t + \sum_{g=1}^{2} \gamma_{2g}\frac{1-(\xi_{2g}+\xi_{3g})L+\xi_{2g}\xi_{3g}L^2}{L^3(1-\lambda_1 L)}\varepsilon_t$$

$$= \sum_{g=1}^{2} \left\{ \gamma_{1g}\sum_{k=0}^{\infty}\vartheta_g^k\varepsilon_{t+3-k} - \gamma_{1g}(\xi_{2g}+\xi_{3g})\sum_{k=0}^{\infty}\vartheta_g^k\varepsilon_{t+2-k} + \gamma_{1g}\xi_{2g}\xi_{3g}\sum_{k=0}^{\infty}\vartheta_g^k\varepsilon_{t+1-k} \right\}$$

$$+ (\gamma_{21}+\gamma_{22})\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t+3-k} - [\gamma_{21}(\xi_{21}+\xi_{31})+\gamma_{22}(\xi_{22}+\xi_{32})]\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t+2-k}$$

$$+ (\gamma_{21}\xi_{21}\xi_{31}+\gamma_{22}\xi_{22}\xi_{32})\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t+1-k}$$

$$= \sum_{g=1}^{2} \left\{ \beta_{1g} \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+3-k} + \beta_{2g} \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+2-k} + \beta_{3g} \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+1-k} \right\}$$

$$+ \beta_4 \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+3-k} + \beta_5 \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+2-k} + \beta_6 \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+1-k}$$

where $\xi_{0g} = \rho\lambda_1^2(1-\lambda_1\vartheta_g)$, $-\xi_{0g}(\xi_{1g}+\xi_{2g}+\xi_{3g}) = \lambda_1(1-\lambda_1\vartheta_g)[\rho-(1-\rho)\lambda_1]$, $\xi_{0g}(\xi_{1g}\xi_{2g}+\xi_{1g}\xi_{3g}+\xi_{2g}\xi_{3g}) = (1-\lambda_1\vartheta_g)[\rho-(1-\rho)\lambda_1(1+\lambda_1)]$, $-\xi_{0g}\xi_{1g}\xi_{2g}\xi_{3g} = \lambda_1^3 - \vartheta_g[\rho-(1-\rho)\lambda_1(1+\lambda_1+\lambda_1^2)]$, $\gamma_{1g} = \frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}(\vartheta_g-\xi_{1g})}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}$, $\gamma_{2g} = \frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}(\xi_{1g}-\lambda_1)}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}$, $\beta_{1g} = \gamma_{1g}$, $\beta_{2g} = -\gamma_{1g}(\xi_{2g}+\xi_{3g})$, $\beta_{3g} = \gamma_{1g}\xi_{2g}\xi_{3g}$, $\beta_4 = \gamma_{21}+\gamma_{22}$, $\beta_5 = -[\gamma_{21}(\xi_{21}+\xi_{31})+\gamma_{22}(\xi_{22}+\xi_{32})]$, and $\beta_6 = \gamma_{21}\xi_{21}\xi_{31}+\gamma_{22}\xi_{22}\xi_{32}$. Before computing the forecast revision of annual inflation, notice that $\overline{\mathbb{E}}_t^c\pi_{t+k} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+k} = \frac{(\rho-\lambda_1)h_2(\lambda_1)}{\rho\lambda_1^k}\frac{1}{1-\lambda_1 L}\varepsilon_t$. Therefore, the forecast revision of annual inflation is

$$\begin{aligned}\overline{\mathbb{E}}_t^c\pi_{t+3,t} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+3,t} &= (\overline{\mathbb{E}}_t^c\pi_t - \overline{\mathbb{E}}_{t-1}^c\pi_t) + (\overline{\mathbb{E}}_t^c\pi_{t+1} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+1}) + (\overline{\mathbb{E}}_t^c\pi_{t+2} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+2}) + (\overline{\mathbb{E}}_t^c\pi_{t+3} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+3}) \\ &= \frac{(\rho-\lambda_1)h_2(\lambda_1)(1+\lambda_1^{-1}+\lambda_1^{-2}+\lambda_1^{-3})}{\rho}\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t-k} = \Upsilon\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t-k}\end{aligned}$$

where $\Upsilon = \frac{(\rho-\lambda_1)(1+\lambda_1+\lambda_1^2+\lambda_1^3)\sum_{g=1}^{2}\psi_{2g}\frac{\rho-\vartheta_g}{1-\lambda_1\vartheta_g}}{\rho^2\lambda_1^3}$. I now seek to compute the OLS coefficient. The covariance is

$$\mathbb{C}\,(\text{forecast error, revision}) = \left[\sum_{g=1}^{2}\frac{\beta_{1g}\Upsilon\vartheta_g^3 + \beta_{2g}\Upsilon\vartheta_g^2 + \beta_{3g}\Upsilon\vartheta_g}{1-\lambda_1\vartheta_g} + \frac{\beta_4\Upsilon\lambda_1^3 + \beta_5\Upsilon\lambda_1^2 + \beta_6\Upsilon\lambda_1}{1-\lambda_1^2}\right]\sigma_\varepsilon^2$$

The variance is $\mathbb{V}\,(\text{revision}) = \frac{\Upsilon^2}{1-\lambda_2^2}\sigma_\varepsilon^2$. Finally, the OLS coefficient is

$$\begin{aligned}\beta_{CG} &= \frac{\mathbb{C}\,(\text{forecast error, revision})}{\mathbb{V}\,(\text{revision})} = \frac{1}{\Upsilon}\left[\sum_{g=1}^{2}(\beta_{1g}\vartheta_g^3 + \beta_{2g}\vartheta_g^2 + \beta_{3g}\vartheta_g)\frac{1-\lambda_1^2}{1-\lambda_1\vartheta_g} + \beta_4\lambda_1^3 + \beta_5\lambda_1^2 + \beta_6\lambda_1\right] \\ &= \frac{1}{\Upsilon}\left[\sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}\vartheta_g(1-\lambda_1^2)}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)^2(\vartheta_g-\lambda_1)}\xi_{0g}(\vartheta_g-\xi_{1g})(\vartheta_g-\xi_{2g})(\vartheta_g-\xi_{3g})\right. \\ &\quad \left. -\sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}}{\rho^2\lambda_1(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\xi_{0g}(\lambda_1-\xi_{1g})(\lambda_1-\xi_{2g})(\lambda_1-\xi_{3g})\right]\end{aligned}$$

$$= \frac{1}{\Upsilon}\left[\sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}\lambda_1\vartheta_g(1-\lambda_1^2)(1+\vartheta_g)(1+\vartheta_g^2)(1-\rho\vartheta_g)}{\rho^2(1-\lambda_1\vartheta_g)^2(\vartheta_g-\lambda_1)}\right.$$

$$\left.+\sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}(1+\lambda_1^2)\{(\rho-\lambda_1)[\vartheta_g(1+\lambda_1)-\lambda_1(1-\lambda_1\vartheta_g)]-\rho\lambda_1^2(1+\lambda_1)(1-\lambda_1\vartheta_g)\}}{\rho^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\right]$$

$$= \frac{1}{\Upsilon}\sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}}{\rho^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\left[\frac{\lambda_1\vartheta_g(1-\lambda_1^2)(1+\vartheta_g)(1+\vartheta_g^2)(1-\rho\vartheta_g)}{1-\lambda_1\vartheta_g}\right.$$

$$\left.+(1+\lambda_1^2)\{(\rho-\lambda_1)[\vartheta_g(1+\lambda_1)-\lambda_1(1-\lambda_1\vartheta_g)]-\rho\lambda_1^2(1+\lambda_1)(1-\lambda_1\vartheta_g)\}\right]$$

$$= \frac{\lambda_1^3}{(\rho-\lambda_1)(1+\lambda_1+\lambda_1^2+\lambda_1^3)\sum_{k=1}^{2}\psi_{2g}\frac{\rho-\vartheta_k}{1-\lambda_1\vartheta_k}}\sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}}{(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}$$

$$\times\left[\frac{\lambda_1\vartheta_g(1-\lambda_1^2)(1+\vartheta_g)(1+\vartheta_g^2)(1-\rho\vartheta_g)}{1-\lambda_1\vartheta_g}\right.$$

$$\left.+(1+\lambda_1^2)\{(\rho-\lambda_1)[\vartheta_g(1+\lambda_1)-\lambda_1(1-\lambda_1\vartheta_g)]-\rho\lambda_1^2(1+\lambda_1)(1-\lambda_1\vartheta_g)\}\right]$$

□

Proposition A2. *Beyond FIRE, the time-varying direct share* α_τ *is given by* $\alpha_\tau = \left(\delta_\rho\rho^\tau+\delta_\lambda\lambda_1^\tau+\sum_{j=1}^{2}\delta_{3j}\vartheta_j^\tau\right)/\left[\sum_{j=1}^{2}\psi_{1j}(\rho^{\tau+1}-\vartheta_j^{\tau+1})\right]$, *where*

$$\delta_\rho = -\frac{\rho(1-\lambda)\left[1+\phi_y\sum_{j=1}^{2}\psi_{1j}+(\phi_\pi-\rho)\sum_{j=1}^{2}\psi_{2j}\right]}{\sigma(1-\rho s\beta)}$$

$$\delta_\lambda = \frac{(1-\lambda)\lambda_1}{\sigma(1-\rho s\beta)}\left\{1+\frac{\phi_y(1-\rho s\beta)\lambda_1}{\rho}\sum_{j=1}^{2}\frac{\psi_{1j}(\rho-\vartheta_j)(1-\rho\vartheta_j)}{(\lambda_1-\vartheta_j)(1-\lambda_1\vartheta_j)}+\frac{\phi_\pi(1-\rho s\beta)\lambda_1}{\rho}\sum_{j=1}^{2}\frac{\psi_{2j}(\rho-\vartheta_j)(1-\rho\vartheta_j)}{(\lambda_1-\vartheta_j)(1-\lambda_1\vartheta_j)}\right.$$

$$+\frac{s\beta\phi_y}{\rho}\sum_{j=1}^{2}\frac{\psi_{1j}(\rho-\vartheta_j)\left[\rho\vartheta_j(1-\rho s\beta\lambda_1\vartheta_j)-\rho\lambda_1(1-s\beta\vartheta_j)-\lambda_1\vartheta_j(1-\rho\lambda_1)\right]}{(1-s\beta\vartheta_j)(\lambda_1-\vartheta_j)(1-\vartheta_j\lambda_1)}$$

$$\left.+\frac{(s\beta\phi_\pi-1)}{\rho}\sum_{j=1}^{2}\frac{\psi_{2j}(\rho-\vartheta_j)\left[\rho\vartheta_j(1-\rho s\beta\lambda_1\vartheta_j)-\rho\lambda_1(1-s\beta\vartheta_j)-\lambda_1\vartheta_j(1-\rho\lambda_1)\right]}{(1-s\beta\vartheta_j)(\lambda_1-\vartheta_j)(1-\vartheta_j\lambda_1)}\right\}$$

$$\delta_{\vartheta j} = -\frac{(1-\lambda)(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho\sigma}\sum_{j=1}^{2}\frac{\vartheta_j^2[\phi_y\psi_{1j}+(\phi_\pi-\vartheta_j)\psi_{2j}]}{(1-s\beta\vartheta_j)(\lambda_1-\vartheta_j)(1-\lambda_1\vartheta_j)}.$$

Proof of Proposition A2. The direct effect $\mathrm{DE}_t = -\frac{1-\lambda}{\sigma}\sum_{k=0}^{\infty}(\beta s)^k \overline{\mathbb{E}}_t^c r_{t+k}$ is given by

$$\mathrm{DE}_t = -\frac{1-\lambda}{\sigma(1-\rho\beta s)}\overline{\mathbb{E}}_t^c \nu_t - \frac{(1-\lambda)\phi_y}{\sigma}\overline{\mathbb{E}}_t^c y_t - \frac{(1-\lambda)\beta s\phi_y}{\sigma}\sum_{k=0}^{\infty}(\beta s)^k \overline{\mathbb{E}}_t^c y_{t+k+1}$$
$$-\frac{(1-\lambda)\phi_\pi}{\sigma}\overline{\mathbb{E}}_t^c \pi_t - \frac{(1-\lambda)(\beta s\phi_\pi - 1)}{\sigma}\sum_{k=0}^{\infty}(\beta s)^k \overline{\mathbb{E}}_t^c \pi_{t+k+1}$$

where the forecasts are given by

$$\overline{\mathbb{E}}_t^c \nu_t = \left(1-\frac{\lambda_1}{\rho}\right)\frac{1}{(1-\lambda_1 L)(1-\rho L)}\varepsilon_t = \frac{1}{\rho}\sum_{k=0}^{\infty}(\rho^{k+1}-\lambda_1^{k+1})\varepsilon_{t-k}$$

$$\overline{\mathbb{E}}_t^c y_t = \sum_{j=1}^{2}\psi_{1j}\left(1-\frac{\vartheta_j}{\rho}\right)\overline{\mathbb{E}}_t^c\left[\frac{1}{1-\vartheta_j L}\nu_t\right]$$

$$= \sum_{j=1}^{2}\psi_{1j}\left(1-\frac{\vartheta_j}{\rho}\right)\left[\begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{(1-\vartheta_j L)(1-\rho L)} & 0\end{bmatrix}\begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L^{-1}} \\ \tau_1^{-\frac{1}{2}}\end{bmatrix}\frac{1-\rho L^{-1}}{1-\lambda_1 L^{-1}}\right]_+ \frac{\lambda_1\tau_1}{\rho}\frac{1}{1-\lambda_1 L}\varepsilon_t$$

$$= \sum_{j=1}^{2}\psi_{1j}\left(1-\frac{\vartheta_j}{\rho}\right)\left\{\frac{\rho}{\rho-\vartheta_j}\sum_{k=0}^{\infty}\rho^k\varepsilon_{t-k} - \frac{\vartheta_j^2(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho(\rho-\vartheta_j)(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\sum_{k=0}^{\infty}\vartheta_j^k\varepsilon_{t-k} + \frac{\lambda_1^2(1-\rho\vartheta_j)}{\rho(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t-k}\right\}$$

$$\overline{\mathbb{E}}_t^c \pi_t = \sum_{j=1}^{2}\psi_{2j}\left(1-\frac{\vartheta_j}{\rho}\right)\left\{\frac{\rho}{\rho-\vartheta_j}\sum_{k=0}^{\infty}\rho^k\varepsilon_{t-k} - \frac{\vartheta_j^2(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho(\rho-\vartheta_j)(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\sum_{k=0}^{\infty}\vartheta_j^k\varepsilon_{t-k} + \frac{\lambda_1^2(1-\rho\vartheta_j)}{\rho(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t-k}\right\}$$

$$\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t^c y_{t+k+1} = \overline{\mathbb{E}}_t^c\left(\frac{y_t}{L-s\beta}\right) = \sum_{j=1}^{2}\psi_{1j}\left(1-\frac{\vartheta_j}{\rho}\right)\overline{\mathbb{E}}_t^c\left[\frac{1}{(L-s\beta)(1-\vartheta_j L)}\nu_t\right]$$

$$= \sum_{j=1}^{2}\psi_{1j}\left(1-\frac{\vartheta_j}{\rho}\right)\left[\begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{(L-s\beta)(1-\vartheta_j L)(1-\rho L)} & 0\end{bmatrix}\begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L^{-1}} \\ \tau_1^{-\frac{1}{2}}\end{bmatrix}\frac{1-\rho L^{-1}}{1-\lambda_1 L^{-1}}\right]_+ \frac{\lambda_1\tau_1}{\rho}\frac{1}{1-\lambda_1 L}\varepsilon_t$$

$$= \sum_{j=1}^{2}\psi_{1j}\left(1-\frac{\vartheta_j}{\rho}\right)\left\{\frac{\rho^2}{(1-\rho s\beta)(\rho-\vartheta_j)}\sum_{k=0}^{\infty}\rho^k\varepsilon_{t-k} - \frac{\vartheta_j^3(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho(\rho-\vartheta_j)(1-s\beta\vartheta_j)(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\sum_{k=0}^{\infty}\vartheta_j^k\varepsilon_{t-k}\right.$$
$$\left.+\frac{\lambda_1\left[\rho\lambda_1(1-s\beta\vartheta_j)+\lambda_1\vartheta_j(1-\rho\lambda_1)-\rho\vartheta_j(1-\rho s\beta\lambda_1\vartheta_j)\right]}{\rho(1-\rho s\beta)(1-s\beta\vartheta_j)(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t-k}\right\}$$

$$\sum_{k=0}^{\infty}(\beta s)^k\overline{\mathbb{E}}_t^c \pi_{t+k+1} = \sum_{j=1}^{2}\psi_{2j}\left(1-\frac{\vartheta_j}{\rho}\right)\left\{\frac{\rho^2}{(1-\rho s\beta)(\rho-\vartheta_j)}\sum_{k=0}^{\infty}\rho^k\varepsilon_{t-k} - \frac{\vartheta_j^3(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho(\rho-\vartheta_j)(1-s\beta\vartheta_j)(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\sum_{k=0}^{\infty}\vartheta_j^k\varepsilon_{t-k}\right.$$
$$\left.+\frac{\lambda_1\left[\rho\lambda_1(1-s\beta\vartheta_j)+\lambda_1\vartheta_j(1-\rho\lambda_1)-\rho\vartheta_j(1-\rho s\beta\lambda_1\vartheta_j)\right]}{\rho(1-\rho s\beta)(1-s\beta\vartheta_j)(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\sum_{k=0}^{\infty}\lambda_1^k\varepsilon_{t-k}\right\}$$

Introducing these objects into the DE_t expression above, I can write $\mathrm{direct}_\tau = \partial \mathrm{DE}_\tau / \partial \varepsilon_t$ as

$$
\begin{aligned}
\mathrm{direct}_\tau = & -\frac{(1-\lambda)\left[1+\phi_y \sum_{j=1}^2 \psi_{1j} + (\phi_\pi - \rho)\sum_{j=1}^2 \psi_{2j}\right]}{\sigma(1-\rho s\beta)}\rho^\tau \\
& + \frac{(1-\lambda)\lambda_1}{\rho\sigma(1-\rho s\beta)}\left\{1+\frac{\phi_y(1-\rho s\beta)\lambda_1}{\rho}\sum_{j=1}^{2}\frac{\psi_{1j}(\rho-\vartheta_j)(1-\rho\vartheta_j)}{(\lambda_1-\vartheta_j)(1-\lambda_1\vartheta_j)} + \frac{\phi_\pi(1-\rho s\beta)\lambda_1}{\rho}\sum_{j=1}^{2}\frac{\psi_{2j}(\rho-\vartheta_j)(1-\rho\vartheta_j)}{(\lambda_1-\vartheta_j)(1-\lambda_1\vartheta_j)}\right. \\
& + \frac{s\beta\phi_y}{\rho}\sum_{j=1}^{2}\frac{\psi_{1j}(\rho-\vartheta_j)\left[\rho\vartheta_j(1-\rho s\beta\lambda_1\vartheta_j)-\rho\lambda_1(1-s\beta\vartheta_j)-\lambda_1\vartheta_j(1-\rho\lambda_1)\right]}{(1-s\beta\vartheta_j)(\lambda_1-\vartheta_j)(1-\vartheta_j\lambda_1)} \\
& \left. + \frac{(s\beta\phi_\pi-1)}{\rho}\sum_{j=1}^{2}\frac{\psi_{2j}(\rho-\vartheta_j)\left[\rho\vartheta_j(1-\rho s\beta\lambda_1\vartheta_j)-\rho\lambda_1(1-s\beta\vartheta_j)-\lambda_1\vartheta_j(1-\rho\lambda_1)\right]}{(1-s\beta\vartheta_j)(\lambda_1-\vartheta_j)(1-\vartheta_j\lambda_1)}\right\}\lambda_1^\tau \\
& - \frac{(1-\lambda)(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho^2\sigma}\sum_{j=1}^{2}\frac{\phi_y\psi_{1j}\vartheta_j^2+(\phi_\pi-\vartheta_j)\psi_{2j}\vartheta_j^2}{(1-s\beta\vartheta_j)(\lambda_1-\vartheta_j)(1-\lambda_1\vartheta_j)}\vartheta_j^\tau
\end{aligned}
$$

The aggregate outcome is

$$
y_t = \sum_{g=1}^{2}\psi_{1g}\left(1-\frac{\vartheta_g}{\rho}\right)\frac{1}{(1-\vartheta_g L)(1-\rho L)}\varepsilon_t = \sum_{g=1}^{2}\frac{\psi_{1g}}{\rho}\sum_{k=0}^{\infty}(\rho^{k+1}-\vartheta_g^{k+1})\varepsilon_{t-k}
$$

and hence I can write $\mathrm{total}_\tau = \partial y_\tau/\partial \varepsilon_t = \sum_{j=1}^2 \frac{\psi_{1j}}{\rho}(\rho^{\tau+1}-\vartheta_j^{\tau+1})$. □

Appendix B. Derivation of the Dynamic IS and Phillips Curves

This appendix provides a detailed derivation of the Dynamic IS and the Phillips curves, following the sketch outlined in the main text.

B.1. The Dynamic IS Curve

Consider first the intertemporal budget constraint. By conditions (1)-(2), the following relation holds in all periods and all states: $\sum_{k=0}^{\infty}\prod_{j=1}^{k}\frac{1}{R_{t+j-1}}\left[s^k C_{i,t+k}^S + (1-s)^k C_{i,t+k}^H\right] = B_{it}^S + \sum_{k=0}^{\infty}\prod_{j=1}^{k}\frac{1}{R_{t+j-1}}\left[s^k Y_{i,t+k}^S + (1-s)^k Y_{i,t+k}^H\right]$. Taking the log-linear approximation of the above intertemporal budget constraint around the steady state, I obtain

$$\sum_{k=0}^{\infty}(\beta s)^k c^S_{i,t+k} + \sum_{k=0}^{\infty}[\beta(1-s)]^k c^H_{i,t+k} = b^S_{it} + \sum_{k=0}^{\infty}(\beta s)^k y^S_{i,t+k} + \sum_{k=0}^{\infty}[\beta(1-s)]^k y^H_{i,t+k}, \tag{B.1}$$

where $b^S_{it} = R_{t-1}B^S_{it}$ denotes the household's initial asset position at period t for an unconstrained household. The Euler condition (3) can be log-linearized to

$$c^S_{it} = -\sigma^{-1}\mathbb{E}_{it}r_t + s\mathbb{E}_{it}c^S_{i,t+1} + (1-s)\mathbb{E}_{it}c^H_{i,t+1}. \tag{B.2}$$

Iterating forward the individual Euler condition (B.2), I can write $c^S_{it} = s^k\mathbb{E}_{it}c^S_{i,t+k} - \sigma^{-1}\sum_{j=0}^{k-1} s^j\mathbb{E}_{it}r_{t+j} + (1-s)\sum_{j=0}^{k-1} s^j\mathbb{E}_{it}c^H_{i,t+j+1}$. Taking expectations on (B.1) and inserting this last expression, I obtain

(B.3)

$$c^S_{it} = (1-\beta)b^S_{it} - \frac{\beta}{\sigma}\sum_{k=0}^{\infty}(\beta s)^k\mathbb{E}_{it}r_{t+k} + \beta(1-s)\sum_{k=0}^{\infty}(\beta s)^k\mathbb{E}_{it}y^H_{i,t+k+1} + (1-\beta)\sum_{k=0}^{\infty}(\beta s)^k\mathbb{E}_{it}y^S_{i,t+k}.$$

where I have used the assumption that HtM households consume their entire income every period to simplify the expression.

Combining the log-linearized labor supply conditions,

$$w_{it} = \sigma c^S_{it} + \varphi n^S_{it} = \sigma c^H_{it} + \varphi n^H_{it}, \tag{B.4}$$

with the log-linearized the budget constraints (2),

$$c^S_{it} = w_{it} + n^S_{it} + e^S_{it} = y^S_{it} \quad \text{and} \quad c^H_{it} = w_{it} + n^H_{it} + e^H_{it} = y^H_{it}, \tag{B.5}$$

I can write both sets of individual consumption functions in terms of the individual real wage and transfers: $c^S_{it} = \frac{1+\varphi}{\varphi+\sigma}w_{it} + \frac{\varphi}{\varphi+\sigma}\frac{1-\tau_D}{1-\lambda}d_{it}$ and $c^H_{it} = \frac{1+\varphi}{\varphi+\sigma}w_{it} + \frac{\varphi}{\varphi+\sigma}\frac{\tau_D}{\lambda}d_{it}$.

Denote aggregate consumption and aggregate labor supply for the unconstrained (constrained) household as $C^S_t = \int C^S_{it}\,di$,$N^S_t = \int N^S_{it}\,di$ ($C^H_t = \int C^H_{it}\,di$ and $N^H_t = \int N^H_{it}\,di$). Equilibrium in the goods market requires that consumption of unconstrained and constrained households equals total consumption $C_t = \lambda C^H_t + (1-\lambda)C^S_t$. Since I consider a closed economy without investment and government spending, the resource constraint implies $Y_t = C_t$. Equilibrium in the labor market requires $N_t = \lambda N^H_t + (1-\lambda)N^S_t$. With uniform steady state hours ($N^S = N^H = N$), and the fiscal policy inducing $C^S = C^H = C$, the above log-linearized market clearing conditions yield

$$y_t = c_t = n_t = \lambda c_t^H + (1-\lambda)c_t^S = \lambda n_t^H + (1-\lambda)n_t^S. \tag{B.6}$$

As is common in NK models without nominal wage rigidities, profits—and dividends and transfers received by firms—are countercyclical, resulting in $d_t = -w_t$. Combining the optimal labor supply conditions (B.4), the aggregate budget constraints (B.5), and the labor and goods market clearing conditions (B.6), I can write $y_t^H = \chi y_t$ and $y_t^S = (1-\lambda\chi)/(1-\lambda)\, y_t$. Inserting these objects into (B.3), using the fact that assets average to zero and aggregating across households, $c_t^S = -\frac{\beta}{\sigma}\sum_{k=0}^{\infty}(\beta s)^k \overline{\mathbb{E}}_t^c r_{t+k} + \beta(1-s)\chi \sum_{k=0}^{\infty}(\beta s)^k \overline{\mathbb{E}}_t^c y_{t+k+1} + (1-\beta)(1-\lambda\chi)/(1-\lambda)\sum_{k=0}^{\infty}(\beta s)^k \overline{\mathbb{E}}_t^c y_{t+k}$, where $\overline{\mathbb{E}}_t^c(\cdot) = \int_0^1 \mathbb{E}_{it}(\cdot)\ di$ is the cross-sectional average forecast across households. Using the market clearing condition (B.6), I can write the aggregate consumption function (9).

B.2. The Phillips Curve

Log-linearizing the firms' FOC around the zero inflation steady state, I obtain the familiar price-setting rule

$$p_{jt}^* = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\left(\psi_{j,t+k|t} + \mu\right), \tag{B.7}$$

where $\psi_{j,t+k|t} = \log \Psi_{j,t+k|t}$ and $\mu = \log \mathcal{M}$. The (log) marginal cost for firm j that last reset its price at time t is $\psi_{j,t+k|t} = w_{j,t+k}$. Let $\psi_t \equiv \int_{\mathcal{J}_f} \psi_{jt}$ denote the (log) average marginal cost. I can then write $\psi_t = w_t$ and $\mathbb{E}_{jt}\psi_{j,t+k|t} = \mathbb{E}_{jt}\psi_{t+k}$. Introducing this last identity into (B.7), I can rewrite the firm price-setting condition as $p_{jt}^* = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\left(p_{t+k} - \mu_{t+k}\right)$, where $\mu_t = -(\psi_t - p_t)$.

I can restate the above condition as $p_{jt}^* - p_{t-1} = -(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\mu_{t+k} + \sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\pi_{t+k}$. Define the firm-specific inflation rate as $\pi_{jt} = (1-\theta)(p_{jt}^* - p_{t-1})$. Then I can write the above expression as $\pi_{jt} = -(1-\theta)(1-\beta\theta)\mathbb{E}_{jt}\mu_t + (1-\theta)\mathbb{E}_{jt}\pi_t + \beta\theta\mathbb{E}_{jt}\pi_{j,t+1}$, where $\pi_t = \int_{\mathcal{J}_f}\pi_{jt}\, dj$. Note that I can write the deviation between average and desired markups as $\widehat{\mu}_t = p_t - \psi_t = p_t - w_t = -(\sigma y_t + \varphi n_t) = -(\sigma+\varphi)\, y_t$. Introducing this in the previous expression, I obtain (12).