HANK Beyond FIRE: Amplification, Forward Guidance and Belief Shocks

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The transmission channel of monetary policy in the benchmark New Keynesian (NK) framework relies on the counterfactual Full Information Rational Expectations (FIRE) assumption, particularly at the general equilibrium (GE) dimension. I relax the Full Information assumption and build a Heterogeneous-Agents NK model under financial frictions and dispersed information. I find that the amplification multiplier of monetary policy is dampened by the lessened role of GE effects. I then conduct the standard full-fledged NK analysis: the determinacy region is widened as a result of *as if* aggregate myopia, and the framework beyond FIRE does not suffer from the forward guidance puzzle. Finally, I find that transitory "animal spirits" shocks generate persistent effects.

Keywords: Imperfect Information; New Keynesian; Heterogeneous Agents; Monetary Policy.

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1. Introduction

Evidence suggests that inequality and information frictions play significant roles in shaping the transmission of aggregate shocks. The proportion of households that are financially restricted is 34% in the U.S., in an upward trend since 2001, and around 31% in Europe with some countries exhibiting values greater than 40% (Kaplan et al. 2014; Almgren et al. 2022).[1] Recent theoretical and empirical studies suggest that economies with higher levels of inequality respond more to fiscal and monetary shocks.[2] Additionally, belief frictions also play an important role in the transmission of shocks. Coibion and Gorodnichenko (2015) provide evidence of forecast underreaction to news in surveys of expectations to consumers, firms, professional forecasters, and central bankers. Empirical evidence suggests that households' and firms' underreaction to shocks reduces their effect, increases their persistence, and that the role of general equilibrium (GE) effects after a monetary policy shock is initially dampened (Angeletos et al. 2021; Holm et al. 2021; Gallegos 2023).

To understand transparently the mechanism of the interaction of these two forces, financial and belief frictions, I build a tractable Heterogeneous-Agents New Keynesian (HANK) model, based on Bilbiie (2021). This framework incorporates key micro-heterogeneity inputs of the quantitative literature: cyclical inequality, idiosyncratic risk, and precautionary savings, which together generate heterogeneous marginal propensities to consume (MPCs). In the benchmark Full Information Rational Expectations (FIRE) setup, more unequal economies react more to exogenous shocks under plausible assumptions. This amplification result arises from the higher MPCs of financially constrained households, and depends on the FIRE assumption at the GE dimension. In the FIRE setting, agents face no uncertainty on the exogenous fundamental and, since information sets are homogenous across individuals, on others' actions. In this paper, I accommodate such doubts. I explore the amplification result under an empirically-consistent deviation from the FIRE assumption in which agents have imperfect and dispersed information about the state of nature, following Lucas (1972); Lorenzoni (2009); Angeletos and Huo (2021).[3]. At the individual level, agents need to forecast both

[1] I consider a household to be financially restricted if it has no liquid savings to self-insure against adverse shocks.

[2] See Galí et al. (2007); Brinca et al. (2016) for the fiscal policy case, and Bilbiie (2008, 2021); Almgren et al. (2022) for the monetary policy case.

[3] A dynamic beauty contest is a class of games in which the optimal decision for an individual agent depends on the expectation of the current and future decisions of others. Morris and Shin (2002) and Woodford (2001) are the first to study the economy as a static beauty contest, and Allen et al. (2006);

the exogenous fundamental (the monetary policy shock) and aggregate variables that are endogenous to individual actions (output and inflation). As a result, an agent needs to predict other agents' actions. The economy can be described as a pair of *across–group* dynamic beauty contests between consumers and firms (the inflation-spending NK multiplier), with each group playing a *within-group* dynamic beauty contest (the spending-income multiplier running within the demand block and the strategic complementarity in price-setting running within the supply block). I study how the PE vs. the GE dynamics are affected by higher-order beliefs in the beyond FIRE framework, muting the amplification effect. The framework is consistent with available evidence on the aggregate underreaction to news (Coibion and Gorodnichenko 2015) and the lagged response of GE effects after a monetary policy shock (Holm et al. 2021).

I use this setting to study determinacy with interest rate rules, where imperfect information relaxes the lower bound on the monetary authority dovishness. I also solve the forward-guidance puzzle (FGP) and study the different effects of a pure monetary policy shock vs. an "animal spirits" shock.

Amplification. As laid out by Bilbiie (2021, 2008); Galí et al. (2007), as well as richer models by Gornemann et al. (2016), Werning (2015), Auclert (2019) and Hagedorn et al. (2019), whether aggregate shocks have bigger or smaller effects on aggregate consumption, compared to the representative agent framework, is ambiguous. In a model that combines the tractability of TANK models with the most important elements of heterogeneous agent models, Bilbiie (2021) shows that the output response to shocks is amplified if the income elasticity of constrained agents with respect to aggregate income is larger than one.[4] He refers to this case as *cyclical income inequality*; a channel which is strengthened if a larger fraction of agents is constrained.[5] Angeletos and Huo (2021) show that dispersed information attenuates the GE effects associated with the Keynesian multiplier and the inflation-spending feedback in a RANK economy. I extend their setup by including financial constraints and HtM agents, and study the implications of dispersed information for the amplification multiplier.

The magnitude of the amplification multiplier is dampened in the dispersed information framework, in which PE effects dominate GE effects in the first year after the

Bacchetta and Van Wincoop (2006); Morris and Shin (2006); Nimark (2008) extend the economy to a dynamic beauty contest.

[4]Almgren et al. (2022); Patterson (2022) find empirical evidence for this assumption.

[5]In models that focus on the cyclicality of income risk, e.g., Werning (2015), the amplification of aggregate shocks is caused by an increase in the probability of becoming constrained for the unconstrained, which leads the latter to save more and consume less.

shock, compared to the FIRE case in which the PE vs. GE share is constant over time. In this private and dispersed information economy, agents need to forecast the exogenous fundamental and aggregate inflation and output. The forecast of the fundamental does not give rise to higher-order beliefs, since the realization does not depend on others' actions and agents do not need to predict others' beliefs about the fundamental. However, forecasting aggregate output and inflation has the additional complication of dealing with higher-order beliefs. In the standard framework, first-order and higher-order beliefs coincide, whereas in this case higher-order beliefs differ from first-order beliefs, and move less than lower-order beliefs (more anchored to priors). As a result, the expectations of endogenous aggregate variables adjust less to news, and are more anchored to priors, attenuating the GE effect. Aggregate dynamics are initially driven by PE effects, consistent with the empirical findings in Holm et al. (2021). Over time, the aggregate dynamics rely more on GE effects, until the PE share converges to the full information benchmark.[6] I find that (*i*) the peak response of output is about 1/3 of that in the FIRE case, consistent with empirical evidence (Ramey 2016); (*ii*) impulse responses are hump-shaped, which the standard FIRE framework can only produce if there is habit formation, price indexation, and lumpy investment;[7] and (*iii*) when income inequality is countercyclical (the case studied in Bilbiie 2021), the response of output after a monetary policy shock is amplified by 7.72%, compared to 10.28% in the FIRE model. That is, dispersed information reduces the amplification multiplier and the overall effect of monetary policy.

Forward Guidance. I also study forward guidance in the beyond FIRE framework. In the NK framework, the determinacy region is ultimately linked to the forward-looking behavior of the model equations. The Taylor rule provides an essential stabilization role, and an excessively dovish monetary authority ends up creating explosive dynamics in the model equations. Del Negro et al. (2012); McKay et al. (2016); Andrade et al. (2019); Hagedorn et al. (2019); Angeletos and Lian (2018) have contributed to a growing

[6] Formally, imperfect information reduces the degree of complementarity of actions across agents and partially mutes the amplification multiplier mechanism that critically relies on them.

[7] Havranek et al. (2017) present a meta-analysis of the different estimates of habits in the macro literature and the available micro-estimates. In general, macro models take values around 0.75, whereas micro-estimates suggest a value around 0.4. Groth and Khan (2010) conduct a similar analysis for the investment adjustment frictions case, finding that the microeconomic estimates an order of magnitude below the ones used in the empirical macro literature, in which they are estimated to minimize the distance between model dynamics and empirical IRFs. Finally, the price-indexation model suggests that every price is changed every period, which is inconsistent with the micro-data estimates provided by Nakamura and Steinsson (2008).

literature that tries to find an explanation for the FGP from different angles, my approach combining those of Hagedorn et al. (2019) and Angeletos and Lian (2018). I find that, although there is compounding at the aggregate DIS curve arising from countercyclical income inequality, higher-order uncertainty induces enough anchoring to cure the FGP, a failure of the standard NK framework. Because expectations play a key role in the determination of aggregate variables, anchoring in expectations translates into intrinsic persistence in endogenous aggregate variables and myopia towards the future. These two results, taken together, enlarge the determinacy region of interest rate rules and solve the FGP, a result consistent with the cognitive discounting framework in Gabaix (2020).

Beliefs Shocks. The last contribution is to study expectation shocks. I consider the case of public information, and I show that although the non-fundamental shock is only transitory, its effects are persistent, which aligns with the findings in Lorenzoni (2009). Because agents cannot fully disentangle whether the shock to the signal that they have observed comes from the fundamental monetary policy rule or the non-fundamental noise part, the "animal spirits" shock partially inherits the properties of the pure monetary shock, which in turn explains its persistent consequences. In a second extension, I consider both public and private information. I find that monetary policy is more effective than in the public information case, and the effect of belief shocks is lessened, as a result of effectively reducing the degree of information friction by including an additional signal.

Roadmap. The paper proceeds as follows. In section 2 I describe the reduced-form theoretical framework, focusing on both household financial heterogeneity and dispersed information, and derive the equilibrium dynamics. In section 3 I discuss the different implications and insights provided by our HANK model beyond FIRE: the amplification multiplier, the role of the PE vs. GE share, the FGP, and "animal spirits" shocks. In section 4 I extend the theoretical setup to include firms, whose actions affect households, and study the applications and insights in the extended framework. Section 5 concludes the paper.

2. The Analytical HANK Beyond FIRE Model

The HANK framework described in this section is a reduced-form version of the standard incomplete markets (SIM) model, based on Bilbiie (2021). Households face an idiosyncratic risk of not being able to access asset markets, instead of risky labor income. This simplifying assumption allows me to solve the model in paper and pencil, and still provides the desired precautionary savings motive that two-agents New Keynesian (TANK) models lack.

On top of household heterogeneity concerning their market participation, agents face uncertainty about the state of nature. They receive idiosyncratic signals about the true state, which endogenously generates heterogeneous information sets. Since agents rely on different information, their beliefs and forecasts will differ. This aspect will be crucial for forecasts of endogenous aggregate variables like output or inflation. This gives rise to higher-order beliefs: to forecast these endogenous outcomes, an agent needs to forecast the action of other agents, and other agents need to forecast the action of others, *ad infinitum*.

For simplicity, I consider only the demand side of the economy in this section. I extend the model to firms in section 4.

2.1. Households

Households save in one-period (liquid) bonds and consume. They have access to financial income, labor income, firm profits, and government transfers.

Financial frictions. Financial frictions are exogenous to individual behavior. In every period, a household is either financially constrained or not. If the household is financially constrained, it is unable to save and loses access to the firm profits, but keeps access to previous-period savings.[8] I denote constrained households as Hand-to-Mouth (HtM). In contrast, unconstrained households benefit from having access to asset markets and firm profits. To insure against the risk of becoming constrained, which entails losing access to part of their resources (firm profits) and the ability to borrow, unconstrained households save in bonds (precautionary savings).

In every period there is a realized idiosyncratic shock. The household then knows if it will be financially constrained or not in that period. The exogenous shock takes the form of a Markov chain. Denote by s the probability of remaining unconstrained, denote

[8] This will be innocuous for the analysis since assets are in zero net supply.

by h the probability of remaining constrained, and denote by $1-s$ and $1-h$ the respective transition probabilities. For simplicity, I assume that the Markov process induces a stationary distribution. Formally, the share of HtM agents λ is given by $\lambda = (1-s)/(2-s-h)$. Notice that this analytical HANK framework nests the TANK model when $s = h = 1$ (i.e., in the first period the state of each household is revealed and will never change), and the RANK economy when $\lambda = 0$, which makes a comparison between the three settings conveniently easy.

Information frictions. Households can observe their current private variables (their wage, the consumption and saving decisions they make, the transfers they receive) but not aggregate variables. For instance, they observe all goods prices and are thus able to see the (current) aggregate price index, but they do not observe the output, inflation, the nominal interest rate, or the monetary policy shock.[9] Instead, households observe an imperfectly correlated signal, introduced in section 2.3. This information structure produces heterogeneous information sets across households since each of them has observed a different reality over time.

2.1.1. Household problem

There is a measure-1 continuum of ex-ante identical consumers in the economy, indexed by $i \in \mathcal{I}_c = [0, 1]$. Household i maximizes an infinite stream of its expected utility over consumption and its dis-utility over labor supply, $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_{it} u(C_{it}, N_{it})$, where C_{it} denotes household i's consumption decision at time t, and N_{it} denotes its labor supply choice. Notice that, differently from standard FIRE models, there is an i subscript in the expectation operator, as a result of the heterogeneity in information sets and forecasts.

Unconstrained households. A share $(1-\lambda)$ of unconstrained households have access to financial income B_{it}; they also have access to labor income $W_t^r N_{it}$, where W_t^r is the aggregate real wage rate. Finally, they receive the untaxed share of firm profits $(1-\tau_D)/(1-\lambda)E_t$, where τ_D is the profit tax rate and E_t. With these resources, an unconstrained household can either consume or save in bonds B_{it} for tomorrow. The solution to their problem, derived in Appendix A, is given by an individual Euler condition, $C_{it}^{-\sigma} \geq$

[9] I assume that agents observe the price level, but do not use this piece of information to form expectations. Vives and Yang (2016) motivates this through bounded rationality and inattention, while Angeletos and Huo (2021) argue that inflation contains little statistical information about real variables. Huo and Pedroni (2021) allow for endogenous information, but such a choice complicates the dynamics and a closed-form solution is not feasible.

$\beta\mathbb{E}_{it}\left(R_t C_{it+1}^{-\sigma}\right)$, where I have assumed that utility takes a CRRA form, with σ denoting the intertemporal elasticity of substitution and φ the inverse Frisch elasticity. Opening up the expectation operator, depending on which state the household can potentially go to (Markov structure), the condition can be written as

$$(C_{it}^S)^{-\sigma} = \beta\mathbb{E}_{it}\left\{R_t\left[s\left(C_{it+1}^S\right)^{-\sigma} + (1-s)\left(C_{it+1}^H\right)^{-\sigma}\right]\right\} \tag{1}$$

Notice that this setting preserves the standard *individual* Euler condition. However, at the *aggregate* level, there will be a discounted Euler condition.

The intratemporal optimality condition of the household $i \in S$ problem is

$$\mathbb{E}_{it}W_t^r = \left(C_{it}^S\right)^{\sigma}\left(N_{it}^S\right)^{\varphi} \tag{2}$$

which is the optimal labor supply decision.

Constrained households. In contrast, a share λ of households is financially constrained. They are banned from asset markets and do not have access to firm dividends, but they still have an intratemporal decision on how much labor to supply, and receive the taxed share of firm profits as government transfers, $\frac{\tau_D}{\lambda}E_t$. Formally, household $i \in H$ only faces an intratemporal labor decision,

$$\mathbb{E}_{it}W_t^r = \left(C_{it}^H\right)^{-\sigma}\left(N_{it}^H\right)^{\varphi} \tag{3}$$

which is the optimal labor supply decision.

2.1.2. Aggregate Consumption Function

The following proposition summarizes the aggregate consumption function for each household type.

PROPOSITION 1. *The log-linearized aggregate consumption functions for households of type S and H at time t are*

$$\begin{aligned} c_t^S = &-\frac{\beta}{\sigma}\sum_{k=0}^{\infty}\beta^k\overline{\mathbb{E}}_t^c r_{t+k} - (1-s)\frac{\varphi}{\varphi+\sigma}\left(\frac{1-\tau_D}{1-\lambda} - \frac{\tau_D}{\lambda}\right)\sum_{k=1}^{\infty}\beta^k\overline{\mathbb{E}}_t^c e_{t+k} \\ &+ (1-\beta)\sum_{k=0}^{\infty}\beta^k\mathbb{E}_{it}\left[\frac{1+\varphi}{\varphi+\sigma}\overline{\mathbb{E}}_t^c w_{t+k}^r + \frac{\varphi}{\varphi+\sigma}\frac{1-\tau_D}{1-\lambda}\overline{\mathbb{E}}_t^c e_{t+k}\right] \end{aligned} \tag{4}$$

$$c_t^H = \frac{1+\varphi}{\varphi+\sigma}\overline{\mathbb{E}}_t^c w_t^r + \frac{\varphi}{\varphi+\sigma}\frac{\tau_D}{\lambda}\overline{\mathbb{E}}_t^c e_t \tag{5}$$

where a lower case variable denotes the logarithm of the capital letter variable, $x_t = \log X_t$, *and* $\overline{\mathbb{E}}_t^c(\cdot) = \int_0^1 \mathbb{E}_{it}(\cdot)\ di$ *is the cross-sectional average forecast across households.*

PROOF. See Appendix A. □

I can write the current aggregate consumption of the S type as a function of future streams of the real interest rate and future aggregate income of the S and H types. On the other hand, the consumption function of the H type depends on the current aggregate wage rate and the current share of transfers they receive.

Conditions (4)-(5) have been derived without assuming a particular belief structure, I have simply not applied the Law of Iterated Expectations (LIE) at the aggregate level. Therefore, it should be interpreted as a *general* aggregate consumption function. Notice also that I have replaced the standard FIRE expectation operator by $\overline{\mathbb{E}}_t^c(\cdot)$, the average expectation operator for households.

2.2. Closing the Model

Fiscal and Monetary Policy. I assume that the government, which conducts fiscal and monetary policy, does not face any information friction. In fiscal terms, on top of the aforementioned optimal production subsidy, it conducts a redistribution scheme: it taxes profits from unconstrained households and rebates the proceedings to the constrained. In log-linear terms, $e_t^S = \frac{1-\tau_D}{1-\lambda}e_t$ and $e_t^H = \frac{\tau_D}{\lambda}e_t$. Furthermore, in the only demand-side setup, monetary policy is conducted in reduced-form via an exogenous AR(1) real interest rate process

$$r_t = \rho r_{t-1} + \sigma_\varepsilon \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,1) \tag{6}$$

The Dynamic IS Curve. As in the textbook NK, the demand curve can be summarized as a single equation; but it cannot be collapsed into a first-order expectational difference equation. The hierarchy of beliefs prevents the LIE from holding at the aggregate level, and the system representation is given by the following proposition.

PROPOSITION 2. *The individual average-household-level DIS curve is given by*

$$c_{it} = -\frac{\beta}{\sigma}(1-\lambda)\mathbb{E}_{it}r_t + [1-\beta(1-\lambda\chi)]\mathbb{E}_{it}y_t + \beta[\delta(1-\lambda\chi)-1]\mathbb{E}_{it}c_{t+1} + \beta\mathbb{E}_{it}c_{i,t+1} \tag{7}$$

and the aggregate DIS curve can be written as

$$(8) \qquad y_t = -\frac{\beta}{\sigma}(1-\lambda)\sum_{k=0}^{\infty}\beta^k\overline{\mathbb{E}}_t r_{t+k} + [1-\beta(1-\lambda\chi)]\overline{\mathbb{E}}_t y_t + (\delta-\beta)(1-\lambda\chi)\sum_{k=1}^{\infty}\beta^k\overline{\mathbb{E}}_t y_{t+k}$$

where $\chi = 1+\varphi\left(1-\frac{\tau_D}{\lambda}\right)$ *measures the degree of amplification with respect to RANK (if* $\chi > 1$ *there is an amplification and if* $\chi < 1$ *there is lessening), and* $\delta = 1+\frac{(\chi-1)(1-s)}{1-\lambda\chi}$ *measures the degree of compounding at the consumer's Euler condition (if* $\delta > 1$ *there is compounding and if* $\delta < 1$ *there is discounting).*

PROOF. See Appendix A. □

Again, conditions (7)-(8) are derived under a general information structure, in which I relax the assumption that the aggregate household expectation operator satisfies the LIE and where agents do not observe aggregate variables. Each household's decision (7) can be described as a beauty contest in which it needs to forecast current real interest rates and future output, which in turn depend on each other households' actions.

Note that, given that the inverse of the Frisch elasticity is strictly positive ($\varphi > 0$), $\chi > 1$ if $\tau_D < \lambda$. As I will show below, there is an amplification of the effects of real interest rate changes if $\chi > 1$ or if income inequality is countercyclical ($\tau_D < \lambda$), and a dampening otherwise. Almgren et al. (2022) find empirical evidence for the amplification effects of real interest rate, and I, therefore, focus on the case $\chi > 1$, which in turn implies $\delta > 1$. Under FIRE, $\delta > 1$ (coming from the precautionary savings motive) induces compounding in the aggregate DIS curve.

Absent information frictions, first-order beliefs coincide with higher-order beliefs and one can simplify the above expression by making use of the LIE and obtain

$$(9) \qquad y_t = -\frac{1}{\nu}\mathbb{E}_t r_t + \delta\mathbb{E}_t y_{t+1} = -\frac{1}{\nu}\sum_{k=0}^{\infty}\delta^k\mathbb{E}_t r_{t+k}$$

where $\nu = \sigma\frac{1-\lambda\chi}{1-\lambda}$. It can be seen that $\delta > 1$ induces compounding at the aggregate DIS curve. A counterfactual consequence of compounding is that the FGP is exacerbated. In the FIRE benchmark, one cannot have any amplification and cure the FGP simultaneously (without including aggregate risk). This is a situation that Bilbiie (2021) denominates *Catch-22*.

The beyond FIRE frmework solves the *Catch-22*. In this case there is discounting in the aggregate DIS curve even if the individual Euler condition preserves compounding due to precautionary savings. Aggregate outcomes depend on expectations, which move sluggishly due to an endogenous anchoring to priors. This anchoring in expectations translates into both intrinsic persistence in outcomes and myopia about the future. I show in section 3.2 that this myopia is sufficiently large to outweigh the compounding induced by the precautionary savings motive.

2.3. Information Structure and Equilibrium Dynamics

I now describe the information structure. I assume that households do not observe the fundamental shock and are uncertain about the state of nature. Every period, each agent receives a dose of private information on the aggregate fundamental. Formally, there is a collection of private Gaussian signals, one per agent and per period. In particular, the period–t signal received by household i is given by

$$x_{it} = r_t + \sigma_u u_{it}, \qquad u_{it} \sim \mathcal{N}(0,1). \tag{10}$$

where $\sigma_u \geq 0$ parameterizes the noise in the private signal.

Equilibrium Dynamics. The equilibrium dynamics must satisfy the individual-level optimal policy functions (7), and rational expectation formation should be consistent with the real interest rate process (6) and the signal process (10).

In this class of global games in which there is a signal about the stochastic fundamental, the literature has extensively used the Kalman filter to solve for optimal expectation updating, a form of Bayesian learning. A caveat is that it requires knowledge of the dynamics of the forecasted variables. Instead, the Wiener-Hopf filter can be used to solve for the optimal updating solution in closed-form without knowing the equilibrium dynamics of the forecasted variable (Huo and Takayama 2018).[10] I show in Proposition 3 that the solution to the fixed point is simply an AR(2) process.

PROPOSITION 3. *In equilibrium, aggregate output obeys the following law of motion*

$$y_t = \vartheta y_{t-1} - \left(1 - \frac{\vartheta}{\rho}\right) \frac{1}{\nu(1-\rho\delta)} r_t \tag{11}$$

where ϑ *is a scalar that is given by the reciprocal of the largest roots of the polynomial of the*

[10] For a detailed derivation of the Wiener-Hopf filter see Appendix H in Gallegos (2023).

following matrix

$$\mathcal{P}(z) \equiv (\beta - z)(z - \rho)\left(z - \frac{1}{\rho}\right) - \frac{\sigma_\varepsilon^2}{\sigma_u^2 \rho}\beta(1 - \lambda\chi)z(\delta - z)$$

PROOF. See Appendix A. □

In this framework, ϑ governs information frictions. When the signal noise is high enough such that the signals are completely uninformative, ϑ reaches its maximum value of ρ. The beyond FIRE model produces intrinsic persistence without assuming habit formation, and equilibrium dynamics are more persistent, and less sensitive to real interest rate changes, as a result of sluggish expectations and imperfect attention. On the contrary, when the signals are perfectly informative, $\vartheta = 0$. In that case, which is simply the FIRE NK model, the model dynamics are given by $y_t = -\frac{1}{\nu(1-\rho\delta)} r_t$.

3. Applications and Additional Insights

In this section, I study the different implications of the HANK beyond FIRE economy by conducting several policy experiments. I exploit the two main frictions, financial and informational, and explain their joint interaction and consequences. In particular, I explain the key role of PE vs. GE effects and how these are affected by financial frictions, I show that the model solves the FGP, and I obtain the effect of an "animal spirits" shock. In section 4, the model is extended with a supply side and a Taylor rule, and I show that the Taylor Principle is relaxed in the economy beyond FIRE (with the determinacy region widened).

Table 1 reports the parameters used in the different analyses. All these values are standard in the literature. The first block contains the standard RANK parameters. The discount factor β, the Calvo inaction probability θ, the intertemporal rate of substitution σ, the inverse Frisch elasticity φ, and the autocorrelation ρ and the variance of the real interest rate shock σ_ε^2 have standard values in the literature, taken from Bilbiie (2021).

The second block contains the parameters related to household financial heterogeneity. These are taken from Bilbiie (2021) and include the probability of being financially restricted s, set to match the quarterly autocorrelation of the income process in Guvenen et al. (2014), the profit tax rate τ_D and the share of HtM λ, jointly set to match the aggregate MPC and the amplification magnitude in Kaplan et al. (2018).

The third block contains the parameters related to imperfect information. The informational friction in our HANK beyond FIRE setting depends on how precise are

Parameter	Description	Value	Source
β	Discount factor	0.99	Bilbiie (2021)
θ	Calvo probability	0.75	Bilbiie (2021)
σ	Intertemporal elasticity of substitution	1	Bilbiie (2021)
φ	Inverse Frisch elasticity	1	Bilbiie (2021)
σ_ε^2	Variance of shock	1	Bilbiie (2021)
ρ	Autocorrelation of real interest rates shock	0.8	Bilbiie (2021)
τ_D	Profit tax rate	0.19	Bilbiie (2021)
λ	Share of HtM	0.37	Bilbiie (2021)
s	Pr(unconstrained$_{t+1}$|unconstrained$_t$)	0.96	Bilbiie (2021)
σ_u^2	Consumer signal innovation variance	2.98	Coibion and Gorodnichenko (2015)

TABLE 1. Parameter values.

the signals that consumers receive. Coibion and Gorodnichenko (2015) focus on annual inflation (GDP Deflator) expectations and regress the ex-ante average forecast error, computed as the difference between the realized variable at $t+3$ and the expectation at time t of that variable at $t+3$, $\pi_{t+3,t} - \mathbb{F}_t\pi_{t+3,t}$, on the average forecast revision, defined as the change in the forecast of a variable at time $t+3$ formed at time t minus the forecast of that same variable formed at time $t-1$, $\mathbb{F}_t\pi_{t+3,t} - \mathbb{F}_{t-1}\pi_{t+3,t}$,

$$\text{forecast error}_t = \beta_\pi \text{revision}_t + u_t \tag{12}$$

I match the underrevision coefficient of households in Coibion and Gorodnichenko 2015 (using data on forecasts from the Michigan Survey of Consumers), $\widehat{\beta}_\pi = 0.705$.[11] For this purpose, I obtain the model-implied coefficient in our HANK beyond FIRE, $\beta_\pi^{\mathcal{M}}$. The following proposition serves that purpose.

PROPOSITION 4. *In our beyond FIRE framework the regression coefficient $\beta_\pi^{\mathcal{M}}$ is given by*

$$\begin{aligned}\beta_\pi^{\mathcal{M}} &= \frac{\mathbb{C}(\pi_{t+3,t} - \overline{\mathbb{E}}_t^c\pi_{t+3,t}, \overline{\mathbb{E}}_t^c\pi_{t+3,t} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+3,t})}{\mathbb{V}(\overline{\mathbb{E}}_t^c\pi_{t+3,t} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+3,t})} \\ &= \frac{\lambda_u^3}{(\rho-\lambda_u)(1+\lambda_u+\lambda_u^2+\lambda_u^3)(\vartheta-\lambda_u)}\left[\frac{\lambda_u\vartheta(1-\lambda_u^2)(1+\vartheta)(1+\vartheta^2)(1-\rho\vartheta)}{1-\lambda_u\vartheta}\right.\end{aligned}$$

[11]To be precise, Coibion and Gorodnichenko (2015) estimate a variant of the above regression that does not include forecast revisions (because the dataset does not permit the calculation) and include oil price changes in an IV setup. However, they show that for the case of firms, in which they can perform both estimations, the estimated coefficients are nearly identical.

$$+ (1+\lambda_u^2)\{(\rho-\lambda_u)[\vartheta(1+\lambda_u) - \lambda_u(1-\lambda_u\vartheta)] - \rho\lambda_u^2(1+\lambda_u)(1-\lambda_u\vartheta)\}\Bigg] \tag{13}$$

where λ_u *is the inside root of the polynomial* $\mathcal{D}(z) \equiv (1-\rho z)(\rho - z) - \frac{\sigma_\varepsilon^2}{\sigma_u^2} z$.

PROOF. See Appendix A. □

Note that λ_u and ϑ are endogenous to the signal precision σ_u. I calibrate σ_u by minimizing the square distance between the model-implied coefficient $\beta_\pi^{\mathcal{M}}$ and the estimated coefficient in Coibion and Gorodnichenko (2015). This implies that $\sigma_u^2 = 2.9766$.

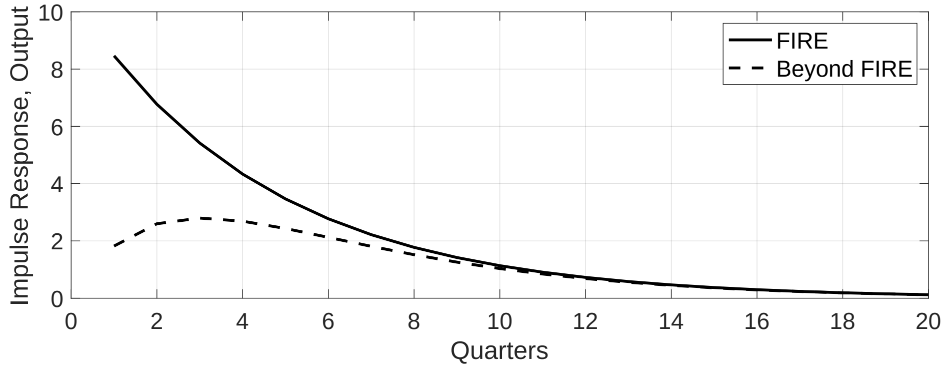
3.1. Response after a Real Interest Rate Shock

The HANK beyond FIRE differs from the textbook NK in two dimensions: household heterogeneity (*HA*) and information frictions. To isolate the effects of both frictions, I study these separately.

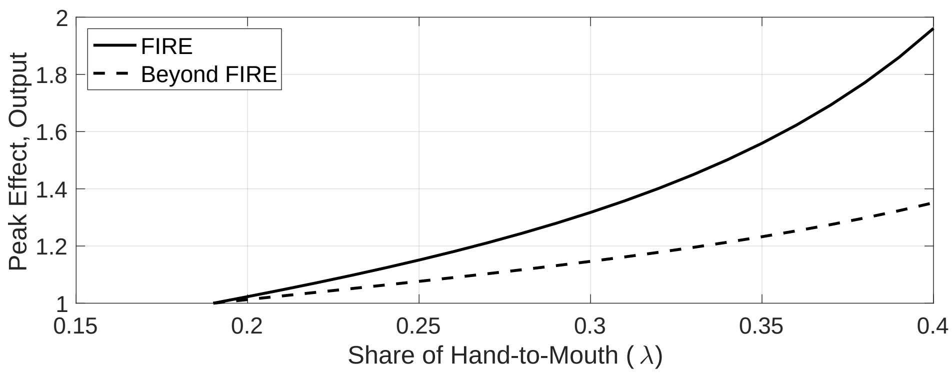
Impulse Response Function. I plot the impulse response of output after a real interest rate shock in the FIRE economy in figure 1A (solid line). The peak response occurs on impact, due to the lack of intrinsic persistence.[12] Once I consider information frictions (dashed line), the IRFs have the hump-shaped dynamics observed in the data (Christiano et al. 2005; Ramey 2016) without compromising the individual (monotonically decreasing) responses to income shocks documented in Fagereng et al. (2019).

Amplification. Bilbiie (2008, 2021) finds that, under plausible parametric assumptions, adding HtM households amplifies the response of aggregate variables to monetary shocks. The proposed transmission mechanism works as follows. Unconstrained households change their consumption choice after a real interest rate shock (according to their individual Euler condition), which in turn affects aggregate demand. Because wages are fully flexible, they adjust to the new schedule. This is how real interest rate

[12] Christiano et al. (2005) show that introducing consumption habits, investment adjustment costs and price indexation helps produce hump-shaped IRFs. The micro estimates for these frictions are lower than those required in the macro literature, usually calibrated to match empirical IRFs. On top of this, empirical evidence suggests that *individual* consumption responses following an income shock have a monotonically decreasing pattern, which makes the consumption habits channel counterfactual (see Fagereng et al. (2019)).



A. Output dynamics after a 100 b.p. real interest rate shock in the FIRE (solid line) and Beyond FIRE (dashed line) frameworks.



B. Amplification multiplier with respect to RANK in the FIRE (solid line) and Beyond FIRE (dashed line) frameworks.

FIGURE 1. Theoretical Dynamics of Output.

shocks affect the HtM. Because they have a unity MPC, they will consume all income change from wages and will magnify any change in aggregate demand.

In figure 1B, I plot the ratio between the output response to a real interest rate shock under a given HtM share, and the output response under no amplification ($\tau_D = \lambda$), for different degrees of HtM shares. Consider first the FIRE benchmark (solid line). The HtM transmission channel is present: output responds *more* to real interest rate shocks the larger the share of HtM agents, λ. For the benchmark calibration $\lambda = 0.37$, the peak output response is 69.28% larger than without financial frictions. Under information frictions (dashed line), the amplification effect of HtM agents is still present but partially muted. A larger degree of financial frictions leads to a larger response of output to real interest rate shocks, but the multiplier is smaller than in the FIRE case. For the benchmark calibration, the peak output response is 27.48% larger than without financial frictions. The HtM mechanism, which operates through general equilibrium dynamics, is partially muted by dispersed information.

PE vs. GE. The amplification effect of HtM agents is present but dampened by information frictions. To interpret this result, it is key to understand that the transmission mechanism proposed by Bilbiie (2008, 2021) relies heavily on GE effects, when constrained agents act. In this framework, agents need to forecast the exogenous fundamental (the real interest rate shock) and aggregate output. While the information friction environment complicates the forecast of the fundamental, it does not give rise to any higher-order beliefs since its realization does not depend on others' beliefs and actions. On the contrary, predicting aggregate output leads to higher-order beliefs: agents need to infer what others believe since its realization hinges on their actions. These higher-order beliefs, which are more anchored to priors at each increasing order, increase the intrinsic persistence of the GE dimension. As a result, aggregate dynamics are driven by PE effects in the initial periods. Over time, agents learn that a (persistent) real interest rate shock has occurred, and the aggregate dynamics rely more and more on GE effects until the PE vs. GE share converges to the FIRE benchmark.

To formalize this result, I decompose the total response in the DIS curve (8) into partial equilibrium (direct) and general equilibrium (indirect) effect components:

$$y_t = \underbrace{-\frac{\beta}{\sigma}(1-\lambda)\sum_{k=0}^{\infty}\beta^k\overline{\mathbb{E}}_t r_{t+k}}_{\text{PE effect}} + \underbrace{[1-\beta(1-\lambda\chi)]\overline{\mathbb{E}}_t y_t + (\delta-\beta)(1-\lambda\chi)\sum_{k=1}^{\infty}\beta^k\overline{\mathbb{E}}_t y_{t+k}}_{\text{GE effect}} \tag{14}$$

In IRF terms, output at time $\tau \in \{t, t+1, t+2, ...\}$ after a real interest rate shock at time t can be written in terms of the two PE and GE components, $\text{IRF}_{t,\tau} = \partial y_\tau / \partial \varepsilon_t = \partial \text{PE}_\tau / \partial \varepsilon_t + \partial \text{GE}_\tau / \partial \varepsilon_t$. Defining the PE share at time τ as $\mu_\tau = \text{PE}_\tau / (\text{PE}_\tau + \text{GE}_\tau)$, the following proposition provides the PE share μ_τ beyond FIRE.

PROPOSITION 5. *Beyond FIRE, the time-varying PE share μ_τ is given by*

$$\mu_\tau = \frac{\beta(1-\lambda\chi)(1-\rho\delta)}{1-\rho\beta}\,\frac{\rho^{\tau+1}-\lambda_u^{\tau+1}}{\rho^{\tau+1}-\vartheta^{\tau+1}}$$

PROOF. See Appendix A □

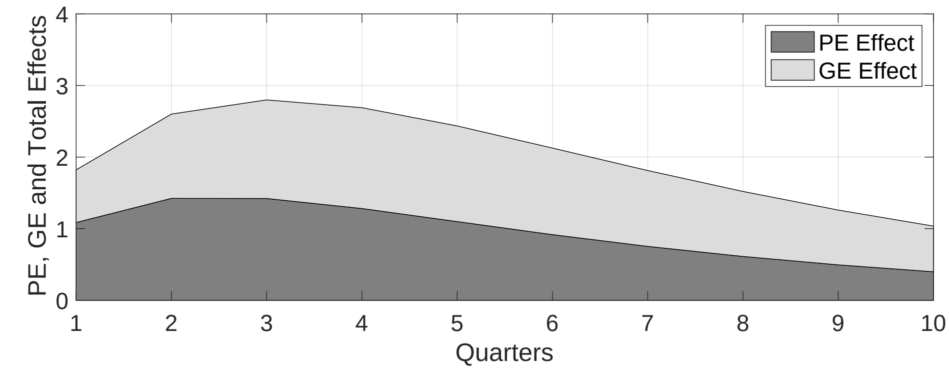
I plot the aggregate output response, the PE response (grey shaded region), and the GE response (light grey shaded region) after a real interest rate shock in figure 2A (figure 2B reports the same dynamics in the FIRE economy). GE effects are arrested in the first periods compared to the FIRE benchmark, consistent with the empirical findings in Holm et al. (2021). Therefore, amplification, which nourishes from the GE dimension, is partially muted. Figure 2C reports the PE share μ_τ (solid line) at each τ period after the real interest rate shock, together with the PE share under no information frictions (dashed line). The GE share beyond FIRE is lower than in FIRE, and mutes the amplification multiplier coming from HtM households.

To summarize, information frictions reduce the degree of complementarity of actions across agents, although the amplification mechanism is still present in the model. Higher-order uncertainty effectively arrests and slows down the GE effect.

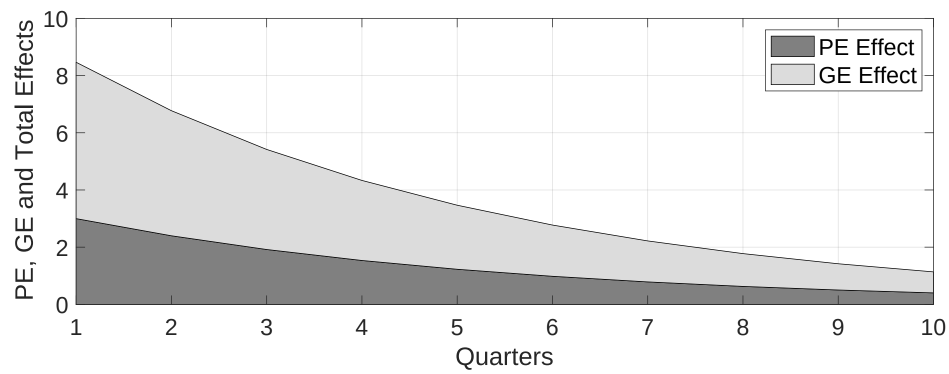
3.2. Forward Guidance

A documented failure of the standard NK model is the *Forward Guidance Puzzle*. Forward guidance is an unconventional monetary policy tool used by central banks in a situation in which the nominal interest rate (their main policy tool) is stuck at zero so that further expansionary conventional policy is unfeasible. The central bank commits to keep the nominal interest rates low (relative to what their Taylor rule would mandate), in the hope of unanchoring the inflation expectations and output. Several central banks made use of it in the recent financial crisis (see Angeletos and Sastry 2020 for a more comprehensive treatment).

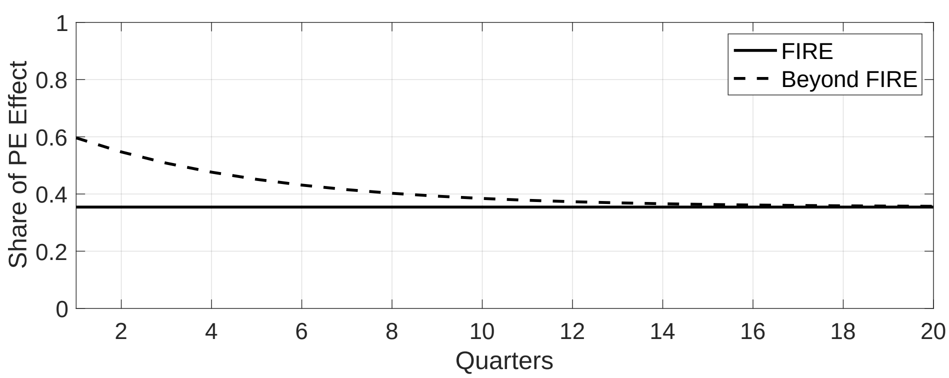
The (excessively) forward-looking standard NK model predicts that a forward guidance τ–shock (i.e., a promise at time t to shock the economy in period $\tau \geq t$ by using the real interest rate) has *the same (or more) effect* the more into the future it is promised. This is easily verified from the FIRE DIS curve (9) iterated forward. In the standard NK, $\nu = \sigma$, $\delta = 1$, and $y_t = -1/\sigma \sum_{k=0}^{\infty} \mathbb{E}_t r_{t+k}$. Any future shock on the real interest rate (a forward guidance shock) has an *identical* impact on today's output, irrespective of when is it realized. This is aggravated in the case of financial constraints, since the precautionary savings motive and amplification induce compounding ($\delta > 1$), making the process explosive: the further into the future that the shock takes place, the larger is the increase in the output gap today (solid line in figure 3). This is the situation that Bilbiie (2021) denominates Catch-22: a realistic amplification of monetary policy effects aggravates the FGP. It is, however, wishful thinking that this policy tool is *so* effective. Del Negro et al. (2012) study this empirically and find that forward guidance is indeed less effective than what the theoretical model suggests.



A. PE, GE and Total effect beyond FIRE.



B. PE, GE and Total effect under FIRE.



C. PE share μ_τ over time.

FIGURE 2. Total, Direct and Indirect Effects.

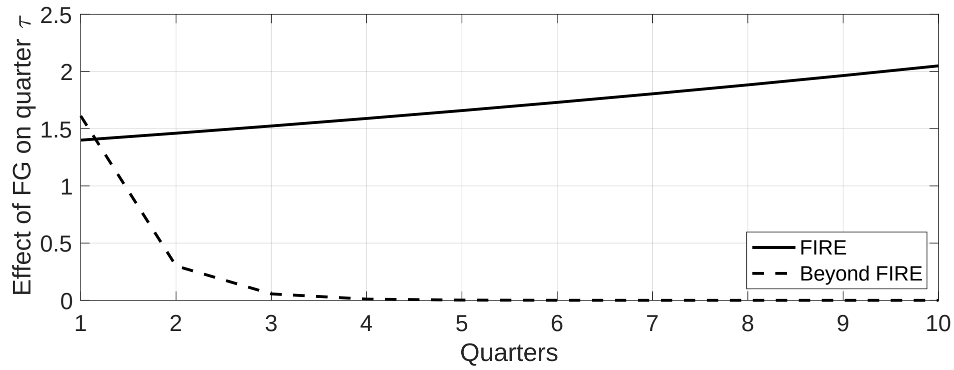


FIGURE 3. The Effect of Forward Guidance on current Output.

Consider a situation in which the economy is stuck in a liquidity trap. Suppose that the zero lower bound (ZLB) for nominal interest rates is binding between periods t and τ, such that $\tau \geq t$. I show in Proposition 6 that information frictions induce intrinsic persistence and myopia at the aggregate level, as discussed in Angeletos and Lian (2018); Angeletos and Huo (2021). This result is sufficient to cure the FGP, whilst the amplification result is maintained.

PROPOSITION 6. (*i*) *The ad-hoc equilibrium dynamics*

$$y_t = \omega_b y_{t-1} + \delta\omega_f \mathbb{E}_t y_{t+1} - \frac{1}{\nu} r_t \tag{15}$$

produce identical dynamics to the dispersed information model if

$$\omega_b = \frac{\rho\vartheta(1-\delta\vartheta)}{\rho^2 - \vartheta^2}, \qquad \omega_f = \frac{\rho^2\delta - \vartheta}{\delta(\rho^2 - \vartheta^2)} \tag{16}$$

(*ii*) *Dispersed information cures the FGP if one of the roots of the polynomial* $\Omega(x) \equiv \delta\omega_f x^2 - x + \omega_b$ *lies outside the unit circle, and the other root lies inside the unit circle. Furthermore, the effect of forward guidance at period* τ *on consumption in period* t *is given by*

$$FG_{t,t+\tau} = \frac{\partial y_t}{\partial \mathbb{E}_t r_{t+\tau}} = -\frac{\zeta}{\omega_b \nu}\left(\zeta\frac{\delta\omega_f}{\omega_b}\right)^{\tau}$$

where $\zeta \in (0,1)$ *is the only inside root of the polynomial* $\Omega(x)$.

PROOF. See Appendix A. □

To study the effect of forward guidance, I first rewrite the equilibrium dynamics under FIRE. Proposition 6 (*i*) delivers the ad-hoc dynamics (15), which under a certain pair (ω_b, ω_f) is observationally equivalent to the beyond FIRE dynamics (11). Dispersed information adds intrinsic persistence and myopia in the DIS curve: compared to (9), intrinsic persistence is added by introducing an additional lagged term, ω_b, and myopia is introduced by the coefficient ω_f.[13] Part (*ii*) derives the output response *today* of an expected real interest rate shock at time $t + \tau$. Notice that the FGP is only solved if $\zeta \in (0, 1)$ and the other root lies is greater than 1. Using the quadratic formula, these two conditions are met when $\delta\omega_f + \omega_b < 1$. Using (16), this can be written as $\rho(\rho - \vartheta) + \vartheta(1 - \vartheta) > \rho\delta(\rho - \vartheta^2)$, which is satisfied if the degree of information frictions is sufficiently large. Under the parameterization in table 1, $\omega_b = 0.705$ and $\omega_f = 0.1552$, which satisfies the restriction. In Figure 9B I plot the impact of a forward guidance shock in period τ on today's output for each τ under FIRE (solid line) and beyond FIRE (dashed line). The FGP is cured, so the further into the future the forward guidance is executed, the lesser the effect.

3.3. Beliefs Shock

What is the effect of an "animal spirits" shock? The benchmark model does not allow for this exercise, since a shock to an individual signal does not have any effect on aggregate variables. In this section, I replace private information with public information and obtain the model dynamics after a shock to the common signal. Instead of the individual signal, all agents receive a common and public noisy signal informing them of the real interest rate shock v_t. Formally, there is a collection of public Gaussian signals, one per period and common across agents. In particular, the period–t signal received by all agents is given by

$$z_t = r_t + \sigma_\epsilon \epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0, 1) \tag{17}$$

where $\sigma_\epsilon \geq 0$ parameterizes the noise in the common signal. The rest of the model is unchanged. The following proposition summarizes the equilibrium dynamics under public information.

[13] In the FIRE NK model, $\omega_b = 0$ and $\omega_f = 1$, and the DIS curve is reduced to (9).

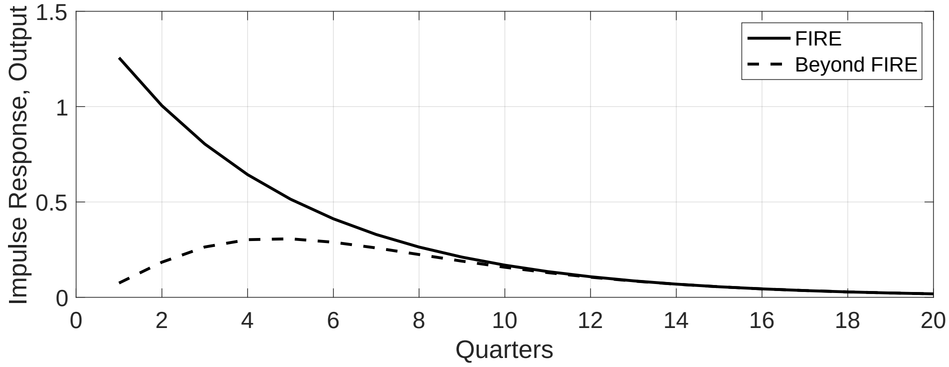


FIGURE 4. Impulse response of output after a 100 b.p. real interest rate shock under FIRE (solid line), beyond FIRE (dashed line), and after a belief shock (dashed-dotted line).

PROPOSITION 7. *In equilibrium, aggregate output obeys the following law of motion*

$$y_t = \widetilde{\vartheta} y_{t-1} - \left(1 - \frac{\widetilde{\vartheta}}{\rho}\right) \frac{1}{\nu(1-\rho\delta)} r_t - \left(1 - \frac{\widetilde{\vartheta}}{\rho}\right) \frac{1}{\nu(1-\rho\delta)} \epsilon_t \tag{18}$$

where $\widetilde{\vartheta}$ *is a scalar that is given by the reciprocal of the largest roots of the polynomial of the following matrix*

$$\mathcal{P}_{public}(z) \equiv (\beta - z)(z - \rho)\left(z - \frac{1}{\rho}\right) - \frac{\sigma_\varepsilon^2}{\sigma_\epsilon^2 \rho} \beta(1 - \lambda\chi) z(\delta - z)$$

PROOF. See Appendix A. □

The new equilibrium dynamics now contain an additional contemporaneous exogenous shock ϵ_t. This term can be interpreted as a belief or "animal spirits" shock. Both shocks have identical effects on impact on aggregate variables, given that agents cannot completely disentangle the noise and the fundamental shock from the signal. However, since the belief shock, ϵ_t is purely transitory, it has fewer long-lasting effects than the real interest rate shock (see figure 4, dashed-dotted line). However, although the belief shock is purely transitory, it produces persistent effects on output over time. This is the result of having imperfectly informed agents, which cannot immediately differentiate between a belief shock and a true real interest rate shock.[14]

[14]To produce this figure I set the public signal noise to $\sigma_\epsilon = \sigma_u$.

4. The Full Analytical HANK Beyond FIRE Model

So far I have only considered the demand side of the economy. Since the real interest rate is assumed to be exogenous, output dynamics are orthogonal to inflation, and I do not need to keep track of firms' decisions. In this section, I decompose the real interest rate into a nominal interest rate part and expected inflation. Additionally, I assume that the nominal interest rate follows a standard Taylor rule, which reacts to inflation and output. Therefore, in this section I explicitly model firms' behavior and revisit the set of results presented in section 3. The economy will be described as a pair of *across–group* dynamic beauty contests between consumers and firms (the inflation-spending NK multiplier), with each group playing a *within-group* dynamic beauty contest (the spending-income multiplier running within the demand block and the strategic complementarity in price-setting running within the supply block).

4.1. Firms and the Phillips Curve

Households consume an aggregate basket of goods $j \in \mathcal{I}_f = [1, 2]$, which takes the form of the standard CES aggregator, $C_t = \left(\int_1^2 C_{jt}^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}}$, where $\epsilon > 1$ is the constant elasticity of substitution between different good varieties. Cost minimization from the final good firm implies that the demand from each good is $C_{jt+k} = \left(P_{jt}/P_{t+k}\right)^{-\epsilon} C_{t+k}$, where P_{jt}/P_t is good j's price in relative terms to the aggregate price index, $P_t = \left(\int_1^2 P_{jt}^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$. Each good is produced by an intermediate monopolistic firm that uses technology linear in labor $Y_{jt} = N_{jt}$.

Aggregate Price Dynamics. As in the benchmark NK model, price rigidities take the form of Calvo-lottery friction. In every period, each firm can reset its price with probability $(1-\theta)$, independent of the time of the last price change. That is, only a measure $(1-\theta)$ of firms can reset their prices in a given period, and the average duration of a price is given by $1/(1-\theta)$. Such an environment implies that the aggregate price dynamics are given (in log-linear terms) by $\pi_t = \int_{\mathcal{I}_f} \pi_{jt}\, dj = (1-\theta)\left[\int_{\mathcal{I}_f} p_{jt}^*\, dj - p_{t-1}\right] = (1-\theta)\left(p_t^* - p_{t-1}\right)$.

Optimal Price Setting. A firm re-optimizing in period t will choose the price P_{jt}^* that maximizes the current market value of the profits generated while the price remains effective. Formally, $P_{jt}^* = \arg\max_{P_{jt}} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_{jt}\left\{\Lambda_{t,t+k}/P_{t+k}\left[P_{jt}Y_{j,t+k|t} - \mathcal{C}_{t+k}(Y_{j,t+j|t})\right]\right\}$, subject to the sequence of the demand schedules $Y_{j,t+k|t} = \left(P_{jt}/P_{t+k}\right)^{-\epsilon} Y_{t+k}$, where $\Lambda_{t,t+k} \equiv \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}$ is the stochastic discount factor, $\mathcal{C}_t(\cdot)$ is the (nominal) cost function, and $Y_{j,t+k|t}$ denotes output in period $t+k$ for a firm j that last reset its price in period t.

Note that, under flexible prices ($\theta = 0$), $P^*_{jt} = \frac{\epsilon}{\epsilon-1} W_t$. Aggregating over firms I obtain the standard result that the aggregate price level is greater than the aggregate marginal cost, due to the markup of monopolistic firms: $P_t = \frac{\epsilon}{\epsilon-1} W_t$. Aggregating the optimal labor supply condition (2) over households, I obtain $N_t^{\varphi} = W_t C_t^{-\sigma}$. Combining the last two conditions, I can write $N_t^{\varphi} C_t^{\sigma} = W_t = \frac{\epsilon-1}{\epsilon} P_t < P_t^{sp} = W_t$, where P_t^{sp} is the price set by a hypothetical social planner. That is, inequality implies that output and employment are below their efficient levels, which comes as a result of monopolistic competition. To solve this suboptimality, the government implements the standard optimal subsidy that induces marginal cost pricing, so that the model is efficient in equilibrium: with the desired markup defined by $P^*_{jt} = \frac{\epsilon}{\epsilon-1}\frac{1}{1-\tau^s} W_t$, the optimal subsidy is $\tau^s = \frac{1}{\epsilon-1}$. The profit function is $D_{jt} = (1+\tau^s)P_{jt}Y_{jt} - W_t N_{jt} - T_t^f$. The subsidy is financed by taxing firms $T_t^f = \tau^s Y_t$, which gives the total profits $D_t = P_t Y_t - W_t N_t$.

The following proposition summarizes the individual pricing rule and the aggregate Phillips curve.

Proposition 8. *The firm-level Phillips curve is given by*

$$\pi_{jt} = \kappa\theta \mathbb{E}_{jt} y_t + (1-\theta)\mathbb{E}_{jt}\pi_t + \beta\theta\mathbb{E}_{jt}\pi_{j,t+1} \tag{19}$$

where $\pi_{jt} = (1-\theta)\left(p^*_{jt} - p_{t-1}\right)$, $\kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}(\sigma+\varphi)$, *and the aggregate Phillips curve can be written as*

$$\pi_t = \kappa\theta \sum_{k=0}^{\infty} (\beta\theta)^k \overline{\mathbb{E}}_t^f y_{t+k} + (1-\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \overline{\mathbb{E}}_t^f \pi_{t+k} \tag{20}$$

where $\overline{\mathbb{E}}_t^f(\cdot) = \int_0^1 \mathbb{E}_{jt}(\cdot)\, dj$ *is the cross-sectional average forecast across firms.*

Proof. See Appendix A. □

Just as in the households' case, conditions (19)-(20) are derived under a general information structure, in which I relax the assumption that the aggregate firm expectation operator satisfies the LIE. Each firm's decision (19) can be described as a beauty contest in which they need to forecast current output and inflation, which in turn depend on each household's and firm's actions and their future optimal action.

4.2. Closing the Model

Fiscal and Monetary Policy. As in section 2, I assume that the government does not face any information friction. On top of the aforementioned optimal subsidy and redistribution scheme, monetary policy is conducted following a Taylor rule of the form

$$i_t = \phi_\pi \pi_t + \phi_y y_t + \nu_t \tag{21}$$

$$\nu_t = \rho \nu_{t-1} + \sigma_\varepsilon \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,1) \tag{22}$$

where the monetary policy shock ν_t follows an AR(1) process, to match the empirically observed inertia in the interest rate.

4.3. Information Structure and Equilibrium Dynamics

Both types of agents, households and firms, are subject to information frictions: they do not observe the fundamental shock and are uncertain about the state of nature. Every period, each agent receives a dose of private information on the aggregate fundamental. Formally, there is a collection of private Gaussian signals, one per agent and per period. In particular, the period–t signal received by agent l in group g is given by

$$x_{lgt} = \nu_t + \sigma_g u_{lgt}, \qquad u_{lgt} \sim \mathcal{N}(0,1) \tag{23}$$

where $g = \{\text{household}, \text{firm}\}$, $\sigma_g \geq 0$ parameterizes the noise in group g. Notice that, by allowing σ_g to differ by g, I accommodate rich information heterogeneity (for example, firms could on average be more informed than households.)

Equilibrium Dynamics. The equilibrium dynamics must satisfy the individual-level optimal policy functions (7) and (19), and rational expectation formation should be consistent with the Taylor rule (21), the exogenous monetary shock process (22) and the signal process (23). I show in Proposition 9 that the solutions to the fixed points is a VARX(1), where the exogenous component is the monetary policy shock.

PROPOSITION 9. *In equilibrium, the aggregate outcome obeys the following law of motion*

$$\boldsymbol{x}_t = A(\vartheta_1, \vartheta_2)\boldsymbol{x}_{t-1} + B(\vartheta_1, \vartheta_2)\nu_t \tag{24}$$

where $\boldsymbol{x}_t = \begin{bmatrix} y_t & \pi_t \end{bmatrix}^\intercal$ *is a vector containing output and inflation,* $A(\vartheta_1, \vartheta_2)$ *is a* 2×2 *matrix* *and* $B(\vartheta_1, \vartheta_2)$ *is a* 2×1 *vector*

$$A = \begin{bmatrix} \frac{\psi_{11}\psi_{22}\vartheta_1 - \psi_{12}\psi_{21}\vartheta_2}{\psi_{11}\psi_{22}-\psi_{12}\psi_{21}} & -\frac{\psi_{11}\psi_{12}(\vartheta_1-\vartheta_2)}{\psi_{11}\psi_{22}-\psi_{12}\psi_{21}} \\ \frac{\psi_{21}\psi_{22}(\vartheta_1-\vartheta_2)}{\psi_{11}\psi_{22}-\psi_{12}\psi_{21}} & -\frac{(\psi_{12}\psi_{21}\vartheta_1-\psi_{11}\psi_{22}\vartheta_2)}{\psi_{11}\psi_{22}-\psi_{12}\psi_{21}} \end{bmatrix}, \quad B = \begin{bmatrix} \psi_{11}\left(1-\frac{\vartheta_1}{\rho}\right) + \psi_{12}\left(1-\frac{\vartheta_2}{\rho}\right) \\ \psi_{21}\left(1-\frac{\vartheta_1}{\rho}\right) + \psi_{22}\left(1-\frac{\vartheta_2}{\rho}\right) \end{bmatrix}$$

where $\{\psi_{gk}\}_{g=1,k=1}^{2}$ *are fixed scalars that depend on deep parameters of the model, satisfying*

(25)

$$\sum_{j=1}^{2} \psi_{1j} = -\frac{1-\rho\beta}{(1-\beta\rho)[\nu(1-\delta\rho)+\phi_y]+\kappa(\phi_\pi-\rho)}, \quad \sum_{j=1}^{2} \psi_{2j} = -\frac{\kappa}{(1-\beta\rho)[\nu(1-\delta\rho)+\phi_y]+\kappa(\phi_\pi-\rho)}$$

and $(\vartheta_1, \vartheta_2)$ *are two scalars that are given by the reciprocal of the two largest roots of the characteristic polynomial of the following matrix*

$$\boldsymbol{C}(z) = \begin{bmatrix} C_{11}(z) & C_{12}(z) \\ C_{21}(z) & C_{22}(z) \end{bmatrix}$$

where

$$C_{11}(z) = \lambda_1 \left\{ (\beta - z)\left(z - \frac{1}{\rho}\right)(z-\rho) + \frac{\sigma_\varepsilon^2}{\rho\sigma_1^2}\beta z \left[z\left(1 - \lambda\chi + \frac{\phi_y(1-\lambda)}{\sigma}\right) - \delta(1-\lambda\chi) \right] \right\}$$

$$C_{12}(z) = -\lambda_1 z \frac{\sigma_\varepsilon^2}{\rho\sigma_1^2}\frac{\beta}{\sigma}(1-\lambda)(1-z\phi_\pi)$$

$$C_{21}(z) = -\lambda_2 z^2 \frac{\sigma_\varepsilon^2}{\rho\sigma_2^2}\kappa\theta$$

$$C_{22}(z) = \lambda_2 \left[(\beta\theta - z)\left(z - \frac{1}{\rho}\right)(z-\rho) + \frac{\sigma_\varepsilon^2}{\rho\sigma_2^2}\theta z (z-\beta) \right]$$

where λ_g, $g \in \{1, 2\}$ *is the inside root of the polynomial* $\boldsymbol{D}(z) \equiv (1-\rho z)(\rho - z) - \frac{\sigma_\varepsilon^2}{\sigma_g^2} z$.

PROOF. See Appendix A. □

The equilibrium dynamics (24) follow a VARX(1) process. In this framework, ϑ_1 and ϑ_2 govern information frictions. When the signal noise is high enough such that the signals are completely uninformative, ϑ_1 and ϑ_2 reach their maximum value of ρ. On the other hand, when the signals are perfectly informative, $\vartheta_1 = \vartheta_2 = 0$. The square coefficient matrix $A(\vartheta_1, \vartheta_2)$ is endogenous to ϑ_1 and ϑ_2 (the roots of its characteristic

Parameter	Description	Value	Source
σ^2_ε	Variance of monetary shock	1	Bilbiie (2021)
ϕ_π	Inflation response in Taylor rule	1.5	Christiano et al. (2005)
ϕ_y	Output response in Taylor rule	0.1	Christiano et al. (2005)
ρ	Autocorrelation of monetary shock	0.8	Christiano et al. (2005)
σ^2_1	Consumer signal innovation variance	3.50	Coibion and Gorodnichenko (2015)
σ^2_2	Firm signal innovation variance	3.50	Coibion and Gorodnichenko (2015)

TABLE 2. Parameter values.

polynomial), and we have $A(0,0) = \mathbf{0}$. In that case, which is simply the FIRE NK model, the model dynamics are given by $\boldsymbol{x}_t = B(0,0)\nu_t$.

Two aspects are worth discussing. First, the beyond FIRE model produces intrinsic persistence, in the sense that $A(\vartheta_1, \vartheta_2) \neq \mathbf{0}$, without assuming habits, adjustment costs, or price indexation. Second, the equilibrium dynamics are less sensitive to monetary policy changes. This is easily verified by comparing $B(\vartheta_1, \vartheta_2)$ and $B(0,0)$: each element in $B(\vartheta_1, \vartheta_2)$ is smaller than each element in $B(0,0)$ (in absolute terms), given that $\{\vartheta_1, \vartheta_2\} \in [0, \rho]^2$.

4.4. Applications and Additional Insights

I study the different implications of the HANK beyond FIRE economy by conducting several policy experiments, revisiting the results in section 3. I exploit the two main frictions, financial and informational, and explain their joint interaction and consequences. In particular, I show that the Taylor Principle is satisfied in the economy beyond FIRE (with the determinacy region widened), I explain the key role of PE vs. GE effects and how these are affected by financial frictions, I show that the model solves the FGP, and I obtain the effect of an "animal spirits" shock.

Table 2 reports the additional parameters used in the different policy analyses. The first block contains the monetary policy parameters. The variance of the monetary policy shock σ^2_ε, taken from Bilbiie (2021), the autocorrelation ρ set to match the empirically observed inertia in the Taylor rule, and the Taylor rule coefficients ϕ_y and ϕ_π to the values used in Christiano et al. (2005).

The second block contains the parameters related to imperfect information. Although the framework is flexible to accommodate heterogeneous signals precision, I restrict attention to households' inflation forecasts and set $\sigma_1 = \sigma_2$ to match the under-revision coefficient of households. In this case, the model-implied coefficient in the full

HANK beyond FIRE, $\beta^{\mathcal{M}}_{c\pi}$ is given by the following proposition.

PROPOSITION 10. *In our beyond FIRE framework the regression coefficient* $\beta^{\mathcal{M}}_{c\pi}$ *is given by*

$$
\begin{aligned}
\beta^{\mathcal{M}}_{c\pi} &= \frac{\mathbb{C}(\pi_{t+3,t} - \overline{\mathbb{E}}^{c}_{t}\pi_{t+3,t}, \overline{\mathbb{E}}^{c}_{t}\pi_{t+3,t} - \overline{\mathbb{E}}^{c}_{t-1}\pi_{t+3,t})}{\mathbb{V}(\overline{\mathbb{E}}^{c}_{t}\pi_{t+3,t} - \overline{\mathbb{E}}^{c}_{t-1}\pi_{t+3,t})} \\
&= \frac{\lambda_1^3}{(\rho - \lambda_1)(1 + \lambda_1 + \lambda_1^2 + \lambda_1^3)\sum_{k=1}^{2}\psi_{2g}\frac{\rho - \vartheta_k}{1 - \lambda_1\vartheta_k}} \times \\
&\times \sum_{g=1}^{2} \frac{(\rho - \vartheta_g)\psi_{2g}}{(1 - \lambda_1\vartheta_g)(\vartheta_g - \lambda_1)} \left[\frac{\lambda_1\vartheta_g(1 - \lambda_1^2)(1 + \vartheta_g)(1 + \vartheta_g^2)(1 - \rho\vartheta_g)}{1 - \lambda_1\vartheta_g} \right. \\
&\left. + (1 + \lambda_1^2)\{(\rho - \lambda_1)[\vartheta_g(1 + \lambda_1) - \lambda_1(1 - \lambda_1\vartheta_g)] - \rho\lambda_1^2(1 + \lambda_1)(1 - \lambda_1\vartheta_g)\} \right]
\end{aligned} \tag{26}
$$

PROOF. See Appendix A. □

Note that the set $(\lambda_1, \vartheta_1, \vartheta_2, \psi_{21}, \psi_{22})$ is endogenous to the signals' precisions σ_1 and σ_2. I calibrate the pair (σ_1, σ_2) by minimizing the square distance between the model-implied coefficient $\beta^{\mathcal{M}}_{c\pi}$ and the estimated coefficient in Coibion and Gorodnichenko (2015). This implies that $\sigma_1^2 = \sigma_2^2 = 3.4989$.

4.4.1. Response after a Monetary Policy Shock

The HANK beyond FIRE differs from the textbook NK in two dimensions: household heterogeneity (*HA*) and information frictions. To isolate the effects of both frictions, I study these separately.

Impulse Response Function. I plot the impulse response of output after a monetary policy shock in the FIRE economy in figure 5A (solid line). The peak response occurs on impact, due to the lack of intrinsic persistence since $\boldsymbol{x}_t = B(0, 0)\nu_t$. Two problems arise. First, the finding that output increases by 1.25 p.p. after a 100 b.p. monetary policy shock is excessive. The empirical macro literature generally presents results in the range of 0.2 – 0.8 b.p. (see e.g. Ramey (2016) for a literature review.) Second, the peak of the IRF occurs on impact, while empirical evidence suggests a hump-shaped IRF. I show in the sequel that information frictions solve these puzzles, reconciling the micro- and macro-econometric evidence. An additional counterfactual prediction of the FIRE framework is that the policy rate increases after an expansionary monetary policy shock (see figure 5C, solid line), as opposed to the decrease found by empirical evidence (Ramey 2016).

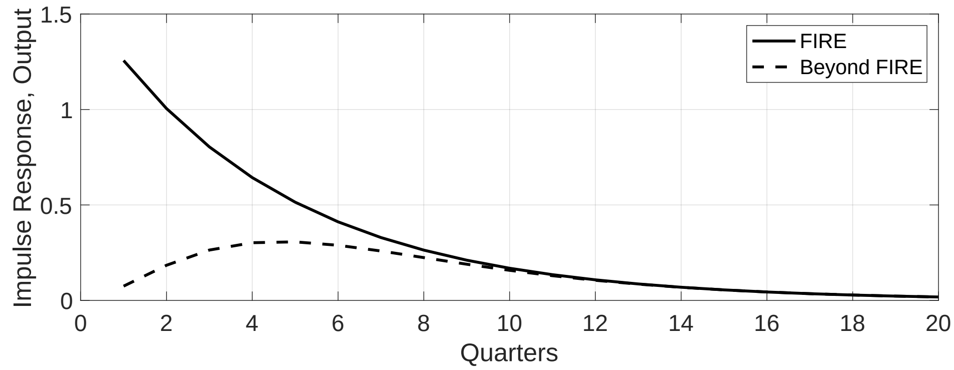
Once I consider information frictions (dashed line), the peak effect in output is around 1/4 of that of the standard framework, around 0.3 p.p. and in line with the findings in Ramey (2016), and the IRFs have the hump-shaped dynamics observed in the data (Christiano et al. 2005; Ramey 2016) without compromising the individual (monotonically decreasing) responses to income shocks documented in Fagereng et al. (2019). Finally, the nominal interest rate decreases after an expansionary monetary policy shock.

Amplification. As argued in section 3, HtM households amplify the response of aggregate variables to monetary shocks.[15] In figure 5B I plot the ratio between the output response to a monetary policy shock under a given HtM share, and the output response under RANK, for different degrees of HtM shares (solid line). The HtM transmission channel is present: output responds *more* to monetary policy shocks the larger the share of HtM agents, λ. For the benchmark calibration $\lambda = 0.37$, the peak output response is 10.28% larger than without financial frictions.

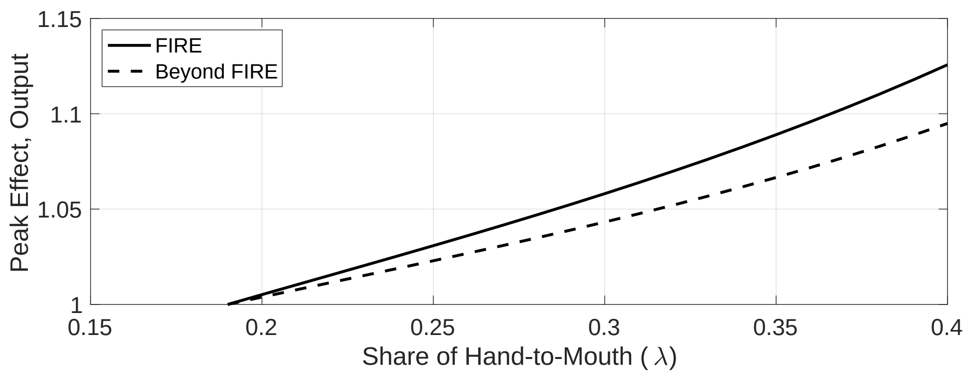
Under information frictions, the amplification effect of HtM agents is still present but partially muted (dashed line). A larger degree of financial frictions leads to a larger response of output to monetary shocks, but the multiplier is smaller than in the FIRE case. For the benchmark calibration $\lambda = 0.37$, the peak output response is 7.72% larger than without financial frictions. The HtM mechanism, which operates through general equilibrium dynamics, is partially muted by dispersed information.

PE vs. GE. The amplification effect of HtM agents is present but dampened by information frictions, which mute the GE dimension. Following (14), I decompose the total response in the DIS curve into partial equilibrium (direct) and general equilibrium (indirect) effect components, with the caveat that what I used to call PE effects are now composed of pure PE effects coming from the monetary shock, the stabilization role of the Taylor rule (21) and inflation expectations. The following proposition provides the PE share μ_τ in the full HANK economy.

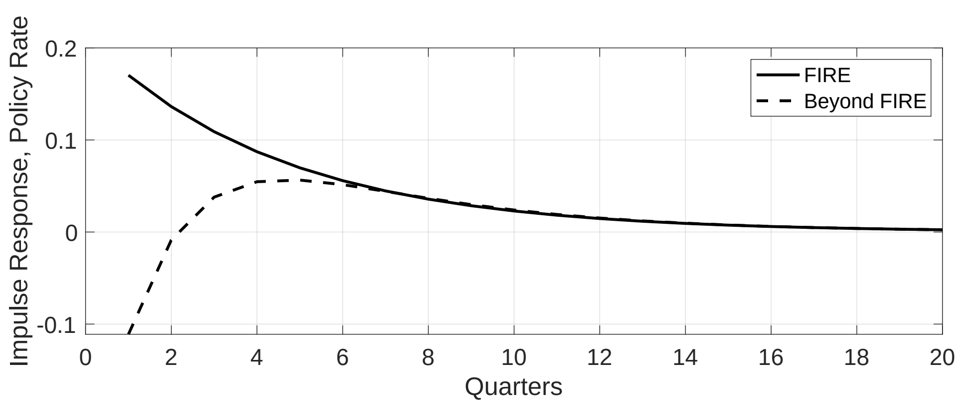
[15] Almgren et al. (2022) test this mechanism in the data. Focusing on euro area economies, which are subject to the same monetary policy shock, find that monetary policy has heterogeneous effects across countries and that the HtM channel drives these results: the larger the share of HtM households in an economy, the larger the effects of monetary policy.



A. Output dynamics after a 100 b.p. monetary policy shock in the FIRE (solid line) and Beyond FIRE (dashed line) frameworks.



B. Amplification multiplier with respect to RANK in the FIRE (solid line) and Beyond FIRE (dashed line) frameworks.



C. Policy rate dynamics after a 100 b.p. monetary policy shock in the FIRE (solid line) and Beyond FIRE (dashed line) frameworks.

FIGURE 5. Theoretical Dynamics of Output.

Proposition 11. *Beyond FIRE, the time-varying PE share* μ_τ *is given by*

$$\mu_\tau = \frac{\rho\left(\sum_{g=1}^{2}\psi_{1g} - \delta_1\right)\rho^\tau - \rho\delta_2\lambda_1^\tau - \sum_{g=1}^{2}(\psi_{1g}\vartheta_g + \delta_{3j})\vartheta_g^\tau}{\rho\sum_{g=1}^{2}\psi_{1g}\rho^\tau - \sum_{g=1}^{2}\psi_{1g}\vartheta_g\vartheta_g^\tau}$$

where

$$\delta_1 = \left\{[1-\beta(1-\lambda\chi)] + (\delta-\beta)(1-\lambda\chi)\beta\frac{\rho}{1-\rho\beta}\right\}\sum_{j=1}^{2}\psi_{1j}$$

$$\delta_2 = [1-\beta(1-\lambda\chi)]\sum_{j=1}^{2}\psi_{1j}\frac{\lambda_1^2(\rho-\vartheta_j)(1-\rho\vartheta_j)}{\rho^2(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}$$

$$+ (\delta-\beta)(1-\lambda\chi)\beta\sum_{j=1}^{2}\psi_{1j}\frac{\lambda_1(\rho-\vartheta_j)\left[\rho\lambda_1(1-\beta\vartheta_j) + \lambda_1\vartheta_j(1-\rho\lambda_1) - \rho\vartheta_j(1-\rho\beta\lambda_1\vartheta_j)\right]}{\rho^2(1-\rho\beta)(1-\beta\vartheta_j)(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}$$

$$\delta_{3j} = -\frac{\psi_{1j}\vartheta_j^2(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho^2(\vartheta_j-\lambda_1)(1-\vartheta_j\lambda_1)}\left\{[1-\beta(1-\lambda\chi)] + (\delta-\beta)(1-\lambda\chi)\beta\frac{\vartheta_j}{1-\beta\vartheta_j}\right\}$$

Proof. See Appendix A □

I plot the aggregate output response, the PE response (grey shaded region), and the GE response (light grey shaded region) after a monetary policy shock in figure 6A (figure 6B reports the same dynamics in the FIRE economy). GE effects are arrested in the first periods compared to the FIRE benchmark, consistent with the empirical findings in Holm et al. (2021). While GE effects depend on the hierarchy of beliefs, with each higher-order belief creating more intrinsic persistence, PE effects depend partially on the expectations of the fundamental, which do not lead to higher-order beliefs. Therefore, amplification, which nourishes from the GE dimension, is partially muted. Figure 6C reports the PE share μ_τ (solid line) at each τ period after the monetary policy shock, together with the PE share under no information frictions (dashed line). The GE share beyond FIRE is lower than in FIRE (except for the initial period), and mutes the amplification multiplier coming from HtM households. As stressed before, the PE effect is contaminated by higher-order beliefs, which results in a non-monotonic PE share over time. The non-monotonic shape of the PE share depends crucially on the hawkishness of the monetary authority. Suppose instead that the central bank is less aggressive with respect to inflation, such that $\phi_\pi = 1$. Figure 7 presents the dynamics in that case, which are monotonically decreasing and closer to those in section 3. Now, suppose that the monetary authority increases ϕ_π. This increase will dampen GE effects, since nominal interest rates will react more to exogenous shocks to provide the desired stabilization. This reduces the degree of strategic complementarities (and increases strategic substitutability), affecting agents' forecasting. Since strategic complementarities are less important, agents rely more on their private signal, and forecasts become less anchored to priors. As a result, forecasts are closer to the FIRE case, in which case GE effects dominate PE effects.

4.4.2. The Taylor Principle beyond FIRE

Extending the demand-side model in section 2 allows us to study the Taylor Principle. As in the standard NK model, the Taylor Principle boils down to studying the determinacy of the system (8), (20), (21) and (22). The equilibrium is indeterminate when the current outcomes are excessively affected by expectations of the future. One should therefore expect, as discussed in Gabaix (2020), that introducing myopia should widen the determinacy region, making the system (8), (20), (21) and (22) stable for a larger set of (ϕ_π, ϕ_y) combinations.

I start discussing the full-information rational-expectations benchmark. I am interested in isolating the role of financial frictions. These are modeled in reduced form by ν and δ. In the empirically factual case of amplification, both terms are greater than unity. Since $\delta > 1$ generates compounding in the DIS curve, the model becomes *more* forward-looking, and the stability region is reduced. To see this formally, I conduct the standard Blanchard and Kahn (1980) analysis in the FIRE case, summarized by the following proposition.

PROPOSITION 12. *The FIRE equilibrium is determinate if*

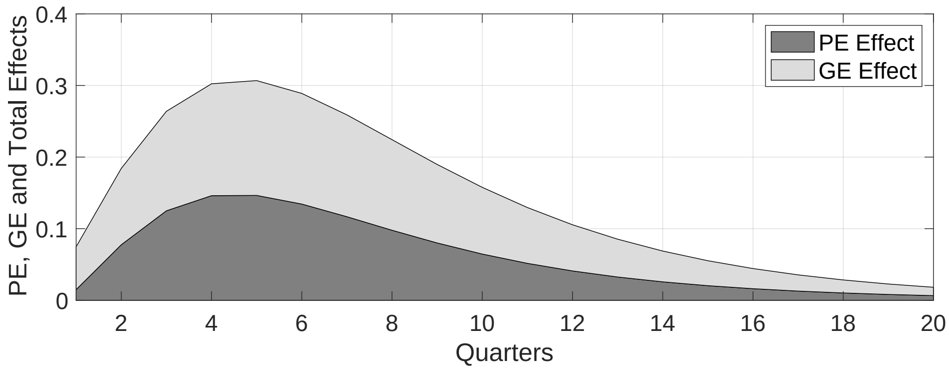
$$(1-\beta\delta) + \frac{1}{\nu}(\kappa\phi_\pi + \phi_y) > 0 \tag{27}$$

$$(1-\beta)(1-\delta) + \frac{1}{\nu}[\kappa(\phi_\pi - 1) + (1-\beta)\phi_y] > 0 \tag{28}$$

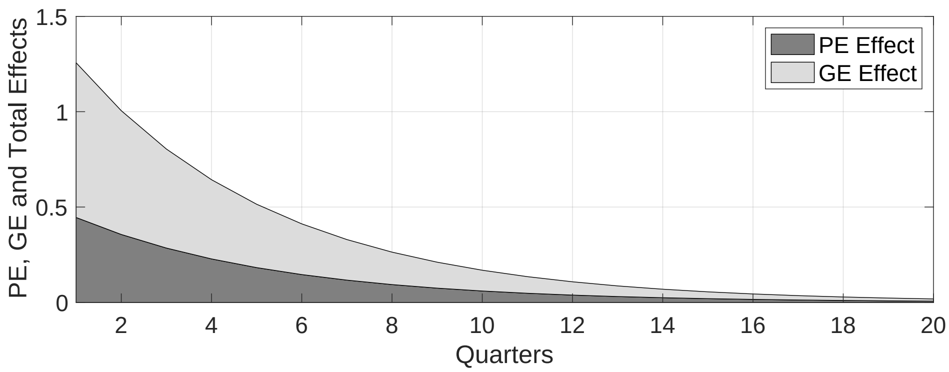
$$(1+\beta)(1+\delta) + \frac{1}{\nu}[\kappa(\phi_\pi + 1) + (1+\beta)\phi_y] > 0 \tag{29}$$

PROOF. See Appendix A. □

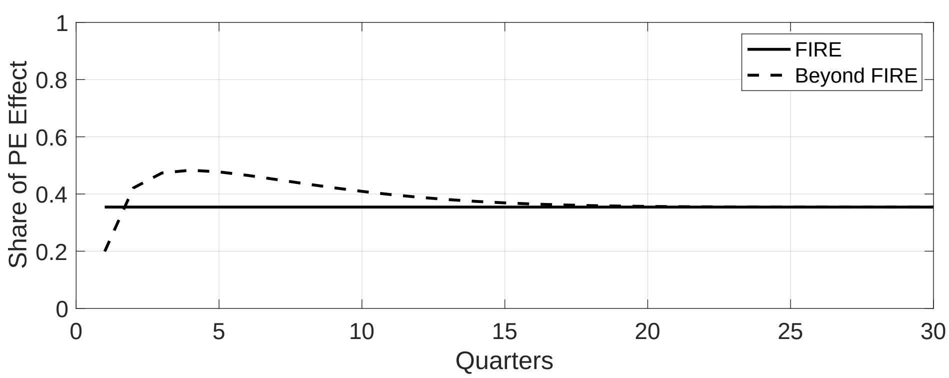
To isolate the role of each of the financial frictions terms, I first compare a TANK model (in which δ is restricted to 1) with the benchmark RANK. In this case, (29) is always satisfied for strictly positive Taylor rule coefficients, and conditions (27)-(28) are



A. PE, GE and Total effect beyond FIRE.

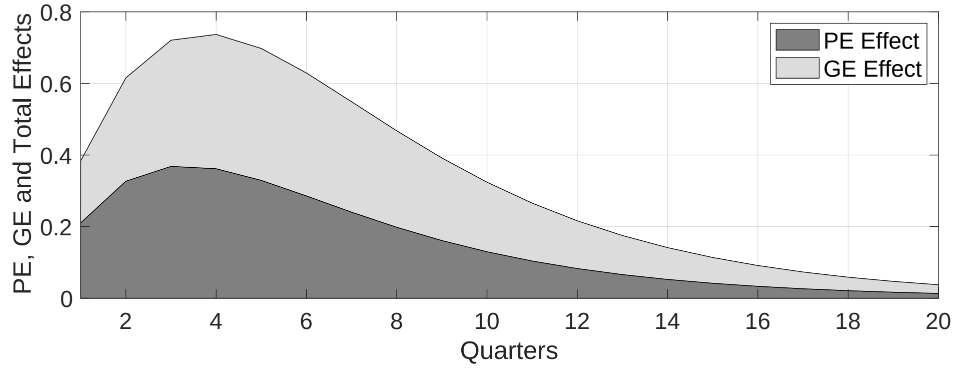


B. PE, GE and Total effect under FIRE.

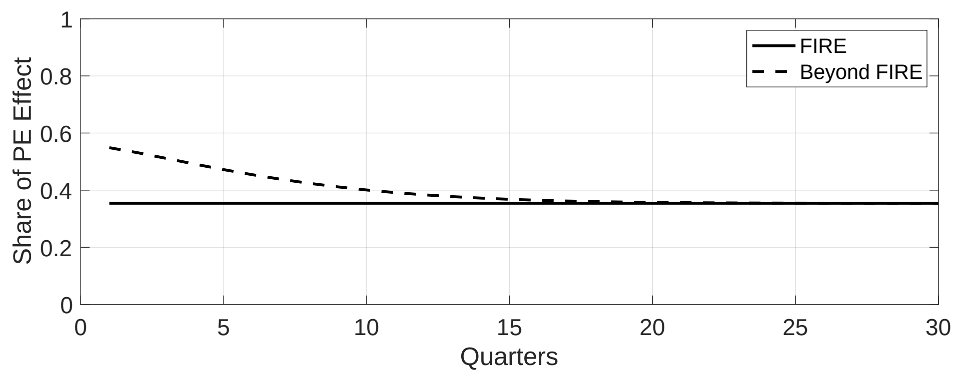


C. PE share μ_τ over time.

FIGURE 6. Total, Direct and Indirect Effects.



A. PE, GE and Total effect beyond FIRE.



B. PE share μ_τ over time

FIGURE 7. Total, Direct and Indirect Effects with $\phi_\pi = 1$.

reduced to

$$1 - \beta + \frac{1}{\nu}(\kappa\phi_\pi + \phi_y) > 0 \tag{30}$$

$$\kappa(\phi_\pi - 1) + (1 - \beta)\phi_y > 0 \tag{31}$$

Condition (31) implies that (30) always holds. In (31), the term ν is completely innocuous for determinacy. As a result, the determinacy region in RANK and TANK is identical. It is ultimately δ, which is the companion of the forward-looking element in (9), that will drive the restrictions on the Taylor Principle. Under HANK (with $\delta > 1$), the determinacy region is reduced. In that case, (29) is always satisfied. As one can see from (27)-(28), $\delta > 1$ is affecting the leftmost term in both equations, making it negative. As a result, the rightmost element on the left-hand side in both conditions needs to be *sufficiently* larger. Precautionary savings are reducing the determinacy region, which I see visually in Figure 8A, through compounding in the individual Euler condition.

I now turn to the beyond FIRE case. Under the parameter values reported in Table 2, I conduct the beyond FIRE equivalent of Blanchard and Kahn (1980), which I summarize in Proposition 13.

PROPOSITION 13. *Equilibrium exists and is unique if*

$$1 - \vartheta_1\vartheta_2 > 0 \tag{32}$$

$$(1 - \vartheta_1)(1 - \vartheta_2) > 0 \tag{33}$$

$$(1 + \vartheta_1)(1 + \vartheta_2) > 0 \tag{34}$$

and ϑ_1 *and* ϑ_2 *are the only two outside roots of polynomial* $\boldsymbol{C}(z)$*, defined in Proposition 9.*

PROOF. See Appendix A. □

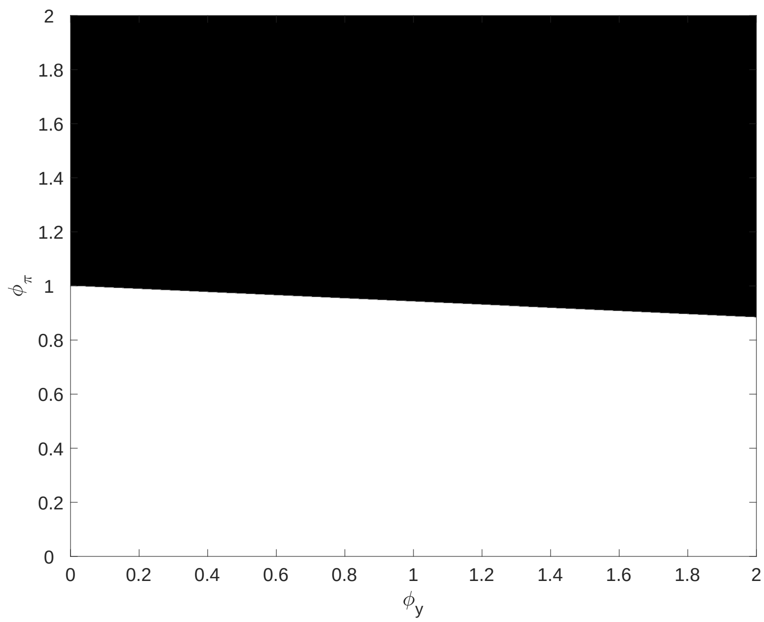
Condition (33) is usually the only one considered in the standard framework since the FIRE equivalent of conditions (32) and (34) is trivially satisfied. In the beyond FIRE framework (32)-(34) are satisfied since $\{\vartheta_g\}_{g=1,2} \in [0, \rho]$. The most restrictive condition is that ϑ_1 and ϑ_2 are the *only* outside roots of polynomial $\boldsymbol{C}(z)$. Note that ϑ_1 and ϑ_2 are endogenously determined by the deep parameters in the model, so that some parameterizations can yield an indeterminacy even if conditions (32)-(34) are met but $C(z)$ contains more than two outside roots. I plot the determinacy regions both beyond FIRE and under FIRE in Figure 8B. Imperfect information widens the determinacy region as a result of aggregate myopia, micro-founded through sluggishness updating of expectations.

4.4.3. Forward Guidance

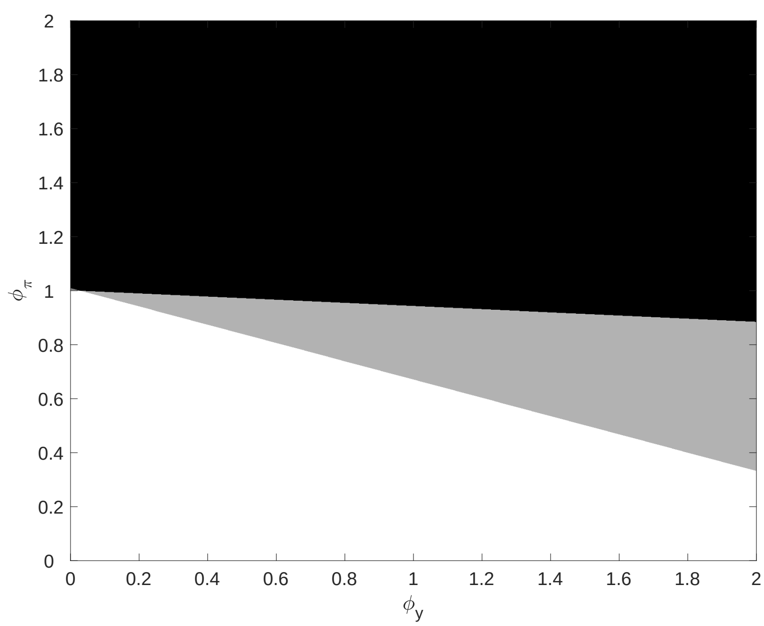
In the benchmark NK model, the Phillips curve is given by

$$\pi_t = \kappa y_t + \beta \mathbb{E}_t \pi_{t+1} \tag{35}$$

the DIS curve is given by (9), and the Taylor rule is given by (21)-(22). Inserting the Taylor rule into the DIS curve, one can write the model as a system of two first-order



A. RANK FIRE vs. HANK FIRE. Determinacy region under FIRE, both RANK and HANK (black), additional determinacy region under RANK (gray), and indeterminacy region (white).



B. HANK FIRE vs. HANK beyond FIRE. Determinacy region under HANK FIRE and HANK beyond FIRE (black), additional determinacy region under HANK beyond FIRE (gray), and indeterminacy region (white).

FIGURE 8. Determinacy regions.

stochastic difference equations, $\widetilde{A}\boldsymbol{x}_t = \widetilde{B}\mathbb{E}_t\boldsymbol{x}_{t+1} + \widetilde{C}v_t$, where

$$\widetilde{A} = \begin{bmatrix} \nu + \phi_y & \phi_\pi \\ -\kappa & 1 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} \nu\delta & 1 \\ 0 & \beta \end{bmatrix}, \quad \text{and} \quad \widetilde{C} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Premultiplying the system by $\widetilde{A}^{-1}$ I obtain

$$\boldsymbol{x}_t = \overline{\boldsymbol{\varphi}}v_t + \overline{\boldsymbol{\delta}}\mathbb{E}_t\boldsymbol{x}_{t+1} \tag{36}$$

where $\overline{\boldsymbol{\delta}} = \widetilde{A}^{-1}\widetilde{B}$ and $\overline{\boldsymbol{\varphi}} = \widetilde{A}^{-1}\widetilde{C}$. I show in Proposition 14 that information frictions induce intrinsic persistence and myopia at the aggregate level, as discussed in Angeletos and Lian (2018); Angeletos and Huo (2021). This result is sufficient to cure the FGP, whilst the amplification result is maintained. Consider a situation in which the economy is stuck in a liquidity trap. Suppose that the ZLB for nominal interest rates is binding between periods t and τ, such that $\tau \geq t$. The following proposition rewrites the DIS curve beyond FIRE in FIRE terms, and proves that there is no FGP anymore.

PROPOSITION 14. (*i*) *The ad-hoc equilibrium dynamics*

$$\boldsymbol{x}_t = \boldsymbol{\omega}_b\boldsymbol{x}_{t-1} + \overline{\boldsymbol{\delta}}\boldsymbol{\omega}_f\mathbb{E}_t\boldsymbol{x}_{t+1} + \overline{\boldsymbol{\varphi}}v_t \tag{37}$$

with $\boldsymbol{\omega}_b = \begin{bmatrix} \omega_{b,11} & \omega_{b,12} \\ \omega_{b,21} & \omega_{b,22} \end{bmatrix}$ *and* $\boldsymbol{\omega}_f = \begin{bmatrix} \omega_{f,11} & \omega_{f,12} \\ \omega_{f,21} & \omega_{f,22} \end{bmatrix}$ *produce identical dynamics to the dispersed information model if* $(\boldsymbol{\omega}_b, \boldsymbol{\omega}_f)$ *satisfy*

$$\begin{aligned} \boldsymbol{\omega}_b &= [I - \overline{\boldsymbol{\delta}}\boldsymbol{\omega}_f A]A \\ B - \overline{\boldsymbol{\varphi}} &= \overline{\boldsymbol{\delta}}\boldsymbol{\omega}_f(A + \rho)B \end{aligned} \tag{38}$$

The DIS curve can be written in FIRE terms as

$$y_t = \omega_{by}y_{t-1} + \omega_{b\pi}\pi_{t-1} - \frac{1}{\nu}(i_t - \mathbb{E}_t\pi_{t+1}) + \omega_{fy}\mathbb{E}_t y_{t+1} + \omega_{f\pi}\mathbb{E}_t\pi_{t+1} \tag{39}$$

where $\omega_{by} = \frac{(\nu+\phi_y)\omega_{b,11}+\phi_\pi\omega_{b,21}}{\nu}$, $\omega_{b\pi} = \frac{(\nu+\phi_y)\omega_{b,12}+\phi_\pi\omega_{b,22}}{\nu}$, $\omega_{fy} = \frac{\nu\delta\omega_{f,11}+\omega_{f,21}}{\nu}$, *and* $\omega_{f\pi} = \frac{\nu\delta\omega_{f,12}+\omega_{f,22}-1}{\nu}$.

(*ii*) *Dispersed information cures the FGP if one of the roots of the polynomial* $\mathcal{Q}(x) \equiv \omega_{fy}x^2 - x + \omega_{by}$ *lies outside the unit circle, and the other root lies inside the unit circle.*

Furthermore, the effect of forward guidance at period τ *on consumption in period t is given by*

$$FG_{t,t+\tau} = \frac{\partial y_t}{\partial \mathbb{E}_t r_{t+\tau}} = -\frac{\zeta}{\omega_{by}} \left(\frac{1}{\nu} + \omega_{f\pi} + \omega_{b\pi}\zeta^2 \frac{\omega_{fy}^2}{\omega_{by}^2} \right) \left(\zeta \frac{\omega_{fy}}{\omega_{by}} \right)^{\tau}$$

where $\zeta \in (0,1)$ *is the only inside root of the polynomial* $\mathcal{Q}(x)$*.*

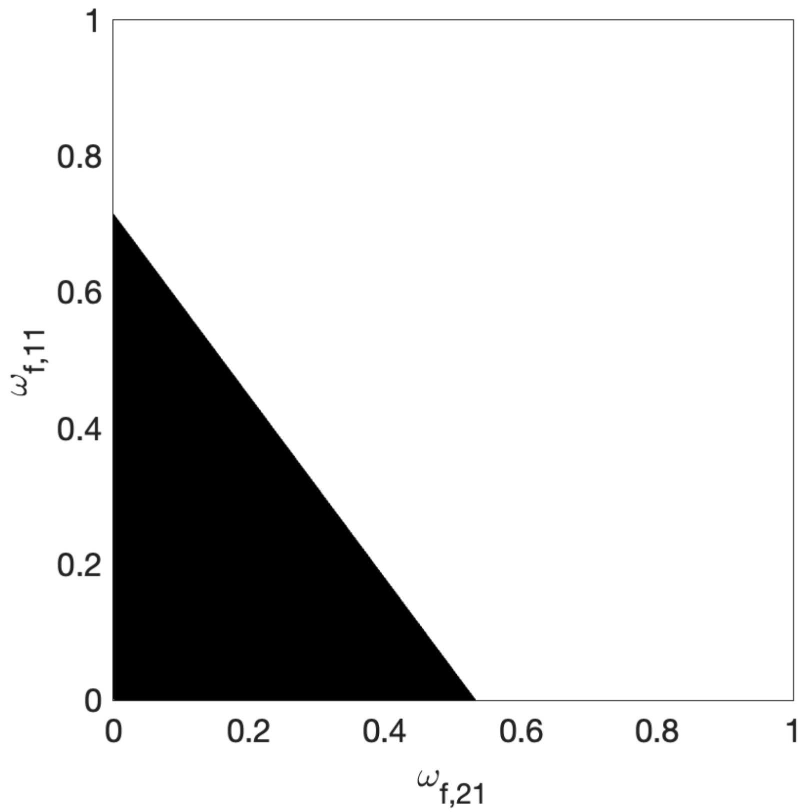
PROOF. See Appendix A. □

In the benchmark NK model with no information frictions, $\omega_{by} = \omega_{b\pi} = \omega_{f\pi} = 0$ and $\omega_{fy} = \delta$, and the DIS curve is reduced to (9). A caveat of the above proposition is that the scalars $\{\omega_{by}, \omega_{b\pi}, \omega_{fy}, \omega_{f\pi}\}$ are not unique, although the dynamics are unique. That is, different weights are consistent with the equilibrium dynamics described by (24). Intuitively, agents' actions can be anchored and/or myopic with respect to aggregate output or inflation, or a combination of both. Hence, to study the dynamics in the Phillips curve and the FGP, the theorist is left with one degree of freedom for each equation. For $\{\omega_{f,11}, \omega_{f,21}\} \in [0,1]^2$, I plot in Figure 9A the space in which the FGP is cured (that is, a polynomial $\mathcal{Q}(x)$ has only one inside root). Only the dark-shaded region is consistent with (38) and cures the FGP.

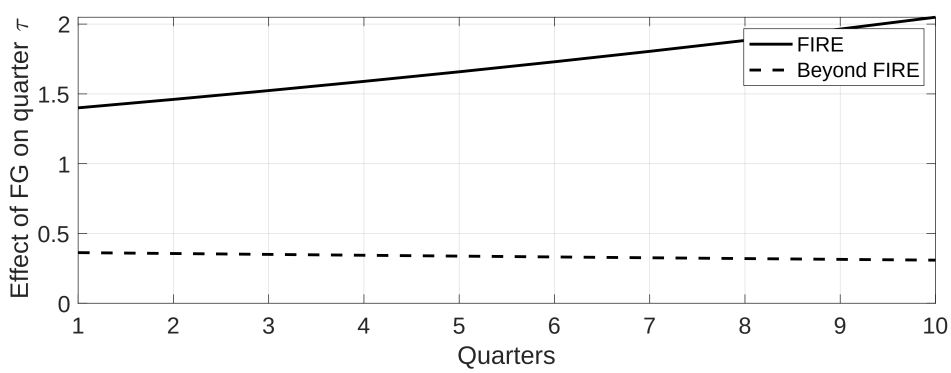
Proposition 14 derives the general DIS curve in FIRE terms. To analyze the effects of forward guidance, consider a situation in which the economy is stuck at the ZLB in which nominal interest rates are binding at the zero constraint for $k \in (t, \tau)$. The ex-ante real interest rate is the (log) inverse of expected inflation, $\mathbb{E}_t r_k = -\mathbb{E}_t \pi_{k+1}$, and the DIS curve (39) becomes

$$y_t = \omega_{by} y_{t-1} + \omega_{b\pi}\pi_{t-1} - \left(\frac{1}{\nu} + \omega_{f\pi} \right) \mathbb{E}_t r_t + \omega_{fy} \mathbb{E}_t y_{t+1} \tag{40}$$

Dispersed information adds intrinsic persistence and myopia in the DIS curve: intrinsic persistence is added both via output and inflation by introducing two additional lagged terms. Myopia is introduced by introducing a term $\omega_{fy} \in (0,1)$. The contemporaneous effect of a real interest rate shock is also diminished since $\omega_{f\pi} < 0$. In Figure 9B I plot the impact of a forward guidance shock in period τ on today's output for each τ under FIRE (solid line) and beyond FIRE (dashed line). The FGP is cured, so the further the forward guidance is implemented, the lesser the effect.



A. FGP cure region (in black) for different values of the degrees of freedom $\{\omega_{f,11}, \omega_{f,21}\} \in [0,1]^2$.



B. The Effect of Forward Guidance on current Output. Results shown for $\omega_{f,11} = \omega_{f,21} = 0.3$.

FIGURE 9. The Effect of Forward Guidance.

4.4.4. Beliefs Shock

Public Information. Consider a collection of public Gaussian signals, one per period and common across agents. In particular, the period–t signal received by all agents, regardless of their group g, is given by

$$z_t = \nu_t + \sigma_\epsilon \epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0,1) \tag{41}$$

where $\sigma_\epsilon \geq 0$ parameterizes the noise in the common signal. The rest of the model is unchanged. The following proposition summarizes the equilibrium dynamics under public information.

PROPOSITION 15. *In equilibrium, the aggregate outcome obeys the following law of motion*

$$\boldsymbol{x}_t = A(\vartheta_1, \vartheta_2)\boldsymbol{x}_{t-1} + B(\vartheta_1, \vartheta_2)\nu_t + B(\vartheta_1, \vartheta_2)\epsilon_t \tag{42}$$

where $(\vartheta_1, \vartheta_2)$ *are two scalars that are given by the reciprocal of the two largest roots of the characteristic polynomial of the following matrix*

$$\boldsymbol{C}(z) = \begin{bmatrix} C_{11}(z) & C_{12}(z) \\ C_{21}(z) & C_{22}(z) \end{bmatrix}$$

where

$$C_{11}(z) = \hat{\lambda}\left\{(\beta - z)\left(z - \frac{1}{\rho}\right)(z-\rho) + \frac{\sigma_\varepsilon^2}{\rho\sigma_\epsilon^2} z\left[z\left(1 + \frac{\phi_y}{\nu}\right) - \delta\right]\right\}$$
$$C_{12}(z) = -\hat{\lambda} z \frac{\sigma_\varepsilon^2}{\rho\sigma_\epsilon^2}\frac{\beta}{\nu}(1 - z\phi_\pi)$$
$$C_{21}(z) = -\hat{\lambda} z^2 \frac{\sigma_\varepsilon^2}{\rho\sigma_\epsilon^2}\kappa\theta$$
$$C_{22}(z) = \hat{\lambda}\left[(\beta\theta - z)\left(z - \frac{1}{\rho}\right)(z-\rho) + \frac{\sigma_\varepsilon^2}{\rho\sigma_\epsilon^2}\theta z(z-\beta)\right]$$

where $\hat{\lambda}$ *is the inside root of the polynomial* $\boldsymbol{D}_\epsilon(z) \equiv (1-\rho z)(\rho - z) - \frac{\sigma_\varepsilon^2}{\sigma_\epsilon^2} z$.

PROOF. See Appendix A. □

The equilibrium dynamics still follow a VARX(1) process, with an additional contemporaneous exogenous shock ϵ_t. This term can be interpreted as a belief or “animal spirits" shock. Both shocks have identical effects on impact on aggregate variables, given that agents cannot completely disentangle the noise and the fundamental shock from the signal. However, since the belief shock, ϵ_t is transitory, it has fewer long-lasting effects than the monetary policy shock (see figure 10A). Although the belief shock is purely transitory, it produces persistent and hump-shaped dynamics of output over time. This is the result of having imperfectly informed agents, which cannot immediately differentiate between a belief shock and a true monetary policy shock. Notice also the different response of the policy rate: after the expansionary monetary policy shock the policy rate decreases. After the non-fundamental belief shock, the central bank raises the interest rates to cool down the economy, which reduces the GE effect of the belief shock (see figure 10B).[16]

Private and Public Information. What if instead of replacing private with public signals, I allow agents to observe both private and public signals? I extend the model to include public information and obtain the model dynamics after a shock to the common signal. On top of the individual signal (23), all agents receive a common and public noisy signal informing them about the monetary policy shock v_t, (41). The following proposition summarizes the equilibrium dynamics under public information.

PROPOSITION 16. *In equilibrium, the aggregate outcome obeys the following law of motion*

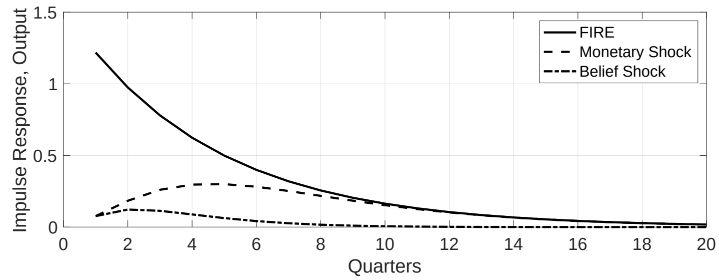
$$\boldsymbol{x}_t = Q_v \sum_{k=0}^{\infty} \Lambda^k \Gamma v_{t-k} + Q_u \sum_{k=0}^{\infty} \Lambda^k \Gamma \epsilon_{t-k} \tag{43}$$

where

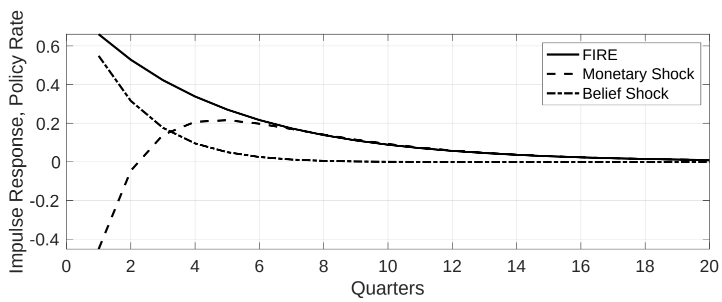
$$Q_v = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}, \quad Q_u = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 - \vartheta_1/\rho \\ 1 - \vartheta_2/\rho \end{bmatrix}$$

where $\{\psi_{gk}, \phi_{gk}\}_{g=1,k=1}^{2}$ *are fixed scalars that depend on deep parameters of the model, and* $(\vartheta_1, \vartheta_2)$ *are two scalars that are given by the reciprocal of the two largest roots of the charac-*

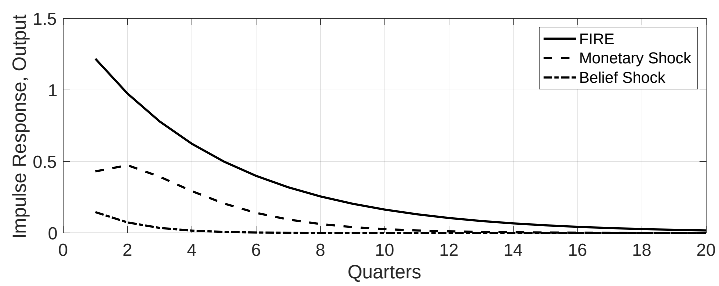
[16] To produce these figures I set the public signal noise to $\sigma_\epsilon = \sigma_1 = \sigma_2$.



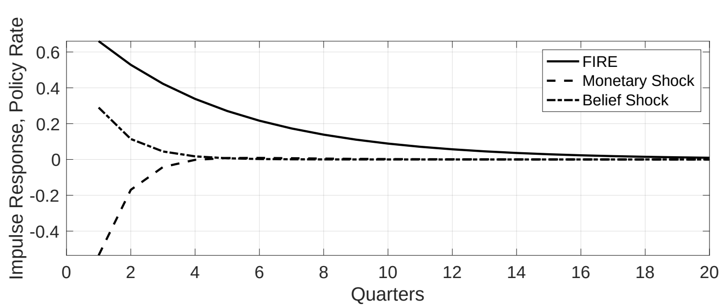
A. Impulse response of output after a 100 b.p. monetary policy shock under FIRE (solid line), beyond FIRE (dashed line), and after a belief shock (dashed-dotted line).



B. Impulse response of the policy rate after a 100 b.p. monetary policy shock under FIRE (solid line), beyond FIRE (dashed line), and after a belief shock (dashed-dotted line).



C. Impulse response of output after a 100 b.p. monetary policy shock under FIRE (solid line), beyond FIRE (dashed line), and after a belief shock (dashed-dotted line).



D. Impulse response of the policy rate after a 100 b.p. monetary policy shock under FIRE (solid line), beyond FIRE (dashed line), and after a belief shock (dashed-dotted line).

FIGURE 10. Output and Policy Rate dynamics with Public Information (figures 10A and 10B) and Public and Private Information (figures 10C and 10D).

teristic polynomial of the following matrix

$$\boldsymbol{C}(z) = \begin{bmatrix} C_{11}(z) & C_{12}(z) & C_{13}(z) & C_{14}(z) \\ C_{21}(z) & C_{22}(z) & C_{23}(z) & C_{24}(z) \\ C_{31}(z) & C_{32}(z) & C_{33}(z) & C_{34}(z) \\ C_{41}(z) & C_{42}(z) & C_{43}(z) & C_{44}(z) \end{bmatrix}$$

where $C_{11}(z) = \beta\left[(1-\lambda\chi)\left(1-\frac{\delta\sigma_\varepsilon^2}{z}\right)+\frac{\phi_y(1-\lambda)}{\sigma}\right]$, $C_{12}(z) = -\frac{\lambda_1\sigma_\varepsilon^4\left\{\beta[\delta(1-\lambda\chi)-1]+z\left[1-\beta\left(1-\lambda\chi+\frac{\phi_y(1-\lambda)}{\sigma}\right)\right]\right\}}{(z-\lambda_1)(1-\lambda_1 z)\rho\sigma_\epsilon^2}$, $C_{13}(z) = \frac{\beta(1-\lambda)\sigma_\varepsilon^2}{\sigma}\left(\phi_\pi-\frac{1}{z}\right)$, $C_{14}(z) = -\frac{\lambda_1\sigma_\varepsilon^4\beta(1-\lambda)(1-\phi_\pi z)}{\sigma(z-\lambda_1)(1-\lambda_1 z)\rho\sigma_\epsilon^2}$, $C_{21}(z) = 0$, $C_{22}(z) = 1-\frac{\beta\sigma_\varepsilon^2}{z}+C_{12}(z)\frac{\sigma_\epsilon^2}{\sigma_1^2}$, $C_{23}(z) = 0$, $C_{24}(z) = C_{14}(z)\frac{\sigma_\epsilon^2}{\sigma_1^2}$, $C_{31}(z) = -\sigma_\varepsilon^2\kappa\theta$, $C_{32}(z) = -\frac{\lambda_2\sigma_\varepsilon^4\kappa\theta z}{(z-\lambda_2)(1-\lambda_2 z)\rho\sigma_\epsilon^2}$, $C_{33}(z) = 1-\sigma_\varepsilon^2\left[1-\theta\left(1-\frac{\beta}{z}\right)\right]$, $C_{34}(z) = -\frac{\lambda_2\sigma_\varepsilon^4(1-\theta)z}{(z-\lambda_2)(1-\lambda_2 z)\rho\sigma_\epsilon^2}$, $C_{41}(z) = 0$, $C_{42}(z) = -\frac{\lambda_2\sigma_\varepsilon^4\kappa\theta z}{(z-\lambda_2)(1-\lambda_2 z)\rho\sigma_2^2}$, $C_{43}(z) = 0$, $C_{44}(z) = 1-\sigma_\varepsilon^2\left[\frac{\beta\theta}{z}+\frac{\lambda_2\sigma_\varepsilon^2(1-\theta)z}{(z-\lambda_2)(1-\lambda_2 z)\rho\sigma_2^2}\right]$, *and* λ_g, $g \in \{1,2\}$ *is the inside root of the polynomial* $\boldsymbol{D}_g(z) \equiv (1-\rho z)(\rho-z)-\frac{(\sigma_g^2+\sigma_\epsilon^2)\sigma_\varepsilon^2}{\sigma_g^2\sigma_\epsilon^2}z$.

PROOF. See Appendix A. □

The first aspect to notice is that the equilibrium dynamics do not follow a VARX(1) process anymore unless $Q_v = Q_u$, which is not generally satisfied. In this case, the two exogenous shocks no longer share the impact effect anymore, since agents can partly disentangle them through the two signals. By introducing an additional signal, I am effectively reducing the degree of information friction that agents face. Even if there is an exogenous shock to the common signal, private signals will be unaffected. As a result, agents will not fully react to the "animal spirits" shock. I find that the effect of the belief shock is smaller than before, the monetary policy shock is more powerful and the produced dynamics are closer to the standard FIRE dynamics (see Figure 10C). The policy rate dynamics are qualitatively similar to the only public information case (see Figure 10D).

To summarize, adding public information reduces information frictions, which in turn dampens the effect of any belief shock and enlarges the effect of monetary policy shocks.

5. Conclusion

I study the transmission channel of monetary policy in HANK economies. The amplification result in the FIRE benchmark relies on financially constrained households being immediately affected after a monetary policy shock through the GE effects. By relaxing the FIRE assumption, I show that a framework with dispersed information results in a different PE vs. GE share than in standard FIRE models, and is consistent with recent empirical evidence. By introducing dispersed information, the GE effects are dampened in the initial periods, thus reducing the magnitude of the multiplier.

I use the theory to shed some light on other questions. I find that the framework produces hump-shaped IRFs without resorting to ad-hoc micro-inconsistent adjustment costs in habits, pricing, or investment decisions. Instead, I micro-found aggregate sluggishness through expectation formation sluggishness, for which the literature has found empirical evidence. I also show that dispersed information produces *as if* myopia, which extends the equilibrium determinacy region and is crucial for the solution of the forward guidance puzzle. Finally, I find that purely transitory "animal spirits" shocks can generate large and persistent effects on output.

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Appendix A. Proofs of Propositions

Proof of Proposition 1. An unconstrained agent $i \in S$ chooses consumption, asset holdings, and leisure solving the standard intertemporal problem: $\max E_{i0} \sum_{t=0}^{\infty} \beta^t U\left(C_{it}^S, N_{it}^S\right)$, subject to the sequence of constraints:

(A1)
$$B_{it} + \Omega_{i,t+1} V_t \leq Z_{it} + \Omega_{it}\left(V_t + P_t D_t\right) + W_t N_{it}^S - P_t C_{it}^S$$

where C_{it}^S, N_{it}^S are consumption and hours worked, B_{it} is the nominal value at end of period t of a portfolio of all state-contingent assets held, except for shares in firms. Likewise for Z_{it}, beginning of period wealth. V_t is average market value at time t of shares, D_t their real dividend payoff and Ω_{it} are share holdings. The absence of arbitrage implies that there exists a stochastic discount factor $Q_{i,t,t+1}$ such that the price at t of a portfolio with an uncertain payoff at $t+1$ is (for state-contingent assets and shares respectively, for an agent i who participates in those markets):

(A2)
$$B_{it} = \mathbb{E}_{it}\left[Q_{i,t,t+1} Z_{i,t+1} \frac{P_t}{P_{t+1}}\right] \quad \text{and} \quad 1 = \mathbb{E}_{it}\left[Q_{i,t,t+1}\left(\frac{P_t}{P_{t+1}} \frac{V_{t+1}}{V_t} + \frac{P_t}{V_t} D_{t+1}\right)\right]$$

which iterated forward gives the fundamental pricing equation: $1 = \mathbb{E}_{it}\left[\frac{P_t}{V_t} \sum_{k=1}^{\infty} Q_{i,t,t+k} D_{t+k}\right]$. The riskless gross short-term real interest rate R_t is a solution to $1 = \mathbb{E}_{it}\left(R_t Q_{i,t,t+1}\right)$. Note that for nominal assets, the nominal interest rate satisfies $1 = \mathbb{E}_{it}\left(\frac{P_t}{P_{t+1}} I_t Q_{i,t,t+1}\right)$. Substituting the no-arbitrage conditions (A2) into the wealth dynamics equation (A1) gives the flow budget constraint. Together with the usual no-borrowing limit for each state, and anticipating that in equilibrium all agents will hold a constant fraction of the shares (there is no trade in shares) Ω_i, whose integral across agents is 1, this implies the usual intertemporal budget constraint $\mathbb{E}_{it}\left[\frac{P_t}{P_{t+1}} Q_{i,t,t+1} X_{i,t+1}\right] \leq \mathbb{E}_{it}\left[X_{it} + W_t N_{it}^S - P_t C_{it}^S\right]$, where $\mathbb{E}_{it} X_{it} = \mathbb{E}_{it}\left[Z_{it} + \Omega_i\left(V_t + P_t D_t\right)\right] = \mathbb{E}_{it}\left[Z_{it} + \Omega_i\left(\sum_{k=0}^{\infty} P_t Q_{i,t,t+k} D_{t+k}\right)\right]$ and

(A3)
$$\mathbb{E}_{it} \sum_{k=0}^{\infty} Q_{i,t,t+k} C_{i,t+k}^S \leq \mathbb{E}_{it}\left[\frac{X_{it}}{P_t} + \sum_{k=0}^{\infty} Q_{i,t,t+k} \frac{W_{t+k}}{P_{t+k}} N_{i,t+k}^S\right] = \mathbb{E}_{it} \sum_{k=0}^{\infty} Q_{i,t,t+k} Y_{i,t+k}^S$$

with $Y_{i,t+k}^S = \Omega_i D_{t+k} + \frac{W_{t+k}}{P_{t+k}} N_{i,t+k}^S$ is the income of agent i. Maximizing utility subject to this constraint gives the first-order necessary and sufficient conditions at each date and in each state, $\beta \frac{U_C\left(C_{i,t+1}\right)}{U_C(C_t)} = Q_{i,t,t+1}$, along with (A3) holding with equality (or flow budget constraint holding with equality and transversality conditions ruling out Ponzi games be satisfied: $\lim_{k \to \infty} \mathbb{E}_{it}\left[Q_{i,t,t+k} Z_{i,t+k}\right] = \lim_{k \to \infty} \mathbb{E}_{it}\left[Q_{i,t,t+k} V_{t+k}\right] = 0\Big)$. Using (A3) and the functional form of the utility function the short-term nominal interest rate must obey $1 = \beta \mathbb{E}_{it}\left[R_t \frac{U_C\left(C_{i,t+1}^S\right)}{U_C\left(C_{it}^S\right)}\right]$. Denote by small letter log deviations from steady-state, except for rates of return (where they denote absolute deviations). Notice that $Q_{t,t+k} = \beta^k \frac{U_C\left(C_{i,t+k}^S\right)}{U_C\left(C_{it}^S\right)}$ and in steady state: $Q_k = \beta^k$. Thus

I have $q_{i,t,t+k} = \ln \frac{Q^S_{i,t,t+k}}{Q^S_{ik}} = \ln \frac{U_C\left(C^S_{i,t+k}\right)}{U_C\left(C^S_{it}\right)} = -\sigma\left(c^S_{i,t+k} - c^S_{it}\right)$ where $q_{i,t,t+k} = q_{i,t,t+1} + q_{i,t+1,t+2} + \ldots + q_{i,t+k-1,t+k}$. Using the stochastic discount factor notation, I can write the unconstrained Euler condition as $\frac{1}{\sigma}q^S_{t,t+1} = c^S_{it} - s\mathbb{E}_{it}c^S_{i,t+1} - (1-s)\mathbb{E}_{it}c^H_{i,t+1}$. Iterating forward the above condition,

$$c^S_{it} = s^k\mathbb{E}_{it}c^S_{t+k} - \sum_{j=0}^{k-1}\left[\frac{1}{\sigma}\mathbb{E}_{it}q^S_{t,t+1} + (1-s)\mathbb{E}_{it}c^H_{i,t+j}\right] \tag{A4}$$

Using the definition of the stochastic discount factor, I can write $\frac{1}{\sigma}q^S_{t,t+k} = c^S_{it} - s\mathbb{E}_{it}c^S_{i,t+1} - (1-s)\mathbb{E}_{it}c^H_{i,t+1} + c^S_{i,t+1} - s\mathbb{E}_{it}c^S_{i,t+2} - (1-s)\mathbb{E}_{it}c^H_{i,t+2} + \cdots + c^S_{i,t+k-1} - s\mathbb{E}_{it}c^S_{i,t+k} - (1-s)\mathbb{E}_{it}c^H_{i,t+k}$, and I can thus write $\frac{1}{\sigma}\mathbb{E}_{it}q^S_{t,t+k} = c^S_{it} + (1-s)\mathbb{E}_{it}\sum_{j=1}^{k}(c^S_{i,t+k} - c^H_{i,t+k})$.

Log-linearizing the intertemporal budget constraint around a steady state with no shocks nor information frictions, zero profits, and no inequality, $C^S = C^H$

$$\sum_{k=0}^{\infty}\beta^k c^S_{it+k} = \sum_{k=0}^{\infty}\beta^k y^S_{i,t+k} \tag{A5}$$

Adding $\sigma^{-1}\mathbb{E}_{it}q^S_{t,t+k}$ on each side

$$\sum_{k=0}^{\infty}\beta^k\mathbb{E}\left[\frac{1}{\sigma}q^S_{t,t+k} + c^S_{it+k}\right] = \sum_{k=0}^{\infty}\beta^k\mathbb{E}_{it}\left[\frac{1}{\sigma}q^S_{t,t+k} + y^S_{it+k}\right] \tag{A6}$$

Using the iterated Euler condition (A4), the LHS is reduced to

$$\frac{1}{1-\beta}c^S_{it} + \frac{1-s}{1-\beta}\sum_{k=1}^{\infty}\beta^k\mathbb{E}_{it}(c^S_{i,t+k} - c^H_{i,t+k}) = \frac{1}{\sigma}\sum_{k=0}^{\infty}\beta^k\mathbb{E}_{it}q^S_{t,t+k} + \sum_{k=0}^{\infty}\beta^k\mathbb{E}_{it}y^S_{it+k} \tag{A7}$$

I can also write $\sum_{k=0}^{\infty}\beta^k\mathbb{E}_{it}q^S_{t,t+k} = -\sum_{k=1}^{\infty}\beta^k\sum_{j=0}^{k-1}\mathbb{E}_{it}r_{t+k} = -\frac{\beta}{1-\beta}\sum_{k=0}^{\infty}\beta^k\mathbb{E}_{it}r_{t+k}$ Hence, I can write the consumption policy function as

$$c^S_{it} = -(1-s)\sum_{k=1}^{\infty}\beta^k\mathbb{E}_{it}(c^S_{i,t+k} - c^H_{i,t+k}) - \frac{\beta}{\sigma}\sum_{k=0}^{\infty}\beta^k\mathbb{E}_{it}r_{t+k} + (1-\beta)\sum_{k=0}^{\infty}\beta^k\mathbb{E}_{it}y^S_{it+k} \tag{A8}$$

I assume that the government implements an optimal steady-state subsidy such that there are zero profits and perfect consumption insurance in steady state, $\tau^S = (\epsilon-1)^{-1}$, and that the government implements a redistribution scheme by taxing profits, τ_D. Log-linearizing the budget constraints

$$c^S_{it} = w_t + n^S_{it} + \frac{1-\tau_D}{1-\lambda}e_t = y^S_{it} \tag{A9}$$

$$c^H_{it} = w_t + n^H_{it} + \frac{\tau_D}{\lambda}e_t = y^H_{it} \tag{A10}$$

Using the intratemporal labor supply conditions

$$\mathbb{E}_{it} w_t^r = \sigma c_{it}^S + \varphi n_{it}^S \tag{A11}$$

$$\mathbb{E}_{it} w_t^r = \sigma c_{it}^H + \varphi n_{it}^H \tag{A12}$$

Combining (A9)-(A12), I can write

$$c_{it}^S = \frac{1+\varphi}{\varphi+\sigma}\mathbb{E}_{it} w_t + \frac{\varphi}{\varphi+\sigma}\frac{1-\tau_D}{1-\lambda}\mathbb{E}_{it} e_t \tag{A13}$$

$$c_{it}^H = \frac{1+\varphi}{\varphi+\sigma}\mathbb{E}_{it} w_t + \frac{\varphi}{\varphi+\sigma}\frac{\tau_D}{\lambda}\mathbb{E}_{it} e_t \tag{A14}$$

Hence, I can rewrite the consumption function (A8) as

$$\begin{aligned} c_{it}^S = &-(1-s)\sum_{k=1}^{\infty}\beta^k\left[\frac{\varphi}{\varphi+\sigma}\left(\frac{1-\tau_D}{1-\lambda}-\frac{\tau_D}{\lambda}\right)\mathbb{E}_{it} e_{t+k}\right] - \frac{\beta}{\sigma}\sum_{k=0}^{\infty}\beta^k \mathbb{E}_{it} r_{t+k} \\ &+ (1-\beta)\sum_{k=0}^{\infty}\beta^k \mathbb{E}_{it}\left[\frac{1+\varphi}{\varphi+\sigma}\mathbb{E}_{it} w_{t+k} + \frac{\varphi}{\varphi+\sigma}\frac{1-\tau_D}{1-\lambda}\mathbb{E}_{it} e_{t+k}\right] \end{aligned} \tag{A15}$$

Aggregating across $i \in S$ agents, I can write

$$\begin{aligned} c_t^S = &-(1-s)\sum_{k=1}^{\infty}\beta^k\left[\frac{\varphi}{\varphi+\sigma}\left(\frac{1-\tau_D}{1-\lambda}-\frac{\tau_D}{\lambda}\right)\overline{\mathbb{E}}_t e_{t+k}\right] - \frac{\beta}{\sigma}\sum_{k=0}^{\infty}\beta^k \overline{\mathbb{E}}_t r_{t+k} \\ &+ (1-\beta)\sum_{k=0}^{\infty}\beta^k \mathbb{E}_{it}\left[\frac{1+\varphi}{\varphi+\sigma}\overline{\mathbb{E}}_t w_{t+k} + \frac{\varphi}{\varphi+\sigma}\frac{1-\tau_D}{1-\lambda}\overline{\mathbb{E}}_t e_{t+k}\right] \end{aligned} \tag{A16}$$

□

Proof of Proposition 2. Denote aggregate consumption and aggregate labor supply for the unconstrained household as $C_t^S = \int C_{it}^S \, di, \quad N_t^S = \int N_{it}^S \, di$ and aggregate consumption and aggregate labor supply for the constrained household given by $C_t^H = \int C_{it}^H \, di, \quad N_t^H = \int N_{it}^H \, di$. Equilibrium in the goods market requires that consumption of unconstrained and constrained households equals total consumption $C_t = \lambda C_t^H + (1-\lambda) C_t^S$. Since I consider a closed economy without investment and government spending, the resource constraint is $Y_t = C_t$. Equilibrium in the labor market requires $N_t = \lambda N_t^H + (1-\lambda) N_t^S$. With uniform steady-state hours by normalization ($N^S = N^H = N$), and the fiscal policy inducing $C^S = C^H = C$, the above log-linearized market clearing conditions yields

$$y_t = c_t = \lambda c_t^H + (1-\lambda) c_t^S \tag{A17}$$

$$n_t = \lambda n_t^H + (1-\lambda) n_t^S \tag{A18}$$

Finally, because the final good sector is competitive and observes all relevant prices p_{jt}, I have $p_t = \int p_{jt} \, dj$,

$y_t = \int y_{jt}\, dj = \int n_{jt}\, dj$, and

$$y_t = n_t = \int n_{it}\, di \tag{A19}$$

$$y_t = c_t = \int c_{it}\, di \tag{A20}$$

Combining the (expectation augmented) optimal labor supply condition of unconstrained households (A11) and that of constrained households (A12), and the labor and goods market clearing conditions (A17)-(A18), I can write

$$\overline{\mathbb{E}}_t^c w_t = \sigma \overline{\mathbb{E}}_t^c c_t + \varphi \overline{\mathbb{E}}_t^c n_t = (\varphi + \sigma) \overline{\mathbb{E}}_t^c y_t \tag{A21}$$

where I have used the aggregate market clearing condition in the goods and labor sectors. As is common in NK models without nominal wage rigidities, profits are countercyclical. This results in dividends (and transfers received by firms) being countercyclical. Using the fact that $e_t = -w_t$, I can write (A14) as $c_t^H = \frac{1}{\varphi+\sigma}\left[1 + \varphi\left(1 - \frac{\tau^D}{\lambda}\right)\right] \overline{\mathbb{E}}_t^c w_t = \left[1 + \varphi\left(1 - \frac{\tau^D}{\lambda}\right)\right] \overline{\mathbb{E}}_t^c y_t = \chi \overline{\mathbb{E}}_t^c y_t$. Hence, I can finally write the aggregate consumption function as

$$c_t = (1-\lambda) c_t^S + \lambda c_t^H = -\frac{\beta}{\sigma}(1-\lambda) \sum_{k=0}^{\infty} \beta^k \overline{\mathbb{E}}_t^c r_{t+k} + [1 - \beta(1-\lambda\chi)] \overline{\mathbb{E}}_t^c y_t + (\delta - \beta)(1-\lambda\chi) \sum_{k=1}^{\infty} \beta^k \overline{\mathbb{E}}_t^c c_{t+k} \tag{A22}$$

where $\delta = 1 + \frac{(\chi-1)(1-s)}{1-\lambda\chi}$ and $\nu = \sigma\frac{1-\lambda\chi}{1-\lambda}$. Finally, notice that this is implied by the following beauty-contest game for a representative household i, $c_{it} = -\frac{\beta}{\sigma}(1-\lambda)\mathbb{E}_{it} r_t + [1 - \beta(1-\lambda\chi)]\mathbb{E}_{it} y_t + \beta[\delta(1-\lambda\chi) - 1]\mathbb{E}_t c_{t+1} + \beta \mathbb{E}_{it} c_{i,t+1}$, is equivalent to (A22) provided that $\lim_{T\to\infty} \beta^T \mathbb{E}_{it} c_{i,t+T}$, which is broadly assumed in the literature given $\beta < 1$. □

Proof of Proposition 3. The best response of household i is specified as follows

$$a_{it} = \varphi_u \mathbb{E}_{it} v_t + \beta_u \mathbb{E}_{it} a_{igt+1} + \gamma_u \mathbb{E}_{it} a_t + \alpha_u \mathbb{E}_{it} a_{jt+1} \tag{A23}$$

Parameters $\{\beta_u\}, \{\gamma_u\}, \{\alpha_u\}$ help parameterize PE and GE considerations. Parameter $\{\varphi_u\}$ captures the direct exposure of household i to the exogenous shock. Iterating forward, $a_{it} = \varphi_u \sum_{k=0}^{\infty} \beta_u^k \mathbb{E}_{it} v_{t+k} + \gamma_u \mathbb{E}_{it} a_{jt} + (\beta_u \gamma_u + \alpha_u) \sum_{k=0}^{\infty} \beta_u^k \mathbb{E}_{it} a_{t+k+1}$. The aggregate action for household i is

$$a_t = \varphi_u \sum_{k=0}^{\infty} \beta_u^k \overline{\mathbb{E}}_t v_{t+k} + \sum_{j=1}^{2} \gamma_u \overline{\mathbb{E}}_t a_{jt} + (\beta_u \gamma_u + \alpha_u) \sum_{k=0}^{\infty} \beta_u^k \overline{\mathbb{E}}_t a_{t+k+1} \tag{A24}$$

Notice that (A24) is equivalent to (8) if $a_t = y_t$, $v_t = r_t$, $\overline{\mathbb{E}}_t(\cdot) = \overline{\mathbb{E}}_t^c(\cdot)$, and the following parametric restrictions are satisfied: $\varphi_u = -\frac{\beta(1-\lambda)}{\sigma}$, $\beta_u = \beta$, $\gamma_u = 1 - \beta(1-\lambda\chi)$, and $\alpha_u = \beta[\delta(1-\lambda\chi) - 1]$.

I now turn to solve the expectation terms. I can write the fundamental representation of the signal process as a system containing (6) and (10), which admits the following state-space representation:

$$\boldsymbol{Z}_t = \boldsymbol{F}\boldsymbol{Z}_{t-1} + \boldsymbol{\Phi}\boldsymbol{s}_{it}, \qquad x_{it} = \boldsymbol{H}\boldsymbol{Z}_t + \boldsymbol{\Psi}\boldsymbol{s}_{it} \tag{A25}$$

with $\boldsymbol{F} = \rho$, $\boldsymbol{\Phi} = \begin{bmatrix}\sigma_\varepsilon & 0\end{bmatrix}$, $\boldsymbol{Z}_t = r_t$, $\boldsymbol{s}_{it} = \begin{bmatrix}\varepsilon_t^r & u_{it}\end{bmatrix}^\top$, $\boldsymbol{H} = 1$, and $\boldsymbol{\Psi} = \begin{bmatrix}0 & \sigma_u\end{bmatrix}$. It is convenient to rewrite the uncertainty parameters in terms of precision: define $\tau_\varepsilon \equiv \frac{1}{\sigma_\varepsilon^2}$, $\tau_u \equiv \frac{1}{\sigma_u^2}$, and $\tau_{\varepsilon u} = \frac{\tau_u}{\tau_\varepsilon}$. The signal system can be written as

$$x_{it} = \frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t^r + \sigma_u u_{it} = \begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & \tau_u^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}\varepsilon_t^r \\ u_{it}\end{bmatrix} = \boldsymbol{M}(L)\boldsymbol{s}_{it}, \qquad \boldsymbol{s}_{it} \sim \mathcal{N}(0, I) \tag{A26}$$

The Wold theorem states that there exists another representation of the signal process (A26), $x_{it} = \boldsymbol{B}(L)\boldsymbol{w}_{it}$ such that $\boldsymbol{B}(z)$ is invertible and $\boldsymbol{w}_{it} \sim (0, \boldsymbol{V})$ is white noise. Hence, I can write the following equivalence:

$$x_{it} = \boldsymbol{M}(L)\boldsymbol{s}_{it} = \boldsymbol{B}(L)\boldsymbol{w}_{it} \tag{A27}$$

In the Wold representation of x_{it}, observing $\{x_{it}\}$ is equivalent to observing $\{\boldsymbol{w}_{it}\}$, and $\{x_i^t\}$ and $\{\boldsymbol{w}_i^t\}$ contain the same information. Furthermore, note that the Wold representation has the property that both processes share the autocovariance generating function, $\rho_{xx}(z) = \boldsymbol{M}(z)\boldsymbol{M}^\top(z^{-1}) = \boldsymbol{B}(z)\boldsymbol{V}\boldsymbol{B}^\top(z^{-1})$. Given the state-space representation of the signal process (A25), optimal expectations of the exogenous fundamental take the form of a Kalman filter $\mathbb{E}_{it}v_t = \lambda_u \mathbb{E}_{it-1}v_{t-1} + \boldsymbol{K}x_{it}$, where $\lambda_u = (I - \boldsymbol{KH})\boldsymbol{F}$, and $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = \boldsymbol{P}\boldsymbol{H}^\top\boldsymbol{V}^{-1} \tag{A28}$$

$$\boldsymbol{P} = \boldsymbol{F}[\boldsymbol{P} - \boldsymbol{P}\boldsymbol{H}^\top\boldsymbol{V}^{-1}\boldsymbol{H}\boldsymbol{P}]\boldsymbol{F} + \boldsymbol{\Phi}\boldsymbol{\Phi}^\top \tag{A29}$$

I still need to find the unknowns $\boldsymbol{B}(z)$ and $\boldsymbol{V}$. Propositions 13.1-13.4 in Hamilton (1994) provide us with these objects. Unknowns $\boldsymbol{B}(z)$ and $\boldsymbol{V}$ satisfy $\boldsymbol{B}(z) = I + \boldsymbol{H}(I - \boldsymbol{F}z)^{-1}\boldsymbol{F}\boldsymbol{K}$ and $\boldsymbol{V} = \boldsymbol{H}\boldsymbol{P}\boldsymbol{H}^\top + \boldsymbol{\Psi}\boldsymbol{\Psi}^\top$. I can write (A48) as

$$\boldsymbol{P}^2 + \boldsymbol{P}[(1-\rho^2)\sigma_u^2 - \sigma_\varepsilon^2] - \sigma_\varepsilon^2\sigma_u^2 = 0 \tag{A30}$$

from which I can infer that $\boldsymbol{P}$ is a scalar. Denote $k = \boldsymbol{P}^{-1}$ and rewrite (A30) as

$$\sigma_u^2\sigma_\varepsilon^2 k^2 = [(1-\rho^2)\sigma_u^2 - \sigma_\varepsilon^2]k + 1 \implies k = \frac{\tau_\varepsilon}{2}\left\{1 - \rho^2 - \tau_{\varepsilon u} \pm \sqrt{\left[\tau_{\varepsilon u} - (1-\rho^2)\right]^2 + 4\tau_{\varepsilon u}}\right\}$$

I also need to find $\boldsymbol{K}$. Now that I have found $\boldsymbol{P}$ in terms of model primitives, I can obtain $\boldsymbol{K}$ using condition (A28), $\boldsymbol{K} = \frac{1}{1+k\sigma_u^2}$. I can finally write λ_u as

$$\lambda_u = \frac{k\sigma_u^2\rho}{1+k\sigma_u^2} = \frac{1}{2}\left[\frac{1}{\rho} + \rho + \frac{\tau_{\varepsilon u}}{\rho} \pm \sqrt{\left(\frac{1}{\rho} + \rho + \frac{\tau_{\varepsilon u}}{\rho}\right)^2 - 4}\right] \tag{A31}$$

One can show that one of the roots λ_u lies inside the unit circle and the other lies outside as long as $\rho \in (0, 1)$, which guarantees that the Kalman expectation process is stationary and unique. I set λ_u to the root that lies inside the unit circle (the one with the '–' sign). Notice that I can also write $\boldsymbol{V}$ in terms of λ, $\boldsymbol{V} = k^{-1} + \sigma_u^2 = \frac{\rho}{\lambda_u\tau_u}$, where I have used the identity $k = \frac{\lambda_u\tau_u}{\rho-\lambda_u}$. Finally, I can obtain $\boldsymbol{B}(z) = 1 + \frac{\rho z}{(1-\rho z)(1+k\sigma_u^2)} = \frac{1-\lambda_u z}{1-\rho z}$ and therefore one can verify that $\boldsymbol{B}(z)\boldsymbol{V}\boldsymbol{B}^\top(z^{-1}) = \boldsymbol{M}(z)\boldsymbol{M}^\top(z^{-1}) \implies (\rho - \lambda_u)(1 - \rho\lambda_u) = \lambda_u\tau_{\varepsilon u}$.

Let us now move to the forecast of *endogenous* variables. Consider a variable $f_t = A(L)\boldsymbol{s}_{it}$. Applying the Wiener-Hopf prediction filter, I can obtain the forecast as $\mathbb{E}_{it} f_t = \left[A(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it}$, where $[\cdot]_+$ denotes the annihilator operator.

I need to find the $A(z)$ polynomial for each of the forecasted variables. Let us start from the exogenous fundamental v_t to verify that the Kalman and Wiener-Hopf filters result in the same forecast. I can write the fundamental as $v_t = \begin{bmatrix} \frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & 0 \end{bmatrix} \boldsymbol{s}_{it} = A_v(L)\boldsymbol{s}_{it}$. Let me now move to the endogenous variables. Guess that household i's policy function satisfies $a_{it} = h(L)x_{it}$. The aggregate outcome can then be expressed as $a_t = \int a_{it}\, di = \int h(L)x_{it}\, di = h(L)\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t = \begin{bmatrix} h(L)\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & 0 \end{bmatrix} \boldsymbol{s}_{l\,it} = A(L)\boldsymbol{s}_{it}$. Similarly, the own and average future actions can be written as $a_{t+1} = \frac{A(L)}{L}\boldsymbol{s}_{it}$ and $a_{it+1} = h(L)x_{i,t+1} = \begin{bmatrix} \tau_\varepsilon^{-\frac{1}{2}}\frac{h(L)}{L(1-\rho L)} & \tau_u^{-\frac{1}{2}}\frac{h(L)}{L} \end{bmatrix} \boldsymbol{s}_{it} = A_i(L)\boldsymbol{s}_{it}$. I now obtain the forecasts,

$$\begin{aligned}
\mathbb{E}_{it} v_t &= \left[A_v(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it} = \left[\frac{L}{(1-\rho L)(L-\lambda_u)}\right]_+ \frac{\lambda_u\tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left[\frac{\phi_1(L)}{L-\lambda_u}\right]_+ \frac{\lambda_u\tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} = \frac{\phi_1(L)-\phi_1(\lambda_u)}{L-\lambda_u}\frac{\lambda_u\tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it}, \quad \phi_1(z) = \frac{z}{1-\rho z} \\
&= \frac{\lambda_u\tau_{\varepsilon u}}{\rho(1-\rho\lambda_u)}\frac{1}{1-\lambda_u L}x_{it} = \left(1-\frac{\lambda_u}{\rho}\right)\frac{1}{1-\lambda_u L}x_{it} = G_1(L)x_{it} \qquad \text{(A32)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{it} a_{t+1} &= \left[\frac{A(L)}{L}\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it} = \left[\frac{h(L)}{(1-\rho L)(L-\lambda_u)}\right]_+ \frac{\lambda_u\tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left[\frac{\phi_2(L)}{L-\lambda_u}\right]_+ \frac{\lambda_u\tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} = \frac{\phi_2(L)-\phi_2(\lambda_u)}{L-\lambda_u}\frac{\lambda_u\tau_u}{\rho\tau_\varepsilon}\frac{1-\rho L}{1-\lambda_u L}x_{it}, \quad \phi_2(z) = \frac{h(z)}{1-\rho z} \\
&= \frac{\lambda_u\tau_{\varepsilon u}}{\rho}\left[h(L) - h(\lambda_u)\frac{1-\rho L}{1-\rho\lambda_u}\right]\frac{1}{(1-\lambda_u L)(L-\lambda_u)}x_{it} = G_2(L)x_{it} \qquad \text{(A33)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{it} a_t &= \left[A(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it} = \left[\frac{h(L)L}{(1-\rho L)(L-\lambda_u)}\right]_+ \frac{\lambda_u\tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left[\frac{\phi_3(L)}{L-\lambda_u}\right]_+ \frac{\lambda_u\tau_{\varepsilon u}}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} = \frac{\phi_3(L)-\phi_3(\lambda_u)}{L-\lambda_u}\frac{\lambda_u\tau_u}{\rho\tau_\varepsilon}\frac{1-\rho L}{1-\lambda_u L}x_{it}, \quad \phi_3(z) = \frac{h(z)z}{1-\rho z} \\
&= \frac{\lambda_u\tau_{\varepsilon u}}{\rho}\left[h(L)L - h(\lambda_u)\lambda_u\frac{1-\rho L}{1-\rho\lambda_u}\right]\frac{1}{(1-\lambda_u L)(L-\lambda_u)}x_{it} = G_3(L)x_{it} \qquad \text{(A34)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{it} a_{i,t+1} &= \left[A_{ig}(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{it} \\
&= \left[\frac{h(L)}{\tau_\varepsilon(1-\rho L)(L-\lambda_u)} + \frac{h(L)(L-\rho)}{\tau_u L(L-\lambda_u)}\right]_+ \frac{\lambda_u\tau_u}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left\{\left[\frac{h(L)}{\tau_\varepsilon(1-\rho L)(L-\lambda_u)}\right]_+ + \left[\frac{h(L)(L-\rho)}{\tau_u L(L-\lambda_u)}\right]_+\right\}\frac{\lambda_u\tau_u}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left\{\left[\frac{\phi_4(L)}{L-\lambda_u}\right]_+ + \left[\frac{\phi_5(L)}{L(L-\lambda_u)}\right]_+\right\}\frac{\lambda_u\tau_u}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left\{\frac{\phi_4(L)-\phi_4(\lambda_u)}{L-\lambda_u} + \frac{\phi_5(L)-\phi_5(\lambda_u)}{\lambda_u(L-\lambda_u)} - \frac{\phi_5(L)-\phi_5(0)}{\lambda_u L}\right\}\frac{\lambda_u\tau_u}{\rho}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \frac{\lambda_u}{\rho}\left\{\frac{h(L)}{L-\lambda_u}\left[\frac{\tau_u}{\tau_\varepsilon(1-\rho L)} + \frac{L-\rho}{L}\right] - \frac{h(\lambda_u)}{L-\lambda_u}\left[\frac{\tau_u}{\tau_\varepsilon(1-\rho\lambda_u)} + \frac{\lambda_u-\rho}{\lambda_u}\right] - \frac{\rho h(0)}{\lambda_u L}\right\}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&= \left\{\frac{h(L)}{L-\lambda_u}\left[\left(1-\frac{\lambda_u}{\rho}\right)\frac{1-\rho\lambda_u}{1-\rho L} + \frac{\lambda_u(L-\rho)}{\rho L}\right] - \frac{h(0)}{L}\right\}\frac{1-\rho L}{1-\lambda_u L}x_{it}
\end{aligned}$$

$$\text{(A35)} \qquad = G_4(L)x_{it}, \quad \phi_4(z) = \frac{h(z)}{\tau_\varepsilon(1-\rho z)}, \quad \phi_5(z) = \frac{h(z)(z-\rho)}{\tau_u}$$

Inserting our obtained expressions into (A42),

$$\begin{aligned}
h(L)x_{it} &= \varphi_u G_1(L)x_{it} + \beta_u G_4(L)x_{it} + \gamma_u G_3(L)x_{it} + \alpha_u G_2(L)x_{it} \\
&= \varphi_u\left(1-\frac{\lambda_u}{\rho}\right)\frac{1}{1-\lambda_u L}x_{it} + \beta_u\left\{\frac{h(L)}{L-\lambda_u}\left[\left(1-\frac{\lambda_u}{\rho}\right)\frac{1-\rho\lambda_u}{1-\rho L} + \frac{\lambda_u(L-\rho)}{\rho L}\right] - \frac{h(0)}{L}\right\}\frac{1-\rho L}{1-\lambda_u L}x_{it} \\
&+ \gamma_u\frac{\lambda_u\tau_{\varepsilon u}}{\rho}\left[h(L)L - h(\lambda_u)\lambda_u\frac{1-\rho L}{1-\rho\lambda_u}\right]\frac{1}{(1-\lambda_u L)(L-\lambda_u)}x_{it} \\
&+ \alpha_u\frac{\lambda_u\tau_{\varepsilon u}}{\rho}\left[h(L) - h(\lambda_u)\frac{1-\rho L}{1-\rho\lambda_u}\right]\frac{1}{(1-\lambda_u L)(L-\lambda_u)}x_{it}
\end{aligned}$$

Removing the x_{it} terms, and rearranging terms on the LHS and RHS

$$\begin{aligned}
&h(z)\left\{1-\beta_u\left[\left(1-\frac{\lambda_u}{\rho}\right)\frac{1-\rho\lambda_u}{1-\rho z} + \frac{\lambda_u(z-\rho)}{\rho z}\right]\frac{1-\rho z}{(1-\lambda_u z)(L-\lambda_u)}\right\} \\
&- h(z)\frac{(\rho-\lambda_u)(1-\rho\lambda_u)}{\rho}\frac{\gamma_u z + \alpha_u}{(1-\lambda_u z)(z-\lambda_u)} \\
&= \varphi_u\left(1-\frac{\lambda_u}{\rho}\right)\frac{1}{1-\lambda_u z} - \beta_u\frac{1-\rho z}{z(1-\lambda_u z)}h(0) - h(\lambda_u)\left(1-\frac{\lambda_u}{\rho}\right)\frac{\gamma_u\lambda_u + \alpha_u}{(1-\lambda_u z)(z-\lambda_u)}(1-\rho z)
\end{aligned}$$

Multiplying both sides by $z(z-\lambda_u)(1-\lambda_u z)$,

$$\begin{aligned}
&h(z)\left[z(z-\lambda_u)(1-\lambda_u z) - \beta_u(z-\lambda_u)(1-\lambda_u z)\right] - h(z)\frac{(\rho-\lambda_u)(1-\rho\lambda_u)}{\rho}z(\gamma_u z + \alpha_u) \\
&= \varphi_u\left(1-\frac{\lambda_u}{\rho}\right)z(z-\lambda_u) - \beta_u(1-\rho z)(z-\lambda_u)h(0) - h(\lambda_u)\left(1-\frac{\lambda_u}{\rho}\right)(\gamma_u\lambda_u + \alpha_u)z(1-\rho z)
\end{aligned}$$

I can write the above system of equations in terms of $\boldsymbol{h}(L)$ in matrix form $\boldsymbol{C}(z)\boldsymbol{h}(z) = \boldsymbol{d}(z)$ where

$$\begin{aligned}
\boldsymbol{C}(z) &= (z-\beta_u)(z-\lambda_u)(1-\lambda_u z) - \frac{(\rho-\lambda_u)(1-\rho\lambda_u)}{\rho}z(\gamma_u z + \alpha_u) \\
&= \lambda_u\left\{(\beta_u - z)(z-\rho)\left(z-\frac{1}{\rho}\right) - \frac{\tau_{\varepsilon u}}{\rho}z[\alpha_u + \beta_u - (1-\gamma_u)z]\right\}
\end{aligned}$$

I can also write

$$\boldsymbol{d}(z) = \varphi_u\left(1-\frac{\lambda_u}{\rho}\right)z(z-\lambda_u) - \beta_u(1-\rho z)(z-\lambda_u)h(0) - h(\lambda_u)\left(1-\frac{\lambda_u}{\rho}\right)(\gamma_u\lambda_u + \alpha_u)z(1-\rho z)$$

Note that $\boldsymbol{C}(z)$ is a polynomial of degree 3 on z. Denote the inside roots of det $\boldsymbol{C}(z)$ as $\{\zeta_1, \zeta_2\}$, and the outside root as $\{\vartheta^{-1}\}$. Because agents cannot use future signals, the inside roots have to be removed. Note that the number of free constants in $\boldsymbol{d}(z)$ is 2: $\{h(0)\}$ and $\left\{\widetilde{h}(\lambda_u) = h(\lambda_u)\left(1-\frac{\lambda_u}{\rho}\right)(\gamma_u\lambda_u + \alpha_u)\right\}$. With a unique solution, it has to be the case that the number of outside roots is 2. By choosing the appropriate constants, the 2 inside roots will be removed. Therefore, the 2 constants are solutions to $d_1(\zeta_n) = 0$ for $\{\zeta_n\}_{n=1}^2$. Using

$C(z) = -\lambda_u(z-\zeta_1)(z-\zeta_2)(z-\vartheta^{-1})$, the Vieta properties to eliminate the inside roots, and $C(\vartheta^{-1}) = 0$, I obtain,

$$a_t = h(L)\nu_t = \left(1 - \frac{\vartheta}{\rho}\right) \frac{\varphi_u}{1-\gamma_u-\rho(\alpha_u+\beta_u)} \frac{1}{1-\vartheta L}\nu_t$$

□

Proof of Proposition 4. From the proof of proposition 3, I have the following objects

$$\begin{aligned}
\pi_{t+k} &= h(L)\nu_{t+k} \\
\overline{\mathbb{E}}_t^c \pi_{t+k} &= \frac{(\rho-\lambda_u)(1-\rho\lambda_u)}{\rho(L-\lambda_u)(1-\lambda_u L)} \left[L^{1-k}h(L) - \frac{1-\rho L}{1-\rho\lambda_u}\lambda_u^{1-k}h(\lambda_u)\right] \\
\pi_{t+k} - \overline{\mathbb{E}}_t^c \pi_{t+k} &= \frac{\lambda_u}{\rho(L-\lambda_u)(1-\lambda_u L)} \left[(L-\rho)L^{-k}h(L) + (\rho-\lambda_u)\lambda_u^{-k}h(\lambda_u)\right]\varepsilon_t
\end{aligned}$$

The forecast error of annual inflation is

$$\begin{aligned}
\pi_{t+3,t} - \overline{\mathbb{E}}_t^c \pi_{t+3,t} &= (\pi_t - \overline{\mathbb{E}}_t^c \pi_t) + (\pi_{t+1} - \overline{\mathbb{E}}_t^c \pi_{t+1}) + (\pi_{t+2} - \overline{\mathbb{E}}_t^c \pi_{t+2}) + (\pi_{t+3} - \overline{\mathbb{E}}_t^c \pi_{t+3}) \\
&= \frac{\lambda_u}{\rho(L-\lambda_u)(1-\lambda_u L)} \left[(L-\rho)\left(\sum_{k=0}^{3} L^{-k}\right) h(L) + (\rho-\lambda_u)\left(\sum_{k=0}^{3}\lambda_u^{-k}\right) h(\lambda_u)\right]\varepsilon_t \\
&= \frac{(\rho-\vartheta)(1-\rho\vartheta)\lambda_u\psi}{\rho^2(1-\lambda_u\vartheta)(1-\lambda_u L)(1-\vartheta L)}\varepsilon_t \\
&+ \frac{(\rho-\vartheta)\left[\rho(1-\lambda_u\vartheta) - \vartheta(\rho-\lambda_u)L\right]\psi}{\rho^2(1-\lambda_u\vartheta)L(1-\lambda_u L)(1-\vartheta L)}\varepsilon_t \\
&+ \frac{(\rho-\vartheta)\left[\rho\lambda_u(1-\lambda_u\vartheta) + (\rho-\lambda_u)(1-\lambda_u\vartheta)L - \vartheta(\rho-\lambda_u)L^2\right]\psi}{\rho^2\lambda_u(1-\lambda_u\vartheta)L^2(1-\lambda_u L)(1-\vartheta L)}\varepsilon_t \\
&+ \frac{(\rho-\vartheta)\left[(L^2+\lambda_u L+\lambda_u^2)(\rho+\lambda_u\vartheta L) - (L+\lambda_u)[\lambda_u L + (L^2+\lambda_u^2)\rho\vartheta]\right]\psi}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)L^3(1-\lambda_u L)(1-\vartheta L)}\varepsilon_t \\
&= \frac{(\rho-\vartheta)\psi}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)L^3(1-\lambda_u L)(1-\vartheta_1 L)} \times \Big\{\rho\lambda_u^2(1-\lambda_u\vartheta) + \lambda_u(1-\lambda_u\vartheta)[\rho-(1-\rho)\lambda_u]L \\
&+(1-\lambda_u\vartheta)[\rho-(1-\rho)\lambda_u(1+\lambda_u)]L^2 + [\lambda_u^3 - \vartheta(\rho-(1-\rho)\lambda_u(1+\lambda_u+\lambda_u^2))]L^3\Big\}\varepsilon_t \\
&= \frac{(\rho-\vartheta)\psi\xi_0}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)} \frac{(1-\xi_1 L)(1-\xi_2 L)(1-\xi_3 L)}{L^3(1-\lambda_u L)(1-\vartheta L)}\varepsilon_t \\
&= \frac{(\rho-\vartheta)\psi\xi_0}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)} \frac{(1-\xi_2 L)(1-\xi_3 L)}{L^3} \left(\frac{\vartheta-\xi_1}{\vartheta-\lambda_u}\frac{1}{1-\vartheta L} - \frac{\lambda_u-\xi_1}{\vartheta-\lambda_u}\frac{1}{1-\lambda_u L}\right)\varepsilon_t \\
&= \frac{(\rho-\vartheta)\psi\xi_0(\vartheta-\xi_1)}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)(\vartheta-\lambda_u)} \frac{(1-\xi_2 L)(1-\xi_3 L)}{L^3(1-\vartheta L)}\varepsilon_t \\
&+ \frac{(\rho-\vartheta)\psi\xi_0(\xi_1-\lambda_u)}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)(\vartheta-\lambda_u)} \frac{(1-\xi_2 L)(1-\xi_3 L)}{L^3(1-\lambda_u L)}\varepsilon_t \\
&= \gamma_1 \frac{(1-\xi_2 L)(1-\xi_3 L)}{L^3(1-\vartheta L)}\varepsilon_t + \gamma_2 \frac{(1-\xi_2 L)(1-\xi_3 L)}{L^3(1-\lambda_u L)}\varepsilon_t \\
&= \gamma_1 \frac{1-(\xi_2+\xi_3)L+\xi_2\xi_3 L^2}{L^3(1-\vartheta L)}\varepsilon_t + \gamma_2 \frac{1-(\xi_2+\xi_3)L+\xi_2\xi_3 L^2}{L^3(1-\lambda_u L)}\varepsilon_t
\end{aligned}$$

$$
\begin{aligned}
&= \gamma_1 \sum_{k=0}^{\infty} \vartheta^k \varepsilon_{t+3-k} - \gamma_1(\xi_2+\xi_3) \sum_{k=0}^{\infty} \vartheta^k \varepsilon_{t+2-k} + \gamma_1 \xi_2 \xi_3 \sum_{k=0}^{\infty} \vartheta^k \varepsilon_{t+1-k} \\
&+ \gamma_2 \sum_{k=0}^{\infty} \lambda_u^k \varepsilon_{t+3-k} - \gamma_2(\xi_2+\xi_3) \sum_{k=0}^{\infty} \lambda_u^k \varepsilon_{t+2-k} + \gamma_2 \xi_2 \xi_3 \sum_{k=0}^{\infty} \lambda_u^k \varepsilon_{t+1-k} \\
&= \beta_1 \sum_{k=0}^{\infty} \vartheta^k \varepsilon_{t+3-k} + \beta_2 \sum_{k=0}^{\infty} \vartheta^k \varepsilon_{t+2-k} + \beta_3 \sum_{k=0}^{\infty} \vartheta^k \varepsilon_{t+1-k} \\
&+ \beta_4 \sum_{k=0}^{\infty} \lambda_u^k \varepsilon_{t+3-k} + \beta_5 \sum_{k=0}^{\infty} \lambda_u^k \varepsilon_{t+2-k} + \beta_6 \sum_{k=0}^{\infty} \lambda_u^k \varepsilon_{t+1-k}
\end{aligned}
$$

where $\psi = -\frac{1}{\nu(1-\rho\delta)}$, $\xi_0 = \rho\lambda_u^2(1-\lambda_u\vartheta)$, $-\xi_0(\xi_1+\xi_2+\xi_3) = \lambda_u(1-\lambda_u\vartheta)[\rho-(1-\rho)\lambda_u]$, $\xi_0(\xi_1\xi_2+\xi_1\xi_3+\xi_2\xi_3) = (1-\lambda_u\vartheta)[\rho-(1-\rho)\lambda_u(1+\lambda_u)]$, $-\xi_0\xi_1\xi_2\xi_3 = \lambda_u^3-\vartheta[\rho-(1-\rho)\lambda_u(1+\lambda_u+\lambda_u^2)]$, $\gamma_1 = \frac{(\rho-\vartheta)\psi\xi_0(\vartheta-\xi_1)}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)(\vartheta-\lambda_u)}$, $\gamma_2 = \frac{(\rho-\vartheta)\psi\xi_0(\xi_1-\lambda_u)}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)(\vartheta-\lambda_u)}$, $\beta_1 = \gamma_1$, $\beta_2 = -\gamma_1(\xi_2+\xi_3)$, $\beta_3 = \gamma_1\xi_2\xi_3$, $\beta_4 = \gamma_2$, $\beta_5 = -\gamma_2(\xi_2+\xi_3)$, and $\beta_6 = \gamma_2\xi_2\xi_3$. Before computing the forecast revision of annual inflation, notice that

$$
\overline{\mathbb{E}}_t^c \pi_{t+k} - \overline{\mathbb{E}}_{t-1}^c \pi_{t+k} = \frac{(\rho-\lambda_u)h(\lambda_u)}{\rho\lambda_u^k} \frac{1}{1-\lambda_u L} \varepsilon_t
$$

Therefore, the forecast revision of annual inflation is

$$
\begin{aligned}
\overline{\mathbb{E}}_t^c \pi_{t+3,t} - \overline{\mathbb{E}}_{t-1}^c \pi_{t+3,t} &= (\overline{\mathbb{E}}_t^c \pi_t - \overline{\mathbb{E}}_{t-1}^c \pi_t) + (\overline{\mathbb{E}}_t^c \pi_{t+1} - \overline{\mathbb{E}}_{t-1}^c \pi_{t+1}) + (\overline{\mathbb{E}}_t^c \pi_{t+2} - \overline{\mathbb{E}}_{t-1}^c \pi_{t+2}) \\
&+ (\overline{\mathbb{E}}_t^c \pi_{t+3} - \overline{\mathbb{E}}_{t-1}^c \pi_{t+3}) \\
&= \frac{(\rho-\lambda_u)h(\lambda_u)(1+\lambda_u^{-1}+\lambda_u^{-2}+\lambda_u^{-3})}{\rho} \sum_{k=0}^{\infty} \lambda_u^k \varepsilon_{t-k} = \alpha_u \sum_{k=0}^{\infty} \lambda_u^k \varepsilon_{t-k}
\end{aligned}
$$

where $\alpha_u = \frac{(\rho-\lambda_u)(1+\lambda_u+\lambda_u^2+\lambda_u^3)\frac{(\rho-\vartheta)\psi}{1-\lambda_u\vartheta}}{\rho^2\lambda_u^3}$. I now seek to compute the OLS coefficient. The covariance is

$$
\mathbb{C}\,(\text{forecast error, revision}) = \left[\frac{\beta_1\alpha_u\vartheta^3 + \beta_2\alpha_u\vartheta^2 + \beta_3\alpha_u\vartheta}{1-\lambda_u\vartheta} + \frac{\beta_4\alpha_u\lambda_u^3 + \beta_5\alpha_u\lambda_u^2 + \beta_6\alpha_u\lambda_u}{1-\lambda_u^2} \right] \sigma_\varepsilon^2
$$

The variance is $\mathbb{V}\,(\text{revision}) = \frac{\alpha_u^2}{1-\lambda_u^2}\sigma_\varepsilon^2$. Finally, the OLS coefficient is

$$
\begin{aligned}
\beta_{CG} &= \frac{\mathbb{C}\,(\text{forecast error, revision})}{\mathbb{V}\,(\text{revision})} \\
&= \frac{1}{\alpha_u} \left[(\beta_1\vartheta^3 + \beta_2\vartheta^2 + \beta_3\vartheta) \frac{1-\lambda_u^2}{1-\lambda_u\vartheta} + \beta_4\lambda_u^3 + \beta_5\lambda_u^2 + \beta_6\lambda_u \right] \\
&= \frac{1}{\alpha_u} \left[\frac{(\rho-\vartheta)\psi\vartheta(1-\lambda_u^2)}{\rho^2\lambda_u^2(1-\lambda_u\vartheta)^2(\vartheta-\lambda_u)} \xi_0(\vartheta-\xi_1)(\vartheta-\xi_2)(\vartheta-\xi_3) \right. \\
&\left. - \frac{(\rho-\vartheta)\psi}{\rho^2\lambda_u(1-\lambda_u\vartheta)(\vartheta-\lambda_u)} \xi_0(\lambda_u-\xi_1)(\lambda_u-\xi_2)(\lambda_u-\xi_3) \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\alpha_u}\left[\frac{(\rho-\vartheta)\psi\lambda_u\vartheta(1-\lambda_u^2)(1+\vartheta)(1+\vartheta^2)(1-\rho\vartheta)}{\rho^2(1-\lambda_u\vartheta)^2(\vartheta-\lambda_u)}\right.\\
&\left.+\frac{(\rho-\vartheta)\psi(1+\lambda_u^2)\{(\rho-\lambda_u)[\vartheta(1+\lambda_u)-\lambda_u(1-\lambda_u\vartheta)]-\rho\lambda_u^2(1+\lambda_u)(1-\lambda_u\vartheta)\}}{\rho^2(1-\lambda_u\vartheta)(\vartheta-\lambda_u)}\right]\\
&= \frac{1}{\alpha_u}\frac{(\rho-\vartheta)\psi}{\rho^2(1-\lambda_u\vartheta)(\vartheta-\lambda_u)}\left[\frac{\lambda_u\vartheta(1-\lambda_u^2)(1+\vartheta)(1+\vartheta^2)(1-\rho\vartheta)}{1-\lambda_u\vartheta}\right.\\
&\left.+(1+\lambda_u^2)\{(\rho-\lambda_u)[\vartheta(1+\lambda_u)-\lambda_u(1-\lambda_u\vartheta)]-\rho\lambda_u^2(1+\lambda_u)(1-\lambda_u\vartheta)\}\right]\\
&= \frac{\lambda_u^3}{(\rho-\lambda_u)(1+\lambda_u+\lambda_u^2+\lambda_u^3)(\vartheta-\lambda_u)}\left[\frac{\lambda_u\vartheta(1-\lambda_u^2)(1+\vartheta)(1+\vartheta^2)(1-\rho\vartheta)}{1-\lambda_u\vartheta}\right.\\
&\left.+(1+\lambda_u^2)\{(\rho-\lambda_u)[\vartheta(1+\lambda_u)-\lambda_u(1-\lambda_u\vartheta)]-\rho\lambda_u^2(1+\lambda_u)(1-\lambda_u\vartheta)\}\right]
\end{aligned}
$$

□

Proof of Proposition 5. The aggregate outcome is

$$
\begin{aligned}
y_t &= \psi\left(1-\frac{\vartheta}{\rho}\right)\frac{1}{1-\vartheta L} = \psi\left(1-\frac{\vartheta}{\rho}\right)\frac{1}{(1-\vartheta L)(1-\rho L)}\varepsilon_t\\
&= \psi\left(1-\frac{\vartheta}{\rho}\right)\left[\frac{\rho}{\rho-\vartheta}\frac{1}{1-\rho L}-\frac{\vartheta}{\rho-\vartheta}\frac{1}{1-\vartheta L}\right]\varepsilon_t = \frac{\psi}{\rho}\sum_{k=0}^{\infty}(\rho^{k+1}-\vartheta^{k+1})\varepsilon_{t-k}
\end{aligned}
$$

The PE component is given by

$$
\begin{aligned}
\mathrm{PE}_t &= -\frac{\beta}{\sigma}(1-\lambda)\sum_{k=0}^{\infty}\beta^k\overline{\mathbb{E}}_t r_{t+k} = -\frac{\beta}{\sigma}(1-\lambda)\sum_{k=0}^{\infty}(\beta\rho)^k\overline{\mathbb{E}}_t r_t = -\frac{\beta(1-\lambda)}{\sigma(1-\rho\beta)}\overline{\mathbb{E}}_t r_t\\
&= -\frac{\beta(1-\lambda)}{\sigma(1-\rho\beta)}\left(1-\frac{\lambda_u}{\rho}\right)\frac{1}{1-\lambda_u L}v_t = -\frac{\beta(1-\lambda)}{\sigma(1-\rho\beta)}\left(1-\frac{\lambda_u}{\rho}\right)\frac{1}{(1-\lambda_u L)(1-\rho L)}\varepsilon_t\\
&= -\frac{\beta(1-\lambda)}{\sigma(1-\rho\beta)}\left(1-\frac{\lambda_u}{\rho}\right)\left[\frac{\rho}{\rho-\lambda_u}\frac{1}{1-\rho L}-\frac{\lambda_u}{\rho-\lambda_u}\frac{1}{1-\lambda_u L}\right]\varepsilon_t = -\frac{\beta(1-\lambda)}{\rho\sigma(1-\rho\beta)}\sum_{k=0}^{\infty}(\rho^{k+1}-\lambda_u^{k+1})\varepsilon_{t-k}
\end{aligned}
$$

Therefore, the PE share μ_τ is given by

$$
\mu_\tau = \frac{\partial \mathrm{PE}_\tau/\partial\varepsilon_t}{\partial \mathrm{TE}_\tau/\partial\varepsilon_t} = \frac{-\frac{\beta(1-\lambda)}{\rho\sigma(1-\rho\beta)}(\rho^{\tau+1}-\lambda_u^{\tau+1})}{\frac{\psi}{\rho}(\rho^{\tau+1}-\vartheta^{\tau+1})} = -\frac{\beta(1-\lambda)}{\psi\sigma(1-\rho\beta)}\frac{\rho^{\tau+1}-\lambda_u^{\tau+1}}{\rho^{\tau+1}-\vartheta^{\tau+1}}
$$

□

Proof of Proposition 6. I first prove (*i*). To show (15) captures the HANK beyond FIRE under certain (ω_f, ω_b), I rely on the Method for Undetermined Coefficients. Both dynamics are observationally equivalent if

$$\begin{aligned}
\vartheta y_{t-1} - \left(1-\frac{\vartheta}{\rho}\right) r_t &= \omega_b y_{t-1} + \delta\omega_f \mathbb{E}_t y_{t+1} - \frac{1}{\nu} r_t \\
&= \omega_b y_{t-1} + \delta\omega_f \mathbb{E}_t \left[\vartheta y_t - \left(1-\frac{\vartheta}{\rho}\right) r_{t+1}\right] - \frac{1}{\nu} r_t \\
&= \omega_b y_{t-1} + \delta\omega_f \vartheta y_t - \delta\omega_f \rho \left(1-\frac{\vartheta}{\rho}\right) r_t - \frac{1}{\nu} r_t \\
&= \omega_b y_{t-1} + \delta\omega_f \vartheta \left[\vartheta y_{t-1} - \left(1-\frac{\vartheta}{\rho}\right) r_t\right] - \delta\omega_f \rho \left(1-\frac{\vartheta}{\rho}\right) r_t - \frac{1}{\nu} r_t \\
&= \omega_b y_{t-1} + \delta\omega_f \vartheta^2 y_{t-1} - \delta\omega_f \vartheta \left(1-\frac{\vartheta}{\rho}\right) r_t - \delta\omega_f \rho \left(1-\frac{\vartheta}{\rho}\right) r_t - \frac{1}{\nu} r_t \\
&= \left(\omega_b + \delta\omega_f \vartheta^2\right) y_{t-1} - \left[\left(1-\frac{\vartheta}{\rho}\right) \frac{\delta\omega_f(\rho+\vartheta)}{\nu(1-\rho\delta)} + \frac{1}{\nu}\right] r_t
\end{aligned}$$

They are thus equivalent when (16) is satisfied.

I now move to (*ii*). Using the lag operator, I can factorize (15)

$$\mathbb{E}_t\left[\frac{1}{\nu} r_t\right] = \mathbb{E}_t\left[\left(\delta\omega_f L^{-2} - L^{-1} + \omega_b\right) y_{t-1}\right] = \mathbb{E}_t\left[\delta\omega_f \left(L^{-1} - \gamma_1^{-1}\right)\left(L^{-1} - \gamma_2^{-1}\right) y_{t-1}\right]$$

where γ_1^{-1} and γ_2^{-1} are the roots of the polynomial $\mathcal{Q}(x) \equiv \delta\omega_f x^2 - x + \omega_b$. Dividing both sides by $(L^{-1} - \gamma_2^{-1})$

$$\delta\omega_f \mathbb{E}_t[(L^{-1} - \gamma_1^{-1}) y_{t-1}] = \mathbb{E}_t\left[\frac{1}{\nu}\frac{1}{L^{-1} - \gamma_2^{-1}} r_t\right] = \mathbb{E}_t\left[-\frac{1}{\nu}\frac{\gamma_2}{1-\gamma_2 L^{-1}} r_t\right]$$

Hence, I can write the dynamics as

$$y_t = \gamma_1^{-1} y_{t-1} - \frac{\gamma_2}{\delta\omega_f \nu}\sum_{k=0}^{\infty} \gamma_2^k \mathbb{E}_t r_{t+k} = \gamma_1^{-1} y_{t-1} - \frac{1}{\gamma_1\omega_b\nu}\sum_{k=0}^{\infty}\left(\frac{\delta\omega_f}{\gamma_1\omega_b}\right)^k \mathbb{E}_t r_{t+k}$$

where I have applied the Vieta properties. Therefore, the effect of a forward guidance shock promised at time t in period τ is

$$FG_{t,t+\tau} = \frac{\partial y_t}{\partial \mathbb{E}_t r_{t+\tau}} = -\frac{1}{\gamma_1\omega_b\nu}\left(\frac{\delta\omega_f}{\gamma_1\omega_b}\right)^{\tau}$$

which is decreasing in τ provided that $\gamma_1 \in (0,1)$ is the only inside root, $\lim_{\tau\to\infty} FG_{t+\tau} = 0$, and the forward guidance puzzle is solved. □

Proof of Proposition 7. The proof is identical to the proof of Proposition 3, modulo the replacement of σ_u for σ_ϵ. In the public information case, the individual action is given by $a_{it} = h(L)z_t = h(L)(\nu_t + \epsilon_t)$. The policy function is given by $h(z) = -\left(1-\frac{\tilde{\vartheta}}{\rho}\right)\frac{1}{\nu(1-\rho\delta)}\frac{1}{1-\tilde{\vartheta}L}$, and hence I have $a_t = h_g(L)(\nu_t+\epsilon_t) = -\left(1-\frac{\tilde{\vartheta}}{\rho}\right)\frac{1}{\nu(1-\rho\delta)}\frac{1}{1-\tilde{\vartheta}L}(\nu_t+\epsilon_t)$. □

Proof of Proposition 8. The First-Order Condition is $\sum_{k=0}^{\infty}\theta^k \mathbb{E}_{jt}[\Lambda_{t,t+k} Y_{j,t+k|t}\frac{1}{P_{t+k}}(P_{jt}^* - \mathcal{M}\Psi_{j,t+k|t})] = 0$, where $\Psi_{j,t+k|t} \equiv \mathcal{C}'_{t+k}(Y_{j,t+j|t})$ denotes the (nominal) marginal cost for firm j, and $\mathcal{M} = \frac{\epsilon}{\epsilon-1}$. Log-linearizing around the zero inflation steady-state, I obtain the familiar price-setting rule

(A36) $$p^*_{jt} = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\left(\psi_{j,t+k|t} + \mu\right)$$

where $\psi_{j,t+k|t} = \log \Psi_{j,t+k|t}$ and $\mu = \log \mathcal{M}$.

Market clearing in the goods market implies that $Y_{jt} = C_{jt} = \int_{\mathcal{I}_h} C_{ijt}\, di$ for each j good/firm. Aggregating across firms, I obtain the aggregate market clearing condition: since assets are in zero net supply and there is no capital, investment, government consumption, or net exports, production equals consumption, $\int_{\mathcal{I}_f} Y_{jt}\, dj = \int_{\mathcal{I}_h}\int_{\mathcal{I}_f} C_{ijt}\, dj\, di \implies Y_t = C_t$.

Aggregate employment is given by the sum of employment across firms and must meet the aggregate labor supply, $N_t = \int_{\mathcal{I}_h} N_{it}\, di = \int_{\mathcal{I}_f} N_{jt}\, dj$. Using the production function and consumption demand, together with goods market clearing, $N_t = \int_{\mathcal{I}_f} Y_{jt}\, dj = Y_t \int_{\mathcal{I}_f}\left(\frac{P_{jt}}{P_t}\right)^{-\epsilon} dj$. Log-linearizing the above expression yields to

(A37) $$n_t = y_t$$

The (log) marginal cost for firm j at time $t+k|t$ is $\psi_{j,t+k|t} = w_{t+k} - mpn_{j,t+k|t}$ $= w_{t+k}$, where $mpn_{j,t+k|t}$ and $n_{j,t+k|t}$ denote (log) marginal product of labor and (log) employment in period $t+k$ for a firm that last reset its price at time t, respectively. Let $\psi_t \equiv \int_{\mathcal{I}_f} \psi_{jt}$ denote the (log) average marginal cost. I can then write $\psi_t = w_t$. Thus, the following relation holds

(A38) $$\psi_{j,t+k|t} = \psi_{t+k}$$

Introducing (A38) into (A36), I can rewrite the firm price-setting condition as $p^*_{jt} = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\left(p_{t+k} - \hat{\mu}_{t+k}\right)$, where $\hat{\mu} = \mu_t - \mu$ is the deviation between the average and desired markups, where $\mu_t = -(\psi_t - p_t)$.

Suppose that firms observe the aggregate prices up to period $t-1$, p^{t-1}, then I can restate the above condition as $p^*_{jt} - p_{t-1} = -(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\hat{\mu}_{t+k} + \sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\pi_{t+k}$. Define the firm-specific inflation rate as $\pi_{jt} = (1-\theta)(p^*_{jt} - p_{t-1})$. Then I can write the above expression as

$$\begin{aligned}
\pi_{jt} &= -(1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\hat{\mu}_{t+k} + (1-\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{jt}\pi_{t+k} \\
&= (1-\theta)\mathbb{E}_{jt}[\pi_t - (1-\beta\theta)\hat{\mu}_t] + \beta\theta\mathbb{E}_{jt}\left\{(1-\theta)\sum_{k=0}^{\infty}(\beta\theta)^k[\pi_{t+1+k} - (1-\beta\theta)\hat{\mu}_{t+1+k}]\right\} \\
&= (1-\theta)\mathbb{E}_{jt}[\pi_t - (1-\beta\theta)\hat{\mu}_t] + \beta\theta\mathbb{E}_{jt}\left\{(1-\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_{j,t+1}[\pi_{t+1+k} - (1-\beta\theta)\hat{\mu}_{t+1+k}]\right\}
\end{aligned}$$

(A39) $$= -(1-\theta)(1-\beta\theta)\mathbb{E}_{jt}\hat{\mu}_t + (1-\theta)\mathbb{E}_{jt}\pi_t + \beta\theta\mathbb{E}_{jt}\pi_{j,t+1}$$

where $\pi_t = \int_{\mathcal{I}_f} \pi_{jt}\, dj$.

Note that I can write the deviation between average and desired markups as $\mu_t = p_t - \psi_t = p_t - w_t =$

$-(\sigma y_t + \varphi n_t) = -(\sigma + \varphi)\, y_t$. As in the benchmark model, under flexible prices ($\theta = 0$) the average markup is constant and equal to the desired μ. Consider the natural level of output, y_t^n as the equilibrium level under flexible prices and full-information rational expectations. Rewriting the above condition under the natural equilibrium, $\mu = -(\sigma + \varphi)\, y_t^n$, which I can write as $y_t^n = \psi_y$, where $\psi_y = -\frac{\mu}{\sigma+\varphi}$. Therefore, I can write $\hat{\mu}_t = -(\sigma + \varphi)\, y_t$ where $y_t = y_t - y_t^n$ is defined as the output gap. Finally, I can write the individual Phillips curve as

$$\pi_{jt} = \kappa\theta\mathbb{E}_{jt} y_t + (1-\theta)\mathbb{E}_{jt}\pi_t + \beta\theta\mathbb{E}_{jt}\pi_{i,t+1} \tag{A40}$$

where $\kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}(\sigma + \varphi)$, and the aggregate Phillips curve can be written as

$$\pi_t = \kappa\theta\sum_{k=0}^{\infty}(\beta\theta)^k\overline{\mathbb{E}}_t^f\, y_{t+k} + (1-\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\overline{\mathbb{E}}_t^f\,\pi_{t+k} \tag{A41}$$

□

Proof of Proposition 9. The best response of agent l in group g is specified as follows

$$a_{lgt} = \varphi_g\mathbb{E}_{lgt}\nu_t + \beta_g\mathbb{E}_{lgt}a_{igt+1} + \sum_{j=1}^{2}\gamma_{gj}\mathbb{E}_{lgt}a_{jt} + \sum_{j=1}^{2}\alpha_{gj}\mathbb{E}_{lgt}a_{jt+1} \tag{A42}$$

where a_{-gt} is the aggregate action of the other group at time t. Parameters $\{\beta_g\}$, $\{\gamma_{gk}\}$, $\{\alpha_{gk}\}$ help parameterize PE and GE considerations. Notice that GE effects run not only within groups but also across groups (the interaction of the two blocks of the NK model). Parameters $\{\varphi_g\}$ capture the direct exposure of group g to the exogenous shock. Iterating forward, $a_{lgt} = \varphi_g\sum_{k=0}^{\infty}\beta_g^k\mathbb{E}_{lgt}\nu_{t+k} + \sum_{j=1}^{2}\gamma_{gj}\mathbb{E}_{lgt}a_{jt} + (\beta_g\gamma_{gj} + \alpha_{gj})\sum_{k=0}^{\infty}\beta_g^k\mathbb{E}_{lgt}a_{j,t+k+1}$. The aggregate action for group g is

$$a_{gt} = \varphi_g\sum_{k=0}^{\infty}\beta_g^k\overline{\mathbb{E}}_{gt}\nu_{t+k} + \sum_{j=1}^{2}\gamma_{gj}\overline{\mathbb{E}}_{gt}a_{jt} + (\beta_g\gamma_{gj} + \alpha_{gj})\sum_{k=0}^{\infty}\beta_g^k\overline{\mathbb{E}}_{gt}a_{j,t+k+1} \tag{A43}$$

Let $\boldsymbol{a}_t = (a_{gt})$ be a column vector collecting the aggregate actions of all groups (e.g., the vector of aggregate consumption and aggregate inflation), let $\boldsymbol{\varphi} = (\varphi_g)$ be a column vector containing the value of φ_g across groups, let $\boldsymbol{\beta} = \mathrm{diag}(\beta_g)$ be a 2×2 diagonal matrix of discount factors, with off-diagonal elements equal to 0, let $\boldsymbol{\gamma}$ be a 2×2 matrix collecting the (contemporaneous) interaction parameters γ_{gj}, let $\boldsymbol{\alpha} = (\alpha_{gk})$ be a 2×2 matrix collecting the (future) interaction parameters α_{gj}, and finally let $\boldsymbol{\delta} \equiv \boldsymbol{\beta} + \boldsymbol{\alpha}$,

$$\boldsymbol{a}_t = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}, \quad \boldsymbol{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

Notice that (A43) is equivalent to (8) and (20), respectively, if $a_{1t} = y_t$, $a_{2t} = \pi_t$, $\nu_t = \nu_t$, $\overline{\mathbb{E}}_{1t}(\cdot) = \overline{\mathbb{E}}_t^c(\cdot)$, $\overline{\mathbb{E}}_{2t}(\cdot) = \overline{\mathbb{E}}_{ft}(\cdot)$ and the following parametric restrictions are satisfied: $\varphi_1 = -\frac{\beta(1-\lambda)}{\sigma}$, $\beta_1 = \beta$, $\gamma_{11} = 1 - \beta\left[1 - \lambda\chi + \frac{\phi_y}{\sigma}(1-\lambda)\right]$, $\gamma_{12} = -\beta(1-\lambda)\frac{\phi_\pi}{\sigma}$, $\alpha_{11} = \beta[\delta(1-\lambda\chi) - 1]$

$\alpha_{12} = \frac{\beta}{\sigma}(1-\lambda)$, $\varphi_2 = 0$, $\beta_2 = \beta\theta$, $\gamma_{21} = \kappa\theta$, $\gamma_{22} = 1-\theta$, and $\alpha_{21} = \alpha_{22} = 0$.

I now turn to solve the expectation terms. I can write the fundamental representation of the signal process as a system containing (22) and (23), which admits the following state-space representation

$$\text{(A44)} \qquad \boldsymbol{Z}_t = \boldsymbol{F}\boldsymbol{Z}_{t-1} + \boldsymbol{\Phi}\boldsymbol{s}_{lgt}, \qquad x_{lgt} = \boldsymbol{H}\boldsymbol{Z}_t + \boldsymbol{\Psi}_g\boldsymbol{s}_{lgt}$$

with $\boldsymbol{F} = \rho$, $\boldsymbol{\Phi} = \begin{bmatrix}\sigma_\varepsilon & 0\end{bmatrix}$, $\boldsymbol{Z}_t = \nu_t$, $\boldsymbol{s}_{lgt} = \begin{bmatrix}\varepsilon_t^\nu & u_{lgt}\end{bmatrix}^\top$, $\boldsymbol{H} = 1$, and $\boldsymbol{\Psi}_g = \begin{bmatrix}0 & \sigma_{gu}\end{bmatrix}$. It is convenient to rewrite the uncertainty parameters in terms of precision: define $\tau_\varepsilon \equiv \frac{1}{\sigma_\varepsilon^2}$, $\tau_{gu} \equiv \frac{1}{\sigma_{gu}^2}$, and $\tau_g = \frac{\tau_{gu}}{\tau_\varepsilon}$. The signal system can be written as

$$\text{(A45)} \qquad x_{lgt} = \frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t^\nu + \sigma_{gu}u_{lgt} = \begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & \tau_{gu}^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}\varepsilon_t^\nu \\ u_{lgt}\end{bmatrix} = \boldsymbol{M}_g(L)\boldsymbol{s}_{lgt}, \qquad \boldsymbol{s}_{lgt} \sim \mathcal{N}(0, I)$$

The Wold theorem states that there exists another representation of the signal process (A45), $x_{lgt} = \boldsymbol{B}_g(L)\boldsymbol{w}_{lgt}$ such that $\boldsymbol{B}_g(z)$ is invertible and $\boldsymbol{w}_{lgt} \sim (0, \boldsymbol{V}_g)$ is white noise. Hence, I can write the following equivalence:

$$\text{(A46)} \qquad x_{lgt} = \boldsymbol{M}_g(L)\boldsymbol{s}_{lgt} = \boldsymbol{B}_g(L)\boldsymbol{w}_{lgt}$$

In the Wold representation of x_{lgt}, observing $\{x_{lgt}\}$ is equivalent to observing $\{\boldsymbol{w}_{lgt}\}$, and $\{x_{lg}^t\}$ and $\{\boldsymbol{w}_{lg}^t\}$ contain the same information. Furthermore, note that the Wold representation has the property that both processes share the autocovariance generating function, $\rho_{xx}^g(z) = \boldsymbol{M}_g(z)\boldsymbol{M}_g^\top(z^{-1}) = \boldsymbol{B}_g(z)\boldsymbol{V}_g\boldsymbol{B}_g^\top(z^{-1})$. Given the state-space representation of the signal process (A44), optimal expectations of the exogenous fundamental take the form of a Kalman filter $\mathbb{E}_{lgt}\nu_t = \lambda_g\mathbb{E}_{it-1}\nu_{t-1} + \boldsymbol{K}_g x_{lgt}$, where $\lambda_g = (I - \boldsymbol{K}_g\boldsymbol{H})\boldsymbol{F}$, and $\boldsymbol{K}_g$ is given by

$$\text{(A47)} \qquad \boldsymbol{K}_g = \boldsymbol{P}_g\boldsymbol{H}^\top\boldsymbol{V}_g^{-1}$$

$$\text{(A48)} \qquad \boldsymbol{P}_g = \boldsymbol{F}[\boldsymbol{P}_g - \boldsymbol{P}_g\boldsymbol{H}^\top\boldsymbol{V}_g^{-1}\boldsymbol{H}\boldsymbol{P}_g]\boldsymbol{F} + \boldsymbol{\Phi}\boldsymbol{\Phi}^\top$$

I still need to find the unknowns $\boldsymbol{B}_g(z)$ and $\boldsymbol{V}_g$. Propositions 13.1-13.4 in Hamilton (1994) provide us with these objects. Unknowns $\boldsymbol{B}_g(z)$ and $\boldsymbol{V}_g$ satisfy $\boldsymbol{B}_g(z) = I + \boldsymbol{H}(I - \boldsymbol{F}z)^{-1}\boldsymbol{F}\boldsymbol{K}_g$ and $\boldsymbol{V}_g = \boldsymbol{H}\boldsymbol{P}_g\boldsymbol{H}^\top + \boldsymbol{\Psi}_g\boldsymbol{\Psi}_g^\top$. I can write (A48) as

$$\text{(A49)} \qquad \boldsymbol{P}_g^2 + \boldsymbol{P}_g[(1-\rho^2)\sigma_{gu}^2 - \sigma_\varepsilon^2] - \sigma_\varepsilon^2\sigma_{gu}^2 = 0$$

from which I can infer that $\boldsymbol{P}_g$ is a scalar. Denote $k_g = \boldsymbol{P}_g^{-1}$ and rewrite (A49) as

$$\sigma_{gu}^2\sigma_\varepsilon^2 k_g^2 = [(1-\rho^2)\sigma_{gu}^2 - \sigma_\varepsilon^2]k_g + 1 \implies k_g = \frac{\tau_\varepsilon}{2}\left\{1 - \rho^2 - \tau_g \pm \sqrt{\left[\tau_g - (1-\rho^2)\right]^2 + 4\tau_g}\right\}$$

I also need to find $\boldsymbol{K}_g$. Now that I have found $\boldsymbol{P}_g$ in terms of model primitives, I can obtain $\boldsymbol{K}_g$ using condition (A47), $\boldsymbol{K}_g = \frac{1}{1+k_g\sigma_{gu}^2}$. I can finally write λ_g as

$$\lambda_g = \frac{k_g\sigma_{gu}^2\rho}{1+k_g\sigma_{gu}^2} = \frac{1}{2}\left[\frac{1}{\rho}+\rho+\frac{\tau_g}{\rho} \pm \sqrt{\left(\frac{1}{\rho}+\rho+\frac{\tau_g}{\rho}\right)^2 - 4}\right] \tag{A50}$$

One can show that one of the roots $\lambda_{g,[1,2]}$ lies inside the unit circle, and the other lies outside as long as $\rho \in (0,1)$, which guarantees that the Kalman expectation process is stationary and unique. I set λ_g to the root that lies inside the unit circle (the one with the '–' sign). Notice that I can also write $\boldsymbol{V}_g$ in terms of λ_g, $\boldsymbol{V}_g = k^{-1}+\sigma_{gu}^2 = \frac{\rho}{\lambda_g\tau_{gu}}$, where I have used the identity $k_g = \frac{\lambda_g\tau_{gu}}{\rho-\lambda_g}$. Finally, I can obtain $\boldsymbol{B}_g(z) = 1+\frac{\rho z}{(1-\rho z)(1+k\sigma_{gu}^2)} = \frac{1-\lambda_g z}{1-\rho z}$ and therefore one can verify that $\boldsymbol{B}_g(z)\boldsymbol{V}_g\boldsymbol{B}_g^\top(z^{-1}) = \boldsymbol{M}_g(z)\boldsymbol{M}_g^\top(z^{-1}) \implies (\rho-\lambda_g)(1-\rho\lambda_g) = \lambda_g\tau_g$.

Let us now move to the forecast of *endogenous* variables. Consider a variable $f_t = A(L)\boldsymbol{s}_{lgt}$. Applying the Wiener-Hopf prediction filter, I can obtain the forecast as $\mathbb{E}_{lgt}f_t = \left[A(L)\boldsymbol{M}^\top(L^{-1})\boldsymbol{B}(L^{-1})^{-1}\right]_+ \boldsymbol{V}^{-1}\boldsymbol{B}(L)^{-1}x_{lgt}$, where $[\cdot]_+$ denotes the annihilator operator.

I need to find the $A(z)$ polynomial for each of the forecasted variables. Let us start from the exogenous fundamental v_t to verify that the Kalman and Wiener-Hopf filters result in the same forecast. I can write the fundamental as $v_t = \begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & 0\end{bmatrix}\boldsymbol{s}_{it} = A_v(L)\boldsymbol{s}_{it}$. Let me now move to the endogenous variables. Guess that agent $i\times g$'s policy function satisfies $a_{lgt} = h_g(L)x_{lgt}$. The aggregate outcome in group g can then be expressed as $a_{gt} = \int a_{lgt}\,di = \int h_g(L)x_{lgt}\,di = h_g(L)\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t = \begin{bmatrix}h_g(L)\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L} & 0\end{bmatrix}\boldsymbol{s}_{l1t} = A_g(L)\boldsymbol{s}_{lgt}$. Similarly, the own and average future actions can be written as $a_{g,t+1} = \frac{A_g(L)}{L}\boldsymbol{s}_{lgt}$ and $a_{igt+1} = a_{ig,t+1} = h_g(L)x_{ig,t+1} = \begin{bmatrix}\tau_\varepsilon^{-\frac{1}{2}}\frac{h_g(L)}{L(1-\rho L)} & \tau_{gu}^{-\frac{1}{2}}\frac{h_g(L)}{L}\end{bmatrix}\boldsymbol{s}_{lgt} = A_{ig}(L)\boldsymbol{s}_{lgt}$. I now obtain the forecasts,

$$\begin{aligned}
\mathbb{E}_{lgt}v_t &= \left[A_v(L)\boldsymbol{M}_g^\top(L^{-1})\boldsymbol{B}_g(L^{-1})^{-1}\right]_+\boldsymbol{V}_g^{-1}\boldsymbol{B}_g(L)^{-1}x_{lgt} = \left[\frac{L}{(1-\rho L)(L-\lambda_g)}\right]_+\frac{\lambda\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}\\
&= \left[\frac{\phi_1(L)}{L-\lambda_g}\right]_+\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt} = \frac{\phi_1(L)-\phi_1(\lambda_g)}{L-\lambda_g}\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}, \quad \phi_1(z) = \frac{z}{1-\rho z}\\
&= \frac{\lambda_g\tau_g}{\rho(1-\rho\lambda_g)}\frac{1}{1-\lambda_g L}x_{lgt} = \left(1-\frac{\lambda_g}{\rho}\right)\frac{1}{1-\lambda_g L}x_{lgt} = G_{1g}(L)x_{lgt} \qquad \text{(A51)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{lgt}a_{k,t+1} &= \left[\frac{A_k(L)}{L}\boldsymbol{M}_g^\top(L^{-1})\boldsymbol{B}_g(L^{-1})^{-1}\right]_+\boldsymbol{V}_g^{-1}\boldsymbol{B}_g(L)^{-1}x_{lgt} = \left[\frac{h_k(L)}{(1-\rho L)(L-\lambda_g)}\right]_+\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}\\
&= \left[\frac{\phi_2(L)}{L-\lambda_g}\right]_+\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt} = \frac{\phi_2(L)-\phi_2(\lambda_g)}{L-\lambda_g}\frac{\lambda_g\tau_{gu}}{\rho\tau_\varepsilon}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}, \quad \phi_2(z) = \frac{h_k(z)}{1-\rho z}\\
&= \frac{\lambda_g\tau_g}{\rho}\left[h_k(L)-h_k(\lambda_g)\frac{1-\rho L}{1-\rho\lambda_g}\right]\frac{1}{(1-\lambda_g L)(L-\lambda_g)}x_{lgt} = G_{2gk}(L)x_{lgt} \qquad \text{(A52)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{lgt}a_{kt} &= \left[A_k(L)\boldsymbol{M}_g^\top(L^{-1})\boldsymbol{B}_g(L^{-1})^{-1}\right]_+\boldsymbol{V}_g^{-1}\boldsymbol{B}_g(L)^{-1}x_{lgt} = \left[\frac{h_k(L)L}{(1-\rho L)(L-\lambda_g)}\right]_+\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}\\
&= \left[\frac{\phi_3(L)}{L-\lambda_g}\right]_+\frac{\lambda_g\tau_g}{\rho}\frac{1-\rho L}{1-\lambda_g L}x_{lgt} = \frac{\phi_3(L)-\phi_3(\lambda_g)}{L-\lambda_g}\frac{\lambda_g\tau_{gu}}{\rho\tau_\varepsilon}\frac{1-\rho L}{1-\lambda_g L}x_{lgt}, \quad \phi_3(z) = \frac{h_k(z)z}{1-\rho z}\\
&= \frac{\lambda_g\tau_g}{\rho}\left[h_k(L)L-h_k(\lambda_g)\lambda_g\frac{1-\rho L}{1-\rho\lambda_g}\right]\frac{1}{(1-\lambda_g L)(L-\lambda_g)}x_{lgt} = G_{3gk}(L)x_{lgt} \qquad \text{(A53)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{lgt} a_{lg,t+1} &= \left[A_{ig}(L) \boldsymbol{M}_g^\top (L^{-1}) \boldsymbol{B}_g (L^{-1})^{-1} \right]_+ \boldsymbol{V}_g^{-1} \boldsymbol{B}_g(L)^{-1} x_{lgt} \\
&= \left[\frac{h_g(L)}{\tau_\varepsilon (1-\rho L)(L-\lambda_g)} + \frac{h_g(L)(L-\rho)}{\tau_{gu} L (L-\lambda_g)} \right]_+ \frac{\lambda_g \tau_{gu}}{\rho} \frac{1-\rho L}{1-\lambda_g L} x_{lgt} \\
&= \left\{ \left[\frac{h_g(L)}{\tau_\varepsilon (1-\rho L)(L-\lambda_g)} \right]_+ + \left[\frac{h_g(L)(L-\rho)}{\tau_{gu} L (L-\lambda_g)} \right]_+ \right\} \frac{\lambda_g \tau_{gu}}{\rho} \frac{1-\rho L}{1-\lambda_g L} x_{lgt} \\
&= \left\{ \left[\frac{\phi_4(L)}{L-\lambda_g} \right]_+ + \left[\frac{\phi_5(L)}{L(L-\lambda_g)} \right]_+ \right\} \frac{\lambda_g \tau_{gu}}{\rho} \frac{1-\rho L}{1-\lambda_g L} x_{lgt} \\
&= \left\{ \frac{\phi_4(L)-\phi_4(\lambda_g)}{L-\lambda_g} + \frac{\phi_5(L)-\phi_5(\lambda_g)}{\lambda_g (L-\lambda_g)} - \frac{\phi_5(L)-\phi_5(0)}{\lambda_g L} \right\} \frac{\lambda_g \tau_{gu}}{\rho} \frac{1-\rho L}{1-\lambda_g L} x_{lgt} \\
&= \frac{\lambda_g}{\rho} \left\{ \frac{h_g(L)}{L-\lambda_g} \left[\frac{\tau_{gu}}{\tau_\varepsilon (1-\rho L)} + \frac{L-\rho}{L} \right] - \frac{h_g(\lambda_g)}{L-\lambda_g} \left[\frac{\tau_{gu}}{\tau_\varepsilon (1-\rho\lambda_g)} + \frac{\lambda_g-\rho}{\lambda_g} \right] - \frac{\rho h_g(0)}{\lambda_g L} \right\} \frac{1-\rho L}{1-\lambda_g L} x_{lgt} \\
&= \left\{ \frac{h_g(L)}{L-\lambda_g} \left[\left(1-\frac{\lambda_g}{\rho}\right) \frac{1-\rho\lambda_g}{1-\rho L} + \frac{\lambda_g (L-\rho)}{\rho L} \right] - \frac{h_g(0)}{L} \right\} \frac{1-\rho L}{1-\lambda_g L} x_{lgt}
\end{aligned}$$

(A54) $$= G_{4g}(L) x_{lgt}, \quad \phi_4(z) = \frac{h_g(z)}{\tau_\varepsilon (1-\rho z)}, \quad \phi_5(z) = \frac{h_g(z)(z-\rho)}{\tau_{gu}}$$

Inserting our obtained expressions into (A42),

$$\begin{aligned}
h_g(L) x_{lgt} &= \varphi_g G_{1g}(L) x_{lgt} + \beta_g G_{4g}(L) x_{lgt} + \sum_{k=1}^{n} \gamma_{gk} G_{3gk}(L) x_{lgt} + \sum_{k=1}^{n} \alpha_{gk} G_{2gk}(L) x_{lgt} \\
h_g(L) x_{lgt} &= \varphi_g \left(1-\frac{\lambda_g}{\rho}\right) \frac{1}{1-\lambda_g L} x_{lgt} + \beta_g \left\{ \frac{h_g(L)}{L-\lambda_g} \left[\left(1-\frac{\lambda_g}{\rho}\right) \frac{1-\rho\lambda_g}{1-\rho L} + \frac{\lambda_g (L-\rho)}{\rho L} \right] - \frac{h_g(0)}{L} \right\} \frac{1-\rho L}{1-\lambda_g L} x_{lgt} \\
&+ \sum_{k=1}^{2} \gamma_{gk} \frac{\lambda_g \tau_g}{\rho} \left[h_k(L) L - h_k(\lambda_g) \lambda_g \frac{1-\rho L}{1-\rho\lambda_g} \right] \frac{1}{(1-\lambda_g L)(L-\lambda_g)} x_{lgt} \\
&+ \sum_{k=1}^{2} \alpha_{gk} \frac{\lambda_g \tau_g}{\rho} \left[h_k(L) - h_k(\lambda_g) \frac{1-\rho L}{1-\rho\lambda_g} \right] \frac{1}{(1-\lambda_g L)(L-\lambda_g)} x_{lgt}
\end{aligned}$$

Removing the x_{lgt} terms, and rearranging terms on the LHS and RHS

$$\begin{aligned}
& h_g(z) \left\{ 1 - \beta_g \left[\left(1-\frac{\lambda_g}{\rho}\right) \frac{1-\rho\lambda_g}{1-\rho z} + \frac{\lambda_g (z-\rho)}{\rho z} \right] \frac{1-\rho z}{(1-\lambda_g z)(L-\lambda_g)} \right\} \\
& - \sum_{k=1}^{2} h_k(z) \frac{(\rho-\lambda_g)(1-\rho\lambda_g)}{\rho} \frac{\gamma_{gk} z + \alpha_{gk}}{(1-\lambda_g z)(z-\lambda_g)} \\
& = \varphi_g \left(1-\frac{\lambda_g}{\rho}\right) \frac{1}{1-\lambda_g z} - \beta_g \frac{1-\rho z}{z(1-\lambda_g z)} h_g(0) - \sum_{k=1}^{2} h_k(\lambda_g) \left(1-\frac{\lambda_g}{\rho}\right) \frac{\gamma_{gk}\lambda_g + \alpha_{gk}}{(1-\lambda_g z)(z-\lambda_g)} (1-\rho z)
\end{aligned}$$

Multiplying both sides by $z(z-\lambda_g)(1-\lambda_g z)$,

$$h_g(z) \left[z(z-\lambda_g)(1-\lambda_g z) - \beta_g (z-\lambda_g)(1-\lambda_g z) \right] - \sum_{k=1}^{2} h_k(z) \frac{(\rho-\lambda_g)(1-\rho\lambda_g)}{\rho} z(\gamma_{gk} z + \alpha_{gk})$$

$$= \varphi_g \left(1 - \frac{\lambda_g}{\rho}\right) z(z - \lambda_g) - \beta_g(1 - \rho z)(z - \lambda_g) h_g(0) - \sum_{k=1}^{2} h_k(\lambda_g) \left(1 - \frac{\lambda_g}{\rho}\right) (\gamma_{gk}\lambda_g + \alpha_{gk}) z(1 - \rho z)$$

I can write the above system of equations in terms of $\boldsymbol{h}(L)$ in matrix form

(A55) $$\boldsymbol{C}(z)\boldsymbol{h}(z) = \boldsymbol{d}(z)$$

where

$$\boldsymbol{C}(z) \equiv \text{diag}\left\{\lambda_g\right\} \left[(\boldsymbol{\beta} - Iz)\, \text{diag}\left\{ \left(z - \frac{1}{\rho}\right)(z - \rho) \right\} - (\boldsymbol{\beta} - Iz) z\, \text{diag}\left\{\frac{\tau_g}{\rho}\right\} - z\, \text{diag}\left\{\frac{\tau_g}{\rho}\right\} (z\boldsymbol{\gamma} + \boldsymbol{\alpha}) \right]$$

That is, I can write $\boldsymbol{C}(z) = \begin{bmatrix} C_{11}(z) & C_{12}(z) \\ C_{21}(z) & C_{22}(z) \end{bmatrix}$, where

$$C_{11}(z) = \lambda_1 \left[(\beta_1 - z)\left(z - \frac{1}{\rho}\right)(z - \rho) + \frac{\tau_1}{\rho} z \left[z(1 - \gamma_{11}) - \delta_{11}\right] \right], \quad C_{12}(z) = -\lambda_1 z \frac{\tau_1}{\rho}(z\gamma_{12} + \delta_{12})$$
$$C_{22}(z) = \lambda_2 \left[(\beta_2 - z)\left(z - \frac{1}{\rho}\right)(z - \rho) + \frac{\tau_2}{\rho} z \left[z(1 - \gamma_{22}) - \delta_{22}\right] \right], \quad C_{21}(z) = -\lambda_2 z \frac{\tau_2}{\rho}(z\gamma_{21} + \delta_{21})$$

I can also write $\boldsymbol{d}(z) = \begin{bmatrix} d_1[z; h_1(\cdot)] \\ d_2[z; h_2(\cdot)] \end{bmatrix}$, where

$$d_g(z) = \varphi_g \left(1 - \frac{\lambda_g}{\rho}\right) z(z - \lambda_g) - \beta_g(1 - \rho z)(z - \lambda_g) h_g(0) - \sum_{k=1}^{2} h_k(\lambda_g) \left(1 - \frac{\lambda_g}{\rho}\right) (\gamma_{gk}\lambda_g + \alpha_{gk}) z(1 - \rho z)$$

From (A55), the solution to the policy function is given by

$$\boldsymbol{h}(z) = \boldsymbol{C}(z)^{-1}\boldsymbol{d}(z) = \frac{\text{adj}\, \boldsymbol{C}(z)}{\det \boldsymbol{C}(z)} \boldsymbol{d}(z)$$

Hence, I need to obtain $\det \boldsymbol{C}(z)$. Note that the degree of $\det \boldsymbol{C}(z)$ is a polynomial of degree 6 on z. Denote the inside roots of $\det \boldsymbol{C}(z)$ as $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, and the outside roots as $\left\{\vartheta_1^{-1}, \vartheta_2^{-1}\right\}$. Because agents cannot use future signals, the inside roots have to be removed. Note that the number of free constants in $\boldsymbol{d}(z)$ is 4: $\left\{h_g(0)\right\}$ and $\left\{\widetilde{h}(\lambda_g) = \sum_{k=1}^{2} h_k(\lambda_g)\left(1 - \frac{\lambda_g}{\rho}\right)(\gamma_{gk}\lambda_g + \alpha_{gk})\right\}$ for each $g \in \{c, f\}$. With a unique solution, it has to be the case that the number of outside roots is 2. By Cramer's rule, $h_g(L)$ is given by

$$h_1(z) = \frac{\det \begin{bmatrix} d_1(z) & C_{12}(z) \\ d_2(z) & C_{22}(z) \end{bmatrix}}{\det \boldsymbol{C}(z)}, \qquad h_2(z) = \frac{\det \begin{bmatrix} C_{11}(z) & d_1(z) \\ C_{21}(z) & d_2(z) \end{bmatrix}}{\det \boldsymbol{C}(z)}$$

which are the policy function for groups 1 (consumers) and 2 (firms). The degree of the numerator is 5, as the highest degree of $d_g(z)$ is 1 degree less than that of $\boldsymbol{C}(z)$. By choosing the appropriate constants $\left\{h_1(0), \widetilde{h}(\lambda_1), h_2(0), \widetilde{h}(\lambda_2)\right\}$, the 4 inside roots will be removed. Therefore, the 4 constants are solutions to the following system of linear equations

$$\det \begin{bmatrix} d_1(\zeta_n) & C_{12}(\zeta_n) \\ d_2(\zeta_n) & C_{22}(\zeta_n) \end{bmatrix} = 0, \qquad \text{for } \{\zeta_n\}_{n=1}^4$$

After removing the inside roots in the denominator, the degree of the numerator is 1 and the degree of the denominator is 2. The above determinants can be written as a system of 4 equations and 4 unknowns (our free constants). Once I have set the appropriate free constants the policy functions will be $h_g(z) = \frac{\widetilde{\psi}_{g1}+\widetilde{\psi}_{g2}z}{(1-\vartheta_1 z)(1-\vartheta_2 z)}$, and hence I have

$$a_{gt} = h_g(L)\nu_t = \frac{\widetilde{\psi}_{g1} + \widetilde{\psi}_{g2}z}{(1-\vartheta_1 z)(1-\vartheta_2 z)}\nu_t = \sum_{j=1}^{2} \psi_{gj}\left(1 - \frac{\vartheta_j}{\rho}\right)\frac{1}{1-\vartheta_j L}\nu_t = \sum_{j=1}^{2} \psi_{gj}\widetilde{\vartheta}_{jt}$$

I can write

$$\boldsymbol{a}_t = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} = Q\widetilde{\vartheta}_t = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{bmatrix} \widetilde{\vartheta}_{1t} \\ \widetilde{\vartheta}_{2t} \end{bmatrix} = \begin{bmatrix} \psi_{11}\widetilde{\vartheta}_{1t} + \psi_{12}\widetilde{\vartheta}_{2t} \\ \psi_{21}\widetilde{\vartheta}_{1t} + \psi_{22}\widetilde{\vartheta}_{2t} \end{bmatrix}$$

Notice that I can write $\widetilde{\vartheta}_{gt}(1-\vartheta_g L) = \left(1 - \frac{\vartheta_g}{\rho}\right)\nu_t \implies \widetilde{\vartheta}_{gt} = \vartheta_g\widetilde{\vartheta}_{g,t-1} + \left(1-\frac{\vartheta_g}{\rho}\right)\nu_t$, which I can write as a system as $\widetilde{\vartheta}_t = \Lambda\widetilde{\vartheta}_{t-1} + \Gamma\nu_t$, where $\Lambda = \begin{bmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1-\frac{\vartheta_1}{\rho} \\ 1-\frac{\vartheta_2}{\rho} \end{bmatrix}$. Hence, I can write

(A56) $$\boldsymbol{a}_t = Q\widetilde{\theta}_t = Q(\Lambda\widetilde{\theta}_{t-1} + \Gamma\nu_t) = Q\Lambda\widetilde{\theta}_{t-1} + Q\Gamma\nu_t = Q\Lambda Q^{-1}\boldsymbol{a}_{t-1} + Q\Gamma\nu_t = A\boldsymbol{a}_{t-1} + B\nu_t$$

Finally, I need to show that (25) hold. First, notice that in the standard FIRE framework, there is no information friction, $\vartheta_1 = \vartheta_2 = 0$. Therefore, the dynamics follow $\boldsymbol{a}_t = A_{\text{FIRE}}\boldsymbol{a}_{t-1} + B_{\text{FIRE}}\nu_t$ where

$$A_{\text{FIRE}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_{\text{FIRE}} = \begin{bmatrix} \psi_{11} + \psi_{12} \\ \psi_{21} + \psi_{22} \end{bmatrix}$$

Under the standard FIRE case, dynamics are given by (36). To find the solution dynamics under FIRE, I proceed with a guess and verify approach. Assume that $\boldsymbol{a}_t = D\nu_t$. Using the method of undetermined coefficients

$$D\nu_t = \overline{\varphi}\nu_t + \overline{\delta}\mathbb{E}_t D\nu_{t+1} = \overline{\varphi}\nu_t + \overline{\delta}D\rho\nu_t \implies D = \overline{\varphi} + \overline{\delta}D\rho$$

hence, it must be that $D = (\boldsymbol{I} - \overline{\delta}\rho)^{-1}\overline{\varphi}$. Notice that, for consistency, $B_{\text{FIRE}} = D$. As a result, even if I cannot find the analytical form of the individual $(\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})$, I know that conditions (25) hold. □

Proof of Proposition 10. From the proof of proposition 9, I have the following objects

$$\pi_{t+k} = h_2(L)\nu_{t+k}$$

$$\overline{\mathbb{E}}_t^c \pi_{t+k} = \frac{(\rho-\lambda_1)(1-\rho\lambda_1)}{\rho(L-\lambda_1)(1-\lambda_1 L)}\left[L^{1-k}h_2(L) - \frac{1-\rho L}{1-\rho\lambda_1}\lambda_1^{1-k}h_2(\lambda_1)\right]$$

$$\pi_{t+k} - \overline{\mathbb{E}}_t^c \pi_{t+k} = \frac{\lambda_1}{\rho(L-\lambda_1)(1-\lambda_1 L)}\left[(L-\rho)L^{-k}h_2(L) + (\rho-\lambda_1)\lambda_1^{-k}h_2(\lambda_1)\right]\varepsilon_t$$

The forecast error of annual inflation is

$$\begin{aligned}
\pi_{t+3,3} - \overline{\mathbb{E}}_t^c \pi_{t+3,t} &= (\pi_t - \overline{\mathbb{E}}_t^c\pi_t) + (\pi_{t+1} - \overline{\mathbb{E}}_t^c\pi_{t+1}) + (\pi_{t+2} - \overline{\mathbb{E}}_t^c\pi_{t+2}) + (\pi_{t+3} - \overline{\mathbb{E}}_t^c\pi_{t+3}) \\
&= \frac{\lambda_1}{\rho(L-\lambda_1)(1-\lambda_1 L)}\left[(L-\rho)\left(\sum_{k=0}^{3} L^{-k}\right)h_2(L) + (\rho-\lambda_1)\left(\sum_{k=0}^{3}\lambda_1^{-k}\right)h_2(\lambda_1)\right]\varepsilon_t \\
&= \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)(1-\rho\vartheta_g)\lambda_1\psi_{2g}}{\rho^2(1-\lambda_1\vartheta_g)(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t \\
&+ \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\left[\rho(1-\lambda_1\vartheta_g) - \vartheta_g(\rho-\lambda_1)L\right]\psi_{2g}}{\rho^2(1-\lambda_1\vartheta_g)L(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t \\
&+ \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\left[\rho\lambda_1(1-\lambda_1\vartheta_g) + (\rho-\lambda_1)(1-\lambda_1\vartheta_g)L - \vartheta_g(\rho-\lambda_1)L^2\right]\psi_{2g}}{\rho^2\lambda_1(1-\lambda_1\vartheta_g)L^2(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t \\
&+ \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\left[(L^2+\lambda_1 L+\lambda_1^2)(\rho+\lambda_1\vartheta_g L) - (L+\lambda_1)[\lambda_1 L + (L^2+\lambda_1^2)\rho\vartheta_g]\right]\psi_{2g}}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)L^3(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t \\
&= \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)L^3(1-\lambda_1 L)(1-\vartheta_1 L)} \times \Big\{\rho\lambda_1^2(1-\lambda_1\vartheta_g) + \lambda_1(1-\lambda_1\vartheta_g)[\rho-(1-\rho)\lambda_1]L \\
&+ (1-\lambda_1\vartheta_g)[\rho-(1-\rho)\lambda_1(1+\lambda_1)]L^2 + [\lambda_1^3 - \vartheta_g(\rho-(1-\rho)\lambda_1(1+\lambda_1+\lambda_1^2))]L^3\Big\}\varepsilon_t \\
&= \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)}\frac{(1-\xi_{1g}L)(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\lambda_1 L)(1-\vartheta_g L)}\varepsilon_t \\
&= \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)}\frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3}\left(\frac{\vartheta_g-\xi_{1g}}{\vartheta_g-\lambda_1}\frac{1}{1-\vartheta_g L} - \frac{\lambda_1-\xi_{1g}}{\vartheta_g-\lambda_1}\frac{1}{1-\lambda_1 L}\right)\varepsilon_t \\
&= \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}(\vartheta_g-\xi_{1g})}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\vartheta_g L)}\varepsilon_t \\
&+ \sum_{g=1}^{2}\frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}(\xi_{1g}-\lambda_1)}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\lambda_1 L)}\varepsilon_t \\
&= \sum_{g=1}^{2}\gamma_{1g}\frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\vartheta_g L)}\varepsilon_t + \sum_{g=1}^{2}\gamma_{2g}\frac{(1-\xi_{2g}L)(1-\xi_{3g}L)}{L^3(1-\lambda_1 L)}\varepsilon_t \\
&= \sum_{g=1}^{2}\gamma_{1g}\frac{1-(\xi_{2g}+\xi_{3g})L+\xi_{2g}\xi_{3g}L^2}{L^3(1-\vartheta_g L)}\varepsilon_t + \sum_{g=1}^{2}\gamma_{2g}\frac{1-(\xi_{2g}+\xi_{3g})L+\xi_{2g}\xi_{3g}L^2}{L^3(1-\lambda_1 L)}\varepsilon_t
\end{aligned}$$

$$= \sum_{g=1}^{2} \left\{ \gamma_{1g} \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+3-k} - \gamma_{1g}(\xi_{2g} + \xi_{3g}) \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+2-k} + \gamma_{1g}\xi_{2g}\xi_{3g} \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+1-k} \right\}$$

$$+ (\gamma_{21} + \gamma_{22}) \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+3-k} - [\gamma_{21}(\xi_{21} + \xi_{31}) + \gamma_{22}(\xi_{22} + \xi_{32})] \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+2-k}$$

$$+ (\gamma_{21}\xi_{21}\xi_{31} + \gamma_{22}\xi_{22}\xi_{32}) \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+1-k}$$

$$= \sum_{g=1}^{2} \left\{ \beta_{1g} \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+3-k} + \beta_{2g} \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+2-k} + \beta_{3g} \sum_{k=0}^{\infty} \vartheta_g^k \varepsilon_{t+1-k} \right\}$$

$$+ \beta_4 \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+3-k} + \beta_5 \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+2-k} + \beta_6 \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t+1-k}$$

where $\xi_{0g} = \rho\lambda_1^2(1 - \lambda_1\vartheta_g)$, $-\xi_{0g}(\xi_{1g} + \xi_{2g} + \xi_{3g}) = \lambda_1(1 - \lambda_1\vartheta_g)[\rho - (1 - \rho)\lambda_1]$, $\xi_{0g}(\xi_{1g}\xi_{2g} + \xi_{1g}\xi_{3g} + \xi_{2g}\xi_{3g}) = (1 - \lambda_1\vartheta_g)[\rho - (1 - \rho)\lambda_1(1 + \lambda_1)]$, $-\xi_{0g}\xi_{1g}\xi_{2g}\xi_{3g} = \lambda_1^3 - \vartheta_g[\rho - (1 - \rho)\lambda_1(1 + \lambda_1 + \lambda_1^2)]$, $\gamma_{1g} = \frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}(\vartheta_g-\xi_{1g})}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}$, $\gamma_{2g} = \frac{(\rho-\vartheta_g)\psi_{2g}\xi_{0g}(\xi_{1g}-\lambda_1)}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}$, $\beta_{1g} = \gamma_{1g}$, $\beta_{2g} = -\gamma_{1g}(\xi_{2g} + \xi_{3g})$, $\beta_{3g} = \gamma_{1g}\xi_{2g}\xi_{3g}$, $\beta_4 = \gamma_{21} + \gamma_{22}$, $\beta_5 = -[\gamma_{21}(\xi_{21} + \xi_{31}) + \gamma_{22}(\xi_{22} + \xi_{32})]$, and $\beta_6 = \gamma_{21}\xi_{21}\xi_{31} + \gamma_{22}\xi_{22}\xi_{32}$. Before computing the forecast revision of annual inflation, notice that

$$\overline{\mathbb{E}}_t^c\pi_{t+k} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+k} = \frac{(\rho - \lambda_1)h_2(\lambda_1)}{\rho\lambda_1^k} \frac{1}{1 - \lambda_1 L} \varepsilon_t$$

Therefore, the forecast revision of annual inflation is

$$\begin{aligned}\overline{\mathbb{E}}_t^c\pi_{t+3,t} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+3,t} &= (\overline{\mathbb{E}}_t^c\pi_t - \overline{\mathbb{E}}_{t-1}^c\pi_t) + (\overline{\mathbb{E}}_t^c\pi_{t+1} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+1}) + (\overline{\mathbb{E}}_t^c\pi_{t+2} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+2}) \\ &+ (\overline{\mathbb{E}}_t^c\pi_{t+3} - \overline{\mathbb{E}}_{t-1}^c\pi_{t+3}) \\ &= \frac{(\rho - \lambda_1)h_2(\lambda_1)(1 + \lambda_1^{-1} + \lambda_1^{-2} + \lambda_1^{-3})}{\rho} \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t-k} = \alpha \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t-k}\end{aligned}$$

where $\alpha = \frac{(\rho-\lambda_1)(1+\lambda_1+\lambda_1^2+\lambda_1^3)\sum_{g=1}^{2}\psi_{2g}\frac{\rho-\vartheta_g}{1-\lambda_1\vartheta_g}}{\rho^2\lambda_1^3}$. I now seek to compute the OLS coefficient. The covariance is

$$\mathbb{C}\,(\text{forecast error, revision}) = \left[\sum_{g=1}^{2} \frac{\beta_{1g}\alpha\vartheta_g^3 + \beta_{2g}\alpha\vartheta_g^2 + \beta_{3g}\alpha\vartheta_g}{1 - \lambda_1\vartheta_g} + \frac{\beta_4\alpha\lambda_1^3 + \beta_5\alpha\lambda_1^2 + \beta_6\alpha\lambda_1}{1 - \lambda_1^2} \right] \sigma_\varepsilon^2$$

The variance is $\mathbb{V}\,(\text{revision}) = \frac{\alpha^2}{1-\lambda_2^2}\sigma_\varepsilon^2$. Finally, the OLS coefficient is

$$\begin{aligned}\beta_{CG} &= \frac{\mathbb{C}\,(\text{forecast error, revision})}{\mathbb{V}\,(\text{revision})} \\ &= \frac{1}{\alpha} \left[\sum_{g=1}^{2} (\beta_{1g}\vartheta_g^3 + \beta_{2g}\vartheta_g^2 + \beta_{3g}\vartheta_g) \frac{1 - \lambda_1^2}{1 - \lambda_1\vartheta_g} + \beta_4\lambda_1^3 + \beta_5\lambda_1^2 + \beta_6\lambda_1 \right]\end{aligned}$$

$$= \frac{1}{\alpha}\left[\sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}\vartheta_g(1-\lambda_1^2)}{\rho^2\lambda_1^2(1-\lambda_1\vartheta_g)^2(\vartheta_g-\lambda_1)}\xi_{0g}(\vartheta_g-\xi_{1g})(\vartheta_g-\xi_{2g})(\vartheta_g-\xi_{3g})\right.$$

$$\left.-\sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}}{\rho^2\lambda_1(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\xi_{0g}(\lambda_1-\xi_{1g})(\lambda_1-\xi_{2g})(\lambda_1-\xi_{3g})\right]$$

$$= \frac{1}{\alpha}\left[\sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}\lambda_1\vartheta_g(1-\lambda_1^2)(1+\vartheta_g)(1+\vartheta_g^2)(1-\rho\vartheta_g)}{\rho^2(1-\lambda_1\vartheta_g)^2(\vartheta_g-\lambda_1)}\right.$$

$$\left.+\sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}(1+\lambda_1^2)\{(\rho-\lambda_1)[\vartheta_g(1+\lambda_1)-\lambda_1(1-\lambda_1\vartheta_g)]-\rho\lambda_1^2(1+\lambda_1)(1-\lambda_1\vartheta_g)\}}{\rho^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\right]$$

$$= \frac{1}{\alpha}\sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}}{\rho^2(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\left[\frac{\lambda_1\vartheta_g(1-\lambda_1^2)(1+\vartheta_g)(1+\vartheta_g^2)(1-\rho\vartheta_g)}{1-\lambda_1\vartheta_g}\right.$$

$$\left.+(1+\lambda_1^2)\{(\rho-\lambda_1)[\vartheta_g(1+\lambda_1)-\lambda_1(1-\lambda_1\vartheta_g)]-\rho\lambda_1^2(1+\lambda_1)(1-\lambda_1\vartheta_g)\}\right]$$

$$= \frac{\lambda_1^3}{(\rho-\lambda_1)(1+\lambda_1+\lambda_1^2+\lambda_1^3)\sum_{k=1}^{2}\psi_{2g}\frac{\rho-\vartheta_k}{1-\lambda_1\vartheta_k}}\sum_{g=1}^{2} \frac{(\rho-\vartheta_g)\psi_{2g}}{(1-\lambda_1\vartheta_g)(\vartheta_g-\lambda_1)}\left[\frac{\lambda_1\vartheta_g(1-\lambda_1^2)(1+\vartheta_g)(1+\vartheta_g^2)(1-\rho\vartheta_g)}{1-\lambda_1\vartheta_g}\right.$$

$$\left.+(1+\lambda_1^2)\{(\rho-\lambda_1)[\vartheta_g(1+\lambda_1)-\lambda_1(1-\lambda_1\vartheta_g)]-\rho\lambda_1^2(1+\lambda_1)(1-\lambda_1\vartheta_g)\}\right]$$

□

Proof of Proposition 11. The aggregate outcome is

$$y_t = \sum_{g=1}^{2}\psi_{1g}\left(1-\frac{\vartheta_g}{\rho}\right)\frac{1}{1-\vartheta_g L} = \sum_{g=1}^{2}\psi_{1g}\left(1-\frac{\vartheta_g}{\rho}\right)\frac{1}{(1-\vartheta_g L)(1-\rho L)}\varepsilon_t$$

$$= \sum_{g=1}^{2}\psi_{1g}\left(1-\frac{\vartheta_g}{\rho}\right)\left[\frac{\rho}{\rho-\vartheta_g}\frac{1}{1-\rho L}-\frac{\vartheta_g}{\rho-\vartheta_g}\frac{1}{1-\vartheta_g L}\right]\varepsilon_t = \sum_{g=1}^{2}\frac{\psi_{1g}}{\rho}\sum_{k=0}^{\infty}(\rho^{k+1}-\vartheta_g^{k+1})\varepsilon_{t-k}$$

The GE component is given by

$$\text{GE}_t = [1-\beta(1-\lambda\chi)]\overline{\mathbb{E}}_t y_t + (\delta-\beta)(1-\lambda\chi)\beta\sum_{k=0}^{\infty}\beta^k\overline{\mathbb{E}}_t y_{t+k+1}$$

where

$$\overline{\mathbb{E}}_t^c y_t = \sum_{j=1}^{2}\psi_{1j}\left(1-\frac{\vartheta_j}{\rho}\right)\overline{\mathbb{E}}_t^c\left[\frac{1}{1-\vartheta_j L}\nu_t\right]$$

$$= \sum_{j=1}^{2}\psi_{1j}\left(1-\frac{\vartheta_j}{\rho}\right)\left[\begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{(1-\vartheta_j L)(1-\rho L)} & 0\end{bmatrix}\begin{bmatrix}\frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L^{-1}} \\ \tau_1^{-\frac{1}{2}}\end{bmatrix}\frac{1-\rho L^{-1}}{1-\lambda_1 L^{-1}}\right]_+\frac{\lambda_1\tau_1}{\rho}\frac{1}{1-\lambda_1 L}\varepsilon_t$$

$$= \sum_{j=1}^{2} \psi_{1j} \left(1 - \frac{\vartheta_j}{\rho}\right) \left\{ \frac{\rho}{\rho - \vartheta_j} \sum_{k=0}^{\infty} \rho^k \varepsilon_{t-k} - \frac{\vartheta_j^2(\rho - \lambda_1)(1 - \rho\lambda_1)}{\rho(\rho - \vartheta_j)(\vartheta_j - \lambda_1)(1 - \vartheta_j\lambda_1)} \sum_{k=0}^{\infty} \vartheta_j^k \varepsilon_{t-k} \right.$$

$$\left. + \frac{\lambda_1^2(1 - \rho\vartheta_j)}{\rho(\vartheta_j - \lambda_1)(1 - \vartheta_j\lambda_1)} \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t-k} \right\}$$

$$\sum_{k=0}^{\infty} \beta^k \overline{\mathbb{E}}_t^c y_{t+k+1} = \overline{\mathbb{E}}_t^c \frac{y_t}{L - \beta} = \sum_{j=1}^{2} \psi_{1j} \left(1 - \frac{\vartheta_j}{\rho}\right) \overline{\mathbb{E}}_t^c \left[\frac{1}{(L - \beta)(1 - \vartheta_j L)} \nu_t \right]$$

$$= \sum_{j=1}^{2} \psi_{1j} \left(1 - \frac{\vartheta_j}{\rho}\right) \left[\begin{bmatrix} \frac{\tau_\varepsilon^{-\frac{1}{2}}}{(L-\beta)(1-\vartheta_j L)(1-\rho L)} & 0 \end{bmatrix} \begin{bmatrix} \frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L^{-1}} \\ \tau_1^{-\frac{1}{2}} \end{bmatrix} \frac{1 - \rho L^{-1}}{1 - \lambda_1 L^{-1}} \right]_+ \frac{\lambda_1 \tau_1}{\rho} \frac{1}{1 - \lambda_1 L} \varepsilon_t$$

$$= \sum_{j=1}^{2} \psi_{1j} \left(1 - \frac{\vartheta_j}{\rho}\right) \left\{ \frac{\rho^2}{(1 - \rho\beta)(\rho - \vartheta_j)} \sum_{k=0}^{\infty} \rho^k \varepsilon_{t-k} \right.$$

$$- \frac{\vartheta_j^3(\rho - \lambda_1)(1 - \rho\lambda_1)}{\rho(\rho - \vartheta_j)(1 - \beta\vartheta_j)(\vartheta_j - \lambda_1)(1 - \vartheta_j\lambda_1)} \sum_{k=0}^{\infty} \vartheta_j^k \varepsilon_{t-k}$$

$$\left. + \frac{\lambda_1 \left[\rho\lambda_1(1 - \beta\vartheta_j) + \lambda_1\vartheta_j(1 - \rho\lambda_1) - \rho\vartheta_j(1 - \rho\beta\lambda_1\vartheta_j)\right]}{\rho(1 - \rho\beta)(1 - \beta\vartheta_j)(\vartheta_j - \lambda_1)(1 - \vartheta_j\lambda_1)} \sum_{k=0}^{\infty} \lambda_1^k \varepsilon_{t-k} \right\}$$

Hence, I have

$$\mathrm{GE}_t = [1 - \beta(1 - \lambda\chi)]\overline{\mathbb{E}}_t y_t + (\delta - \beta)(1 - \lambda\chi)\beta \sum_{k=0}^{\infty} \beta^k \overline{\mathbb{E}}_t y_{t+k+1}$$

$$= [1 - \beta(1 - \lambda\chi)] \sum_{j=1}^{2} \psi_{1j} \left\{ \sum_{k=0}^{\infty} \rho^k - \frac{\vartheta_j^2(\rho - \lambda_1)(1 - \rho\lambda_1)}{\rho^2(\vartheta_j - \lambda_1)(1 - \vartheta_j\lambda_1)} \sum_{k=0}^{\infty} \vartheta_j^k + \frac{\lambda_1^2(\rho - \vartheta_j)(1 - \rho\vartheta_j)}{\rho^2(\vartheta_j - \lambda_1)(1 - \vartheta_j\lambda_1)} \sum_{k=0}^{\infty} \lambda_1^k \right\} \varepsilon_{t-k}$$

$$+ (\delta - \beta)(1 - \lambda\chi)\beta \sum_{j=1}^{2} \psi_{1j} \left\{ \frac{\rho}{1 - \rho\beta} \sum_{k=0}^{\infty} \rho^k - \frac{\vartheta_j^3(\rho - \lambda_1)(1 - \rho\lambda_1)}{\rho^2(1 - \beta\vartheta_j)(\vartheta_j - \lambda_1)(1 - \vartheta_j\lambda_1)} \sum_{k=0}^{\infty} \vartheta_j^k \right.$$

$$\left. + \frac{\lambda_1(\rho - \vartheta_j) \left[\rho\lambda_1(1 - \beta\vartheta_j) + \lambda_1\vartheta_j(1 - \rho\lambda_1) - \rho\vartheta_j(1 - \rho\beta\lambda_1\vartheta_j)\right]}{\rho^2(1 - \rho\beta)(1 - \beta\vartheta_j)(\vartheta_j - \lambda_1)(1 - \vartheta_j\lambda_1)} \sum_{k=0}^{\infty} \lambda_1^k \right\} \varepsilon_{t-k}$$

$$= \left(\delta_1 \sum_{k=0}^{\infty} \rho^k + \delta_2 \sum_{k=0}^{\infty} \lambda_1^k + \sum_{j=1}^{2} \delta_{3j} \sum_{k=0}^{\infty} \vartheta_j^k \right) \varepsilon_{t-k}$$

Therefore, the PE share μ_τ is given by

$$\mu_\tau = 1 - \frac{\partial \mathrm{GE}_\tau / \partial \varepsilon_t}{\partial \mathrm{TE}_\tau / \partial \varepsilon_t} = 1 - \frac{\delta_1 \rho^\tau + \delta_2 \lambda_1^\tau + \sum_{j=1}^{2} \delta_{3j} \vartheta_j^\tau}{\sum_{g=1}^{2} \frac{\psi_{1g}}{\rho} (\rho^{\tau+1} - \vartheta_g^{\tau+1})}$$

$$= \frac{\rho \left(\sum_{g=1}^{2} \psi_{1g} - \delta_1 \right) \rho^\tau - \rho\delta_2 \lambda_1^\tau - \sum_{g=1}^{2} (\psi_{1g}\vartheta_g + \delta_{3j}) \vartheta_g^\tau}{\rho \sum_{g=1}^{2} \psi_{1g} \rho^\tau - \sum_{g=1}^{2} \psi_{1g} \vartheta_g \vartheta_g^\tau}$$

□

Proof of Proposition 12. Recall that equilibrium dynamics satisfy (36). I need to find the conditions under which the equilibrium process is stationary. This sums up to having all the eigenvalues in the matrix $\bar{\delta}^{-1}$ outside the unit circle. This restriction is satisfied if

$$\det \bar{\delta}^{-1} > 1 \tag{A57}$$

$$\det \bar{\delta}^{-1} - \operatorname{tr} \bar{\delta}^{-1} > -1 \tag{A58}$$

$$\det \bar{\delta}^{-1} + \operatorname{tr} \bar{\delta}^{-1} > -1 \tag{A59}$$

Introducing the respective values in (A57)-(A59), I obtain (27)-(29). □

Proof of Proposition 13. Recall the equilibrium dynamics described by Proposition 9. I need to find the conditions under which the equilibrium process is stationary. This sums up to having all the eigenvalues in matrix A inside the unit circle. This restriction is satisfied if

$$\det A < 1 \tag{A60}$$

$$\det A - \operatorname{tr} A > -1 \tag{A61}$$

$$\det A + \operatorname{tr} A > -1 \tag{A62}$$

Notice that $\operatorname{tr} A = \vartheta_1 + \vartheta_2$, where ϑ_1 and ϑ_2 are the two roots of the characteristic polynomial of A, and $\det A = \vartheta_1 \vartheta_2$. Therefore, the above conditions can be translated to

$$\vartheta_1 \vartheta_2 < 1$$
$$(\vartheta_1 - 1)(\vartheta_2 - 1) > 0$$
$$(\vartheta_1 + 1)(\vartheta_2 + 1) > 0$$

Notice that such a system can only be satisfied if both roots are inside the unit circle. Introducing the respective values in (A60)-(A62), I obtain (32)-(34). □

Proof of Proposition 14. I first prove (i). Guess an ad-hoc system of dynamics, such that

$$\boldsymbol{x}_t = \omega_b \boldsymbol{x}_{t-1} + \bar{\delta} \omega_f \mathbb{E}_t \boldsymbol{x}_{t+1} + \bar{\varphi} v_t \tag{A63}$$

for some arbitrary 2×2 matrices (ω_b, ω_f). To show that the ad-hoc model presented above captures our HANK beyond FIRE under certain (ω_f, ω_b), I rely on the Method for Undetermined Coefficients. Both dynamics are observationally equivalent if

$$\begin{aligned} A\boldsymbol{x}_{t-1} + Bv_t &= \bar{\varphi} v_t + \bar{\delta} \omega_f \mathbb{E}_t \boldsymbol{x}_{t+1} + \omega_b \boldsymbol{x}_{t-1} \\ &= \bar{\varphi} v_t + \bar{\delta} \omega_f \mathbb{E}_t (A\boldsymbol{x}_t + Bv_{t+1}) + \omega_b \boldsymbol{x}_{t-1} \\ &= \bar{\varphi} v_t + \bar{\delta} \omega_f (A\boldsymbol{x}_t + B\mathbb{E}_t v_{t+1}) + \omega_b \boldsymbol{x}_{t-1} \\ &= \bar{\varphi} v_t + \bar{\delta} \omega_f (A\boldsymbol{x}_t + B\rho v_t) + \omega_b \boldsymbol{x}_{t-1} \end{aligned}$$

$$= \overline{\varphi} v_t + \overline{\delta} \omega_f [A(A\boldsymbol{x}_{t-1} + Bv_t) + B\rho v_t] + \omega_b \boldsymbol{x}_{t-1}$$
$$= \left[\overline{\delta}\omega_f AA + \omega_b\right] \boldsymbol{x}_{t-1} + \left[\overline{\varphi} + \overline{\delta}\omega_f (A+\rho)B\right] v_t$$

They are thus equivalent when (38) is satisfied. Now that I have the system dynamics (A63), I just need to multiply the system by $\widetilde{A}$ to back out the DIS curve, which I can write as (39).

I now move to (*ii*). Using the lag operator, I can factorize (40)

$$\mathbb{E}_t\left[\left(\frac{1}{\nu} + \omega_{f\pi}\right) r_t - \omega_{b\pi}\pi_{t-1}\right] = \mathbb{E}_t\left[\left(\omega_{fy}L^{-2} - L^{-1} + \omega_{by}\right) y_{t-1}\right]$$
$$= \mathbb{E}_t\left[\omega_{fy}\left(L^{-1} - \gamma_1^{-1}\right)\left(L^{-1} - \gamma_2^{-1}\right) y_{t-1}\right]$$

where γ_1^{-1} and γ_2^{-1} are the roots of the polynomial $\mathcal{P}(x) \equiv \omega_{fy}x^2 - x + \omega_{by}$. Dividing both sides by $(L^{-1} - \gamma_2^{-1})$

$$\omega_{fy}\mathbb{E}_t[(L^{-1} - \gamma_1^{-1})y_{t-1}] = \mathbb{E}_t\left[\left(\frac{1}{\nu} + \omega_{f\pi}\right)\frac{1}{L^{-1} - \gamma_2^{-1}} r_t - \omega_{b\pi}\frac{1}{L^{-1} - \gamma_2^{-1}}\pi_{t-1}\right]$$
$$= \mathbb{E}_t\left[-\left(\frac{1}{\nu} + \omega_{f\pi}\right)\frac{\gamma_2}{1 - \gamma_2 L^{-1}} r_t + \omega_{b\pi}\frac{\gamma_2}{1 - \gamma_2 L^{-1}}\pi_{t-1}\right]$$

Hence, I can write the dynamics as

$$y_t = \gamma_1^{-1} y_{t-1} + \frac{\gamma_2\omega_{b\pi}}{\omega_{fy}}(\pi_{t-1} + \gamma_2\pi_t) - \frac{\gamma_2}{\omega_{fy}}\left(\frac{1}{\nu} + \omega_{f\pi} + \omega_{b\pi}\gamma_2^2\right)\sum_{k=0}^{\infty}\gamma_2^k \mathbb{E}_t r_{t+k}$$
$$= \gamma_1^{-1} y_{t-1} + \frac{\omega_{b\pi}}{\gamma_1\omega_{by}}\left(\pi_{t-1} + \frac{\omega_{fy}}{\gamma_1\omega_{by}}\pi_t\right) - \frac{1}{\gamma_1\omega_{by}}\left(\frac{1}{\nu} + \omega_{f\pi} + \omega_{b\pi}\frac{\omega_{fy}^2}{\gamma_1^2\omega_{by}^2}\right)\sum_{k=0}^{\infty}\left(\frac{\omega_{fy}}{\gamma_1\omega_{by}}\right)^k \mathbb{E}_t r_{t+k}$$

where I have applied the Vieta properties. Therefore, the effect of a forward guidance shock promised at time t in period τ is

$$FG_{t,t+\tau} = \frac{\partial y_t}{\partial \mathbb{E}_t r_{t+\tau}} = -\frac{1}{\gamma_1\omega_{by}}\left(\frac{1}{\nu} + \omega_{f\pi} + \omega_{b\pi}\frac{\omega_{fy}^2}{\gamma_1^2\omega_{by}^2}\right)\left(\frac{\omega_{fy}}{\gamma_1\omega_{by}}\right)^{\tau}$$

which is decreasing in τ provided that $\gamma_1 \in (0,1)$ is the only inside root, $\lim_{\tau\to\infty} FG_{t+\tau} = 0$, and the forward guidance puzzle is solved. □

Proof of Proposition 15. The proof is identical to the proof of Proposition 9, modulo the replacement of σ_g for σ_ϵ. In the public information case, the individual action is given by $a_{lgt} = h_g(L)z_t = h_g(L)(v_t + \epsilon_t)$. The policy function of an agent in group g is given by $h_g(z) = \frac{\widetilde{\psi}_{g1} + \widetilde{\psi}_{g2}z}{(1-\theta_1 z)(1-\theta_2 z)}$, and hence I have $a_{gt} = h_g(L)(v_t + \epsilon_t) = \frac{\widetilde{\psi}_{g1} + \widetilde{\psi}_{g2}z}{(1-\theta_1 z)(1-\theta_2 z)}(v_t + \epsilon_t) = \psi_{g1}\left(1 - \frac{\theta_1}{\rho}\right)\frac{1}{1-\theta_1 L}(v_t + \epsilon_t) + \psi_{g2}\left(1 - \frac{\theta_2}{\rho}\right)\frac{1}{1-\theta_2 L}(v_t + \epsilon_t) = \psi_{g1}\widetilde{\theta}_{1t} + \psi_{g2}\widetilde{\theta}_{2t}$. I can write

$$\boldsymbol{a}_t = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} = Q\widetilde{\theta}_t = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}\begin{bmatrix} \widetilde{\theta}_{1t} \\ \widetilde{\theta}_{2t} \end{bmatrix} = \begin{bmatrix} \psi_{11}\widetilde{\theta}_{1t} + \psi_{12}\widetilde{\theta}_{2t} \\ \psi_{21}\widetilde{\theta}_{1t} + \psi_{22}\widetilde{\theta}_{2t} \end{bmatrix}$$

Notice that I can write

$$\widetilde{\theta}_{gt}(1-\theta_g L) = \left(1-\frac{\theta_g}{\rho}\right)(\nu_t+\epsilon_t) \implies \widetilde{\theta}_{gt} = \theta_g\widetilde{\theta}_{g,t-1} + \left(1-\frac{\theta_g}{\rho}\right)(\nu_t+\epsilon_t)$$

Which I can write as a system as $\widetilde{\theta}_t = \Lambda\widetilde{\theta}_{t-1} + \Gamma(\nu_t+\epsilon_t)$, where

$$\Lambda = \begin{bmatrix}\theta_1 & 0\\ 0 & \theta_2\end{bmatrix}, \qquad \Gamma = \begin{bmatrix}1-\frac{\theta_1}{\rho}\\ 1-\frac{\theta_2}{\rho}\end{bmatrix}$$

Hence, I can write

$$\begin{aligned}\boldsymbol{a}_t &= Q\widetilde{\theta}_t = Q[\Lambda\widetilde{\theta}_{t-1} + \Gamma(\nu_t+\epsilon_t)] = Q\Lambda\widetilde{\theta}_{t-1} + Q\Gamma(\nu_t+\epsilon_t) = Q\Lambda Q^{-1}\boldsymbol{a}_{t-1} + Q\Gamma(\nu_t+\epsilon_t)\\ &= A\boldsymbol{a}_{t-1} + B\nu_t + B\epsilon_t \end{aligned} \tag{A64}$$

□

Proof of Proposition 16. This proof mimics the proof of Proposition 9 and extends it to allow for a public signal. In this case the fundamental representation of the signal process as a system containing (22), (23) and (41), which admits the state-space representation $\boldsymbol{Z}_t = \boldsymbol{F}\boldsymbol{Z}_{t-1} + \boldsymbol{\Phi}\boldsymbol{s}_{lgt}$ and $\boldsymbol{X}_{gt} = \boldsymbol{H}\boldsymbol{Z}_t + \boldsymbol{\Psi}_g\boldsymbol{s}_{lgt}$, with $\boldsymbol{F} = \rho$, $\boldsymbol{\Phi} = \begin{bmatrix}0 & 0 & \sigma_\varepsilon\end{bmatrix}$, $\boldsymbol{Z}_t = \nu_t$, $\boldsymbol{s}_{lgt} = \begin{bmatrix}\epsilon_t & u_{lgt} & \varepsilon_t^{\nu}\end{bmatrix}^\top$, $\boldsymbol{H} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, and $\boldsymbol{\Psi}_g = \begin{bmatrix}\sigma_\epsilon & 0 & 0\\ 0 & \sigma_g & 0\end{bmatrix}$ and $\boldsymbol{X}_{gt} = \begin{bmatrix}z_t & x_{lgt}\end{bmatrix}^\top$. It is convenient to rewrite the uncertainty parameters in terms of precision: define $\tau_\varepsilon \equiv \frac{1}{\sigma_\varepsilon^2}$, $\tau_g \equiv \frac{1}{\sigma_g^2}$, and $\tau_\epsilon = \frac{1}{\sigma_\epsilon^2}$. The signal system can be written as

$$\boldsymbol{X}_{gt} = \begin{bmatrix}\tau_\epsilon^{-\frac{1}{2}} & 0 & \frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L}\\ 0 & \tau_g^{-\frac{1}{2}} & \frac{\tau_\varepsilon^{-\frac{1}{2}}}{1-\rho L}\end{bmatrix}\begin{bmatrix}\widehat{\varepsilon}_t\\ \widehat{u}_{lgt}\end{bmatrix} = \boldsymbol{M}_g(L)\boldsymbol{s}_{lgt}, \qquad \boldsymbol{s}_{lgt} \sim \mathcal{N}(0, I)$$

Denote λ_g as the inside root of $\det[\boldsymbol{M}_g(L)\boldsymbol{M}_g'(L)]$, which is given by

$$\lambda_g = \frac{1}{2}\left[\frac{1}{\rho} + \rho + \frac{\tau_g+\tau_\epsilon}{\rho\tau_\varepsilon} - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{\tau_g+\tau_\epsilon}{\rho\tau_\varepsilon}\right)^2 - 4}\right] \tag{A65}$$

I can also write

$$V_g^{-1} = \frac{\tau_g\tau_\epsilon}{\rho\tau_\varepsilon(\tau_g+\tau_\epsilon)}\begin{bmatrix}\frac{\rho\tau_g+\lambda_g\tau_\epsilon}{\tau_g} & \lambda_g-\rho\\ \lambda_g-\rho & \frac{\lambda_g\tau_g+\rho\tau_\epsilon}{\tau_\epsilon}\end{bmatrix}, \quad B_g(L)^{-1} = \frac{1}{1-\lambda_g L}\begin{bmatrix}1-\frac{\lambda_g\tau_g+\rho\tau_\epsilon}{\tau_g+\tau_\epsilon}L & \frac{\tau_g(\lambda_g-\rho)}{\tau_g+\tau_\epsilon}L\\ \frac{\tau_\epsilon(\lambda_g-\rho)}{\tau_g+\tau_\epsilon}L & 1-\frac{\rho\tau_g+\lambda_g\tau_\epsilon}{\tau_g+\tau_\epsilon}L\end{bmatrix}$$

Let us now move to the forecasting part. Denote agent i in group g policy function $a_{lgt} = h_{g1}(L)z_t + h_{g2}(L)x_{lgt}$. The aggregate outcome in group g can then be expressed as follows

$$a_{gt} = \int a_{lgt}\,di = \int h_{g1}(L)z_t + h_{g2}(L)x_{lgt}\,di$$

$$= \int h_{g1}(L)\left(\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t + \sigma_\epsilon \epsilon_t\right) + h_{g2}(L)\left(\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t + \sigma_g u_{lgt}\right)\, di$$
$$= [h_{g1}(L) + h_{g2}(L)]\frac{\sigma_\varepsilon}{1-\rho L}\varepsilon_t + h_{g1}(L)\sigma_\epsilon \epsilon_t$$

Hence, the forecasts are

$$\mathbb{E}_{lgt}\nu_t = \frac{\lambda_g}{\rho\tau_\varepsilon(1-\lambda_g\rho)}\frac{1}{1-\lambda_g L}\begin{bmatrix}\tau_\epsilon & \tau_g\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix}$$

$$\begin{aligned}
\mathbb{E}_{lgt}a_{kt+1} = &\begin{bmatrix}\frac{h_{k1}(L)}{L\tau_\varepsilon} + h_{k2}(L)\frac{\lambda_g\tau_\epsilon}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2} & h_{k2}(L)\frac{\lambda_g\tau_g}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix} - \\
&- \frac{\lambda_g(1-\rho L)h_{k2}(\lambda_g)}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}\tau_\epsilon & \tau_g\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix} - \\
&- \frac{\lambda_g h_{k1}(0)}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}\frac{(1-\lambda_g L)\tau_g + (1-\rho L)\tau_\epsilon}{L(\rho-\lambda_g)} & -\tau_g\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{lgt}a_{kt} = &\begin{bmatrix}\frac{h_{k1}(L)}{\tau_\varepsilon} + h_{k2}(L)\frac{L\lambda_g\tau_\epsilon}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2} & h_{k2}(L)\frac{L\lambda_g\tau_g}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix} - \\
&- \frac{\lambda_g^2(1-\rho L)h_{k2}(\lambda_g)}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}\tau_\epsilon & \tau_g\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{lgt}\left(a_{igt+1} - a_{gt+1}\right) = &\frac{\lambda_g h_{g2}(L)}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\begin{bmatrix}-\tau_\epsilon & \frac{(L-\rho)(1-\rho L)\lambda_g\tau_g + (L-\lambda_g)(1-\lambda_g L)\rho\tau_\epsilon}{L(\rho-\lambda_g)(1-\rho\lambda_g)}\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix} - \\
&- \frac{\lambda_g(1-\rho L)h_{g2}(\lambda_g)}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}-\tau_\epsilon & \tau_g\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix} - \\
&- \frac{\lambda_g h_{g2}(0)}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2}\begin{bmatrix}-\tau_\epsilon & \frac{(1-\rho L)\tau_g + (1-\lambda_g L)\tau_\epsilon}{L(\rho-\lambda_g)}\end{bmatrix}\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix}
\end{aligned}$$

Introducing the expectations just calculated into the best response (A42), and rearranging terms,

$$\begin{aligned}
&\Bigg[h_{g1}(L)\left(1-\frac{\beta_g}{L\tau_\varepsilon}\right) - \sum_{k=1}^{2}\frac{h_{k1}(L)}{\tau_\varepsilon}\left(\gamma_{gk} + \frac{\alpha_{gk}}{L}\right) - \sum_{k=1}^{2}\frac{h_{k2}(L)\lambda_g\tau_\epsilon}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\left(\gamma_{gk}L + \alpha_{gk}\right), \\
&\quad h_{g2}(L)\left(1-\frac{\beta_g}{L\tau_\varepsilon}\right) - \sum_{k=1}^{2}\frac{h_{k2}(L)\lambda_g\tau_g}{(L-\lambda_g)(1-\lambda_g L)\rho\tau_\varepsilon^2}\left(\gamma_{gk}L + \alpha_{gk}\right)\Bigg]\begin{bmatrix}z_t\\ x_{lgt}\end{bmatrix} = \\
&= \Bigg[\frac{\varphi_g\lambda_g\tau_\epsilon}{\rho\tau_\varepsilon(1-\rho\lambda_g)(1-\lambda_g L)} - h_{g1}(0)\frac{\beta_g\lambda_g[(1-\lambda_g L)\tau_g + (1-\rho L)\tau_\epsilon]}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2 L(\rho-\lambda_g)} + h_{g2}(0)\frac{\beta_g\lambda_g\tau_\epsilon}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2} - \\
&\quad - \sum_{k=1}^{2}h_{k1}(0)\frac{\alpha_{gk}\lambda_g[(1-\lambda_g L)\tau_g + (1-\rho L)\tau_\epsilon]}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2 L(\rho-\lambda_g)} - \sum_{k=1}^{2}h_{k2}(\lambda_g)\frac{\lambda_g(1-\rho L)\tau_\epsilon}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}(\alpha_{gk} + \lambda_g\gamma_{gk}), \\
&\quad \frac{\varphi_g\lambda_g\tau_g}{\rho\tau_\varepsilon(1-\rho\lambda_g)(1-\lambda_g L)} + h_{g1}(0)\frac{\beta_g\lambda_g\tau_g}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2} - h_{g2}(0)\frac{\beta_g\lambda_g[(1-\rho L)\tau_g + (1-\lambda_g L)\tau_\epsilon]}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2 L(\rho-\lambda_g)} +
\end{aligned}$$

$$+\sum_{k=1}^{2} h_{k1}(0)\frac{\alpha_{gk}\lambda_g\tau_g}{(1-\lambda_g L)(1-\rho\lambda_g)\tau_\varepsilon^2} - \sum_{k=1}^{2} h_{k2}(\lambda_g)\frac{\lambda_g(1-\rho L)\tau_g}{(L-\lambda_g)(1-\lambda_g L)\rho(1-\rho\lambda_g)\tau_\varepsilon^2}(\alpha_{gk}+\lambda_g\gamma_{gk})\Bigg]\begin{bmatrix} z_t \\ x_{lgt}\end{bmatrix}$$

I can write the above system of equations in terms of $\boldsymbol{h}(L)$ in matrix form

$$\boldsymbol{C}(L)\boldsymbol{h}(L) = \boldsymbol{d}[L; h(\lambda), h(0)] \tag{A66}$$

where

$$\boldsymbol{C}(L) = \begin{bmatrix} C_{11}(L) & C_{12}(L) & C_{13}(L) & C_{14}(L) \\ C_{21}(L) & C_{22}(L) & C_{23}(L) & C_{24}(L) \\ C_{31}(L) & C_{32}(L) & C_{33}(L) & C_{34}(L) \\ C_{41}(L) & C_{42}(L) & C_{43}(L) & C_{44}(L) \end{bmatrix}, \quad \boldsymbol{h}(L) = \begin{bmatrix} h_{11}(L) \\ h_{12}(L) \\ h_{21}(L) \\ h_{22}(L) \end{bmatrix}, \quad \boldsymbol{d}[L; h(\lambda), h(0)] = \begin{bmatrix} d_1(L) \\ d_2(L) \\ d_3(L) \\ d_4(L) \end{bmatrix}$$

where

$$C_{11}(L) = 1 - \frac{\beta_1+\alpha_{11}}{L\tau_\varepsilon} - \frac{\gamma_{11}}{\tau_\varepsilon}$$

$$C_{12}(L) = -\frac{\lambda_1\tau_\epsilon(\alpha_{11}+\gamma_{11}L)}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2}$$

$$C_{13}(L) = -\frac{\gamma_{12}}{\tau_\varepsilon} - \frac{\alpha_{12}}{L\tau_\varepsilon}$$

$$C_{14}(L) = -\frac{\lambda_1\tau_\epsilon(\alpha_{12}+\gamma_{12}L)}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2}$$

$$C_{21}(L) = 0$$

$$C_{22}(L) = 1 - \frac{\beta_1}{L\tau_\varepsilon} - \frac{\lambda_1\tau_1(\alpha_{11}+\gamma_{11}L)}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2}$$

$$C_{23}(L) = 0$$

$$C_{24}(L) = -\frac{\lambda_1\tau_1(\alpha_{12}+\gamma_{12}L)}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2}$$

$$C_{31}(L) = -\frac{\gamma_{21}}{\tau_\varepsilon} - \frac{\alpha_{21}}{L\tau_\varepsilon}$$

$$C_{32}(L) = -\frac{\lambda_2\tau_\epsilon(\alpha_{21}+\gamma_{21}L)}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2}$$

$$C_{33}(L) = 1 - \frac{\beta_2+\alpha_{22}}{L\tau_\varepsilon} - \frac{\gamma_{22}}{\tau_\varepsilon}$$

$$C_{34}(L) = -\frac{\lambda_2\tau_\epsilon(\alpha_{22}+\gamma_{22}L)}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2}$$

$$C_{41}(L) = 0$$

$$C_{42}(L) = -\frac{\lambda_2\tau_2(\alpha_{21}+\gamma_{21}L)}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2}$$

$$C_{43}(L) = 0$$

$$C_{44}(L) = 1 - \frac{\beta_2}{L\tau_\varepsilon} - \frac{\lambda_2\tau_2(\alpha_{22}+\gamma_{22}L)}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2}$$

and

$$\begin{aligned} d_1(L) = {} & \frac{\varphi_1\lambda_1\tau_\epsilon}{\rho\tau_\varepsilon(1-\rho\lambda_1)(1-\lambda_1 L)} - h_{11}(0)\frac{(\beta_1+\alpha_{11})\lambda_1[(1-\lambda_1 L)\tau_1 + (1-\rho L)\tau_\epsilon]}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2 L(\rho-\lambda_1)} + \\ & + h_{12}(0)\frac{\beta_1\lambda_1\tau_\epsilon}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2} - h_{21}(0)\frac{\alpha_{12}\lambda_1[(1-\lambda_1 L)\tau_1 + (1-\rho L)\tau_\epsilon]}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2 L(\rho-\lambda_1)} - \\ & - [h_{12}(\lambda_1)(\alpha_{11}+\lambda_1\gamma_{11}) + h_{22}(\lambda_1)(\alpha_{12}+\lambda_1\gamma_{12})]\frac{\lambda_1(1-\rho L)\tau_\epsilon}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2(1-\rho\lambda_1)} \end{aligned}$$

$$\begin{aligned} d_2(L) = {} & \frac{\varphi_1\lambda_1\tau_1}{\rho\tau_\varepsilon(1-\rho\lambda_1)(1-\lambda_1 L)} + h_{11}(0)\frac{(\beta_1+\alpha_{11})\lambda_1\tau_1}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2} - \\ & - h_{12}(0)\frac{\beta_1\lambda_1[(1-\rho L)\tau_1 + (1-\lambda_1 L)\tau_\epsilon]}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2 L(\rho-\lambda_1)} + h_{21}(0)\frac{\alpha_{12}\lambda_1\tau_1}{(1-\lambda_1 L)(1-\rho\lambda_1)\tau_\varepsilon^2} - \\ & - [h_{12}(\lambda_1)(\alpha_{11}+\lambda_1\gamma_{11}) + h_{22}(\lambda_1)(\alpha_{12}+\lambda_1\gamma_{12})]\frac{\lambda_1(1-\rho L)\tau_1}{(L-\lambda_1)(1-\lambda_1 L)\rho\tau_\varepsilon^2(1-\rho\lambda_1)} \end{aligned}$$

$$
\begin{aligned}
d_3(L) = & \frac{\varphi_2\lambda_2\tau_\epsilon}{\rho\tau_\varepsilon(1-\rho\lambda_2)(1-\lambda_2 L)} - h_{11}(0)\frac{\alpha_{21}\lambda_2[(1-\lambda_2 L)\tau_2 + (1-\rho L)\tau_\epsilon]}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2 L(\rho-\lambda_2)} - \\
& - h_{21}(0)\frac{(\beta_2+\alpha_{22})\lambda_2[(1-\lambda_2 L)\tau_2 + (1-\rho L)\tau_\epsilon]}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2 L(\rho-\lambda_2)} + h_{22}(0)\frac{\beta_2\lambda_2\tau_\epsilon}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2} - \\
& - [h_{12}(\lambda_2)(\alpha_{21}+\lambda_2\gamma_{21}) + h_{22}(\lambda_2)(\alpha_{22}+\lambda_1 2\gamma_{22})]\frac{\lambda_2(1-\rho L)\tau_\epsilon}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2(1-\rho\lambda_2)} \\
d_4(L) = & \frac{\varphi_2\lambda_2\tau_2}{\rho\tau_\varepsilon(1-\rho\lambda_2)(1-\lambda_2 L)} + h_{11}(0)\frac{\alpha_{21}\lambda_2\tau_2}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2} + h_{21}(0)\frac{(\beta_2+\alpha_{22})\lambda_2\tau_2}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2} - \\
& - h_{22}(0)\frac{\beta_2\lambda_2[(1-\rho L)\tau_2 + (1-\lambda_2 L)\tau_\epsilon]}{(1-\lambda_2 L)(1-\rho\lambda_2)\tau_\varepsilon^2 L(\rho-\lambda_2)} - \\
& - [h_{12}(\lambda_2)(\alpha_{21}+\lambda_2\gamma_{21}) + h_{22}(\lambda_2)(\alpha_{22}+\lambda_1 2\gamma_{22})]\frac{\lambda_2(1-\rho L)\tau_2}{(L-\lambda_2)(1-\lambda_2 L)\rho\tau_\varepsilon^2(1-\rho\lambda_2)}
\end{aligned}
$$

From (A66), the solution to the policy function is given by $\boldsymbol{h}(L) = \boldsymbol{C}(L)^{-1}\boldsymbol{d}(L) = \frac{\text{adj}\,\boldsymbol{C}(L)}{\det \boldsymbol{C}(L)}\boldsymbol{d}(L)$. Hence, I need to obtain $\det \boldsymbol{C}(L)$. Note that the degree of $\det \boldsymbol{C}(L)$ is a polynomial of degree 8 on L. Denote the inside roots of $\det \boldsymbol{C}(L)$ as $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$, and the outside roots as $\{\vartheta_1^{-1}, \vartheta_2^{-1}\}$. Because agents cannot use future signals, the inside roots have to be removed. Note that the number of free constants in $\boldsymbol{d}(L)$ is 6:

$$
\Big\{h_{11}(0), h_{12}(0), h_{21}(0), h_{22}(0), \underbrace{h_{12}(\lambda_1)(\alpha_{11}+\lambda_1\gamma_{11}) + h_{22}(\lambda_1)(\alpha_{12}+\lambda_1\gamma_{12})}_{h(\lambda_1)},
$$
$$
\underbrace{h_{12}(\lambda_2)(\alpha_{21}+\lambda_2\gamma_{21}) + h_{22}(\lambda_2)(\alpha_{22}+\lambda_2\gamma_{22})}_{h(\lambda_2)}\Big\}
$$

For a unique solution, it has to be the case that the number of outside roots is 2. By Cramer's rule, $h_{11}(L)$ is given by

$$
h_{11}(L) = \frac{\det \begin{bmatrix} d_1(L) & C_{12}(L) & C_{13}(L) & C_{14}(L) \\ d_2(L) & C_{22}(L) & C_{23}(L) & C_{24}(L) \\ d_3(L) & C_{32}(L) & C_{33}(L) & C_{34}(L) \\ d_4(L) & C_{42}(L) & C_{43}(L) & C_{44}(L) \end{bmatrix}}{\det \boldsymbol{C}(L)}
$$

and similarly with the rest of policy functions. The degree of the numerator is 7, as the highest degree of $D_g(L)$ is 1 degree less than that of $\boldsymbol{C}(L)$. By choosing the appropriate constants $\{h_{11}(0), h_{12}(0), h_{21}(0), h_{22}(0), h(\lambda_1), h(\lambda_2)\}$, the 6 inside roots will be removed. Therefore, the 6 constants are solutions to the following system of linear equations

$$
\det \begin{bmatrix} d_1(\zeta_i) & C_{12}(\zeta_i) & C_{13}(\zeta_i) & C_{14}(\zeta_i) \\ d_2(\zeta_i) & C_{22}(\zeta_i) & C_{23}(\zeta_i) & C_{24}(\zeta_i) \\ d_3(\zeta_i) & C_{32}(\zeta_i) & C_{33}(\zeta_i) & C_{34}(\zeta_i) \\ d_4(\zeta_i) & C_{42}(\zeta_i) & C_{43}(\zeta_i) & C_{44}(\zeta_i) \end{bmatrix} = 0
$$

for $i = 1, 2, \ldots, 6$. After removing the inside roots in the denominator, the degree of the numerator is 1 and the degree of the denominator is 2. The policy functions will be

$$h_{g1}(L) = \frac{\widetilde{\psi}_{g1,1} + \widetilde{\psi}_{g2,1}L}{(1-\vartheta_1 L)(1-\vartheta_2 L)}, \quad h_{g2}(L) = \frac{\widetilde{\psi}_{g1,2} + \widetilde{\psi}_{g2,2}L}{(1-\vartheta_1 L)(1-\vartheta_2 L)}$$

and hence I have

$$\begin{aligned} a_{gt} &= [h_{g1}(L) + h_{g2}(L)]\nu_t + h_{g1}(L)\epsilon_t = \frac{(\widetilde{\psi}_{g1,1} + \widetilde{\psi}_{g1,2}) + (\widetilde{\psi}_{g2,1} + \widetilde{\psi}_{g2,2})L}{(1-\vartheta_1 L)(1-\vartheta_2 L)}\nu_t + \frac{\widetilde{\psi}_{g1,2} + \widetilde{\psi}_{g2,2}L}{(1-\vartheta_1 L)(1-\vartheta_2 L)}\epsilon_t \\ &= \sum_{j=1} \psi_{gj}\left(1 - \frac{\vartheta_j}{\rho}\right)\frac{1}{1-\vartheta_j L}\nu_t + \sum_{j=1}^{2} \phi_{gj}\left(1 - \frac{\vartheta_j}{\rho}\right)\frac{1}{1-\vartheta_j L}\epsilon_t = \psi_{g1}\widetilde{\vartheta}^{\nu}_{1t} + \psi_{g2}\widetilde{\vartheta}^{\nu}_{2t} + \phi_{g1}\widetilde{\vartheta}^{\epsilon}_{1t} + \phi_{g2}\widetilde{\vartheta}^{\epsilon}_{2t} \end{aligned}$$

I can write

$$\boldsymbol{a}_t = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} = Q_\nu \widetilde{\vartheta}^{\nu}_t + Q_u \widetilde{\vartheta}^{\epsilon}_t = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{bmatrix} \widetilde{\vartheta}^{\nu}_{1t} \\ \widetilde{\vartheta}^{\nu}_{2t} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} \widetilde{\vartheta}^{\epsilon}_{1t} \\ \widetilde{\vartheta}^{\epsilon}_{2t} \end{bmatrix}$$

Notice that I can write $\widetilde{\vartheta}^x_t = \Lambda \widetilde{\vartheta}^x_{t-1} + \Gamma x_t = (I - \Lambda L)^{-1}\Gamma x_t$ for $x \in \{\nu, \epsilon\}$, where

$$\Lambda = \begin{bmatrix} \vartheta_1 & 0 \\ 0 & \vartheta_2 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 1 - \frac{\vartheta_1}{\rho} \\ 1 - \frac{\vartheta_2}{\rho} \end{bmatrix}$$

Hence, I can write $\boldsymbol{a}_t = Q_\nu(I - \Lambda L)^{-1}\Gamma \nu_t + Q_u(I - \Lambda L)^{-1}\Gamma \epsilon_t = Q_\nu \sum_{k=0}^{\infty} \Lambda^k \Gamma \nu_{t-k} + Q_u \sum_{k=0}^{\infty} \Lambda^k \Gamma u_{t-k}$. □