Lecture Notes: A Step-by-step Derivation of the New Keynesian Small Open Economy Framework*

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1 A small open economy model

1.1 Households

A typical small open economy is inhabited by a representative household who seeks to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{1}$$

The variable C_t is a composite consumption index determined by both home and foreign goods, defined by

$$C_t \equiv \left[(1-\alpha)^{\frac{1}{\eta}} C_{Ht}^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} C_{Ft}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

The parameter α measures the degree of openness in the economy. Equivalently $1-\alpha$ measures the home bias. The closer α is to one, the more open is the economy. If $\alpha = 0$ we get the closed economy case described earlier.

The substitutability between domestic and foreign goods from the viewpoint of domestic consumers is denoted γ.

C_{Ht} is an index of consumption of domestic goods, given by the CES function

$$C_{Ht} = \left(\int_0^1 C_{Ht}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

The parameter ε is the elasticity of substitution between varieties produced within any given country i, including the home country.

C_{Ft} is an index of imported goods given by the CES function

* This document is a mere extension of the work by Drago Bergholt. Full credit goes to him.

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$$C_{Ft} = \left(\int_0^1 C_{it}^{\frac{\gamma-1}{\gamma}} di \right)^{\frac{\gamma}{\gamma-1}}$$

γ denotes the elasticity of substitution between importing countries.
C_{it} is an index of different goods j imported from country i, given by the CES function

$$C_{it} = \left(\int_0^1 C_{it}(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Maximization of (1) is subject to a sequence of budget constraints

$$\int_0^1 P_{Ht}(j) C_{Ht}(j) dj + \int_0^1 \int_0^1 P_{it}(j) C_{it}(j) dj di + \mathbb{E}_t Q_{t,t+1} D_{t+1} \leq D_t + W_t N_t + T_t$$

The domestic price on good j is denoted as $P_{Ht}(j)$ while the price on good j imported from country i is denoted $P_{it}(j)$. D_{t+1} is the nominal payoff in period $t+1$ from a portfolio held at the end of period t. $Q_{t,t+1}$ is the stochastic discount factor for one-period ahead nominal payoffs of the domestic bond.

Aggregate price indices and optimal consumption bundles are given by (these could be derived)

$$C_{Ht}(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}} \right]^{-\varepsilon} C_{Ht} \tag{2}$$

$$C_{it}(j) = \left[\frac{P_{it}(j)}{P_{it}} \right]^{-\varepsilon} C_{it}$$

$$P_{Ht} = \left(\int_0^1 P_{Ht}(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} \tag{3}$$

$$P_{it} = \left(\int_0^1 P_{it}(j)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} \tag{4}$$

$$\int_0^1 P_{Ht}(j) C_{Ht}(j) dj = P_{Ht} C_{Ht}$$

$$\int_0^1 P_{it}(j) C_{it}(j) dj = P_{it} C_{it}$$

$$C_{it} = \left(\frac{P_{it}}{P_{Ft}} \right)^{-\gamma} C_{Ft} \tag{5}$$

$$P_{Ft} = \left(\int_0^1 P_{it}^{1-\gamma} di \right)^{\frac{1}{1-\gamma}} \tag{6}$$

$$\int_0^1 P_{it} C_{it} di = P_{Ft} C_{Ft}$$

$$C_{Ht} = (1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t \tag{7}$$

$$C_{Ft} = \alpha \left(\frac{P_{Ft}}{P_t} \right)^{-\eta} C_t \tag{8}$$

$$P_{Ht} C_{Ht} + P_{Ft} C_{Ft} = P_t C_t$$

rewrite the budget constraint

$$P_t C_t + \mathbb{E}_t Q_{t,t+1} D_{t+1} \leq D_t + W_t N_t + T_t$$

where

$$P_t = \left[(1-\alpha) P_{Ht}^{1-\eta} + \alpha P_{Ft}^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{9}$$

The household must decide on the allocation of total consumption and labor.

We specify utility function to be $U = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\phi}}{1+\phi}$. Solving the household problem yields

$$C_t^{\sigma} N_t^{\phi} = \frac{W_t}{P_t} \tag{10}$$

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}} \frac{P_t}{P_{t+1}} C_{t+1}^{-\sigma} \right] \tag{11}$$

which we log-linearize

$$c_t = \mathbb{E}_t c_{t+1} - \frac{1}{\sigma} (i_t - \mathbb{E}_t \pi_{t+1} - \rho) \tag{12}$$

$$w_t - p_t = \sigma c_t + \phi n_t \tag{13}$$

where $x_t = \log X_t$, $\rho = -\log \beta$, $i_t = -\log Q_t$ and $\pi_{t+1} = p_{t+1} - p_t = \log P_{t+1} - \log P_t$.

1.2 Foreign Sector: Terms of Trade, Domestic Inflation and CPI Inflation

Bilateral terms of trade between the domestic economy and country i is defined as the price of country i's goods in terms of home goods

$$S_{it} = \frac{P_{it}}{P_{Ht}} \tag{14}$$

The effective terms of trade are thus given by (revisar)

$$S_t = \frac{P_{Ft}}{P_{Ht}} = \frac{\left(\int_0^1 P_{it}^{1-\gamma} di\right)^{\frac{1}{1-\gamma}}}{\left(\int_0^1 P_{Ht}(j)^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}} \stackrel{?}{=} \left(\int_0^1 S_{it}(j)^{1-\gamma} di\right)^{\frac{1}{1-\gamma}}$$

A first-order approximation around a symmetric steady-state ($S_{it} = S_i = 1\ \forall i$) gives

$$\begin{aligned} S_t = \left(\int_0^1 S_{it}(j)^{1-\gamma} di\right)^{\frac{1}{1-\gamma}} &\approx \left(\int_0^1 S_i(j)^{1-\gamma} di\right)^{\frac{1}{1-\gamma}} + \frac{1}{1-\gamma}\left(\int_0^1 S_i(j)^{1-\gamma} di\right)^{\frac{\gamma}{1-\gamma}} \int_0^1 (1-\gamma) S_i^{-\gamma}(S_{it} - S_i) di \\ &= 1 + \int_0^1 (S_{it} - 1) di \\ \implies \frac{S_t - 1}{1} &\approx \int_0^1 \frac{S_{it} - 1}{1} di \\ \implies s_t = p_{Ft} - p_{Ht} &\approx \int_0^1 s_{it} di \end{aligned} \tag{15}$$

Log-linearization of the CPI (9) around a symmetric steady-state ($P_H = P_F = P$)

$$\begin{aligned} P_t &\approx [(1-\alpha)P^{1-\eta} + \alpha P^{1-\eta}]^{\frac{1}{1-\eta}} + \frac{1}{1-\eta}[(1-\alpha)P^{1-\eta} + \alpha P^{1-\eta}]^{\frac{\eta}{1-\eta}} \times \\ &\qquad \times [(1-\alpha)(1-\eta)P^{-\eta}(P_{Ht} - P) + \alpha(1-\eta)P^{-\eta}(P_{Ft} - P)] = \\ &= P + [(1-\alpha)(P_{Ht} - P) + \alpha(P_{Ft} - P)] \\ \implies \frac{P_t - P}{P} &\approx (1-\alpha)\frac{P_{Ht} - P}{P} + \alpha\frac{P_{Ft} - P}{P} \\ \implies p_t &\approx (1-\alpha)p_{Ht} + \alpha p_{Ft} = p_{Ht} + \alpha s_t \end{aligned} \tag{16}$$

Domestic inflation is given by $\pi_{Ht} = p_{Ht} - p_{Ht-1}$. Thus, using (16)

$$\pi_t = p_t - p_{t-1} = (p_{Ht} + \alpha s_t) - (p_{Ht-1} + \alpha s_{t-1}) = \pi_{Ht} + \alpha \Delta s_t \tag{17}$$

The gap between domestic inflation and CPI inflation is proportional to the percentage change in terms of trade.

1.3 The Real Exchange Rate

Define $\mathcal{E}_{it}$ as the bilateral nominal exchange rate. Define $P_{it}^i(j)$ as the price of country i's good j in terms of its own currency. Assume that the law of one price holds for individual goods at all times for both import and export prices. Thus, for all goods j in every country i, $P_{it}(j) = \mathcal{E}_{it} P_{it}^i(j)$. Aggregation across all goods using (4) gives

$$P_{it} = \left(\int_0^1 [\mathcal{E}_{it} P^i_{it}(j)]^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}} = \left(\mathcal{E}_{it}^{1-\varepsilon} \int_0^1 P^i_{it}(j)^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}} = \mathcal{E}_{it} \left(\int_0^1 P^i_{it}(j)^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}} = \mathcal{E}_{it} P^i_{it} \tag{18}$$

where $P^i_{it} = \left(\int_0^1 P^i_{it}(j)^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}$ is the aggregate price level in country i in terms of country i currency. Thus, (18) is the law of one price at the country level, where P^i_{it} represents the domestically produced goods in country i, in contrast to P^i_t which represents all goods in country i. Insert (18) into (6)

$$P_{Ft} = \left(\int_0^1 [\mathcal{E}_{it} P^i_{it}]^{1-\gamma} di\right)^{\frac{1}{1-\gamma}}$$

Log-linearizing around a symmetric steady-state

$$\begin{aligned}
P_{Ft} \approx & \left(\int_0^1 [\mathcal{E}_i P^i_i]^{1-\gamma} di\right)^{\frac{1}{1-\gamma}} + \frac{1}{1-\gamma} \left(\int_0^1 [\mathcal{E}_i P^i_i]^{1-\gamma} di\right)^{\frac{\gamma}{1-\gamma}} \int_0^1 (1-\gamma)(\mathcal{E}_i P^i_i)^{-\gamma} P^i_i (\mathcal{E}_{it} - \mathcal{E}_i) di + \\
& + \frac{1}{1-\gamma} \left(\int_0^1 [\mathcal{E}_i P^i_i]^{1-\gamma} di\right)^{\frac{\gamma}{1-\gamma}} \int_0^1 (1-\gamma)(\mathcal{E}_i P^i_i)^{-\gamma} \mathcal{E}_i (P^i_{it} - P^i_i) di \\
P_{Ft} - P = & P^{\gamma} \left[\int_0^1 \mathcal{E}_i^{-\gamma} (P^i_i)^{1-\gamma} (\mathcal{E}_{it} - \mathcal{E}_i) di + \int_0^1 \mathcal{E}_i^{1-\gamma} (P^i_i)^{-\gamma} (P^i_{it} - P^i_i) di\right] \\
p_{Ft} = & P^{\gamma-1} \left[\int_0^1 P_i^{1-\gamma} e_{it} di + \int_0^1 P_i^{1-\gamma} p^i_{it} di\right] \\
= & \int_0^1 e_{it} di + \int_0^1 p^i_{it} di = \\
= & e_t + p^*_t
\end{aligned} \tag{19}$$

The log domestic price index for country i expressed in terms of its own currency is denoted $p^i_{it} = \int_0^1 p^i_{it}(j) di$, the effective exchange nominal exchange rate is denoted $e_t = \int_0^1 e_{it} di$ and the log world price index is denoted $p^*_t = \int_0^1 p^i_{it} di$.

Insert (19) into (15)

$$s_t = p_{Ft} - p_{Ht} = e_t + p^*_t - p_{Ht} \tag{20}$$

which expresses the terms of trade as a linear function of the effective nominal exchange rate, the world price and the price on domestically produced goods.

Define the bilateral exchange rate between the home country and country i as

$$\mathcal{Q}_{it} = \frac{\mathcal{E}_{it} P^i_{it}}{P_t}$$

in logs

$$q_{it} = e_{it} + p^i_{it} - p_t \tag{21}$$

define the log effective real exchange rate as

$$q_t = \int_0^1 q_{it} di \tag{22}$$

Inserting (21), using (20) and (16)

$$q_t = \int_0^1 (e_{it} + p^i_{it} - p_t) di = e_t + p^*_t - p_t = s_t + p_{Ht} - p_t = s_t + p_{Ht} - (p_{Ht} + \alpha s_t) = (1 - \alpha) s_t \tag{23}$$

1.4 International risk sharing

Assuming complete markets for securities traded internationally, a condition analogous to (11) must hold for the representative household in any other country, say country i. Using (18)

$$1 = \beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}} \frac{P_{it}}{P_{it+1}} \left(\frac{C^i_{t+1}}{C^i_t} \right)^{-\sigma} \right] = \beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}} \frac{\mathcal{E}_{it} P^i_{it}}{\mathcal{E}_{it+1} P^i_{it+1}} \left(\frac{C^i_{t+1}}{C^i_t} \right)^{-\sigma} \right] \tag{24}$$

Taking the ratio of (11) and (24),

$$\frac{1}{1} = \frac{\beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}} \frac{P_t}{P_{t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \right]}{\beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}} \frac{\mathcal{E}_{it} P^i_{it}}{\mathcal{E}_{it+1} P^i_{it+1}} \left(\frac{C^i_{t+1}}{C^i_t} \right)^{-\sigma} \right]} = \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{C^i_{t+1}}{C^i_t} \right)^{\sigma} \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{it+1} P^i_{it+1}}{\mathcal{E}_{it} P^i_{it}} \right] \implies$$

$$\implies C_t^{-\sigma} = \mathbb{E}_t \left[(C^i_t)^{-\sigma} \left(\frac{C^i_{t+1}}{C_{t+1}} \right)^{\sigma} \frac{P_t}{\mathcal{E}_{it} P^i_{it}} \frac{\mathcal{E}_{it+1} P^i_{it+1}}{P_{t+1}} \right] = \mathbb{E}_t \left[(C^i_t)^{-\sigma} \left(\frac{C^i_{t+1}}{C_{t+1}} \right)^{\sigma} \frac{\mathcal{Q}_{it+1}}{\mathcal{Q}_{it}} \right]$$

$$\implies C_t = \mathbb{E}_t \left[C^i_t \frac{C_{t+1}}{C^i_{t+1}} \mathcal{Q}_{it+1}^{-\frac{1}{\sigma}} \mathcal{Q}_{it}^{\frac{1}{\sigma}} \right] = \mathbb{E}_t \left(\frac{C_{t+1}}{C^i_{t+1} \mathcal{Q}_{it+1}^{\frac{1}{\sigma}}} \right) C^i_t \mathcal{Q}_{it}^{\frac{1}{\sigma}} = \vartheta_i C^i_t \mathcal{Q}_{it}^{\frac{1}{\sigma}} \tag{25}$$

where ϑ_i is a constant that will depend on initial conditions regarding relative net asset positions.

Assuming symmetric initial conditions (i.e., zero net foreign asset holdings and an ex-ante identical environment) implies $\vartheta_i = \vartheta = 1 \ \forall i$. Taking logs on both sides

$$c_t = c^i_t + \frac{1}{\sigma} q_{it} \tag{26}$$

Equation (26) is at the household level. Note that world consumption is given by

$$c_t^* = \int_0^1 c_t^i di$$

Integrating (26) over all i and using (22) and (23)

$$c_t = \int_0^1 \left(c_t^i + \frac{1}{\sigma} q_{it} \right) di = c_t^* + \frac{1}{\sigma} q_t = c_t^* + \frac{1-\alpha}{\sigma} s_t \tag{27}$$

The assumption of complete markets at the international level leads to a simple relationship linking domestic consumption with world consumption and the terms of trade, where relative home consumption to world consumption is given by $c_t - c_t^*$.

1.5 Uncovered Interest Rate Parity

Allow households to invest both in domestic and foreign bonds; B_t and B_t^*. The budget constraint can be rewritten as

$$P_t C_t + \mathbb{E}_t Q_{t,t+1} B_{t+1} + \mathbb{E}_t Q_{t,t+1}^* B_{t+1}^* \leq B_t + \mathcal{E}_t B_t^* + W_t N_t + T_t$$

The optimality conditions with respect to these assets are

$$1 = \beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right]$$
$$1 = \beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}^*} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right]$$

Take the ratio between them

$$\frac{1}{1} = \frac{\beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right]}{\beta \mathbb{E}_t \left[\frac{1}{Q_{t,t+1}^*} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right]} = \frac{\mathbb{E}_t \left[\frac{1}{Q_{t,t+1}} \right]}{\mathbb{E}_t \left[\frac{1}{Q_{t,t+1}^*} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right]} \implies \frac{Q_{t,t+1}^*}{Q_{t,t+1}} = \mathbb{E}_t \left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right)$$

log-linearizing

$$-i_t^* + i_t = \mathbb{E}_t (e_{t+1} - e_t) = \mathbb{E}_t \Delta e_{t+1} \implies i_t = i_t^* + \mathbb{E}_t \Delta e_{t+1} \tag{28}$$

This is the familiar uncovered interest rate parity equation, which states that the nominal interest rate at home is equal to the world nominal interest rate plus expected rate of depreciation of the home currency. From (20) we have that

$$\mathbb{E}_t s_{t+1} - s_t = \mathbb{E}_t e_{t+1} - e_t + \mathbb{E}_t p^*_{t+1} -^*_t + \mathbb{E}_t p_{Ht+1} - p_{Ht} = \mathbb{E}_t \Delta e_{t+1} + \mathbb{E}_t \pi^*_{t+1} - \mathbb{E}_t \pi_{Ht+1} \implies$$
$$\implies s_t = \mathbb{E}_t s_{t+1} - \mathbb{E}_t \Delta e_{t+1} - \mathbb{E}_t \pi^*_{t+1} + \mathbb{E}_t \pi_{Ht+1}$$

Using (28) we get the following stochastic difference equation

$$s_t = \mathbb{E}_t s_{t+1} - (i_t - i^*_t) - \mathbb{E}_t \pi^*_{t+1} + \mathbb{E}_t \pi_{Ht+1} = (i^*_t - \mathbb{E}_t \pi^*_{t+1}) - (i_t - \mathbb{E}_t \pi_{t+1}) + \mathbb{E}_t s_{t+1}$$

Given that the terms are pinned down uniquely in the perfect foresight steady-state, and given the assumptions of stationarity in the model driving forces and unit relative prices in steady-state, it follows that $\lim_{T\to\infty} \mathbb{E}_t s_T = 0$ (?). Hence, it can be solved forward to obtain

$$s_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \left[(i^*_{t+k} - \pi^*_{t+k}) - (i_{t+k} - \mathbb{E}_t \pi_{t+k}) \right] \right\}$$

which expresses the terms of trade as the expected sum of real interest rate differentials between the world market and the home market.

1.6 Firms and Technologies

Now we turn to the supply side of the economy. Because domestic firms take the business environment as given, including state of affairs in foreign economies, the individual firm still only takes into account its own marginal cost.

The production side is split into two. There is a representative competitive final goods firm which aggregates intermediate inputs according to a CES technology. To the extent to which the intermediates are imperfect substitutes in the CES aggregator, this generates a downward-sloping demand for each intermediate variety, which gives these intermediate producers pricing power. There are a continuum of intermediates, so these producers behave as monopolistically competitive (they treat all prices but their own as given). These firms produce output using labor and are subject to an aggregate productivity shock. They are not freely able to adjust prices each period, in a way that we will discuss in more depth below.

1.6.1 Final Goods Producer

The final output good is a CES aggregate of a continuum of intermediates

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Here $\varepsilon > 1$. The profit maximization problem of the final good firm is

$$\max_{Y_t(j)} \quad P_{Ht}\left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}dj\right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 P_{Ht}(j)Y_t(j)dj$$

The FOC for a typical variety of intermediate j is

$$P_{Ht}\frac{\varepsilon}{\varepsilon-1}\left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}dj\right)^{\frac{1}{\varepsilon-1}}\frac{\varepsilon-1}{\varepsilon}Y_t(j)^{-\frac{1}{\varepsilon}} = P_{Ht}(j) \implies \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}}dj\right)^{-\frac{\varepsilon}{\varepsilon-1}}Y_t(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon}$$

Making note of the definition of the aggregate final good, we have

$$Y_t(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} Y_t \tag{29}$$

This says that the relative demand for intermediate good j is a function of its relative price, the price elasticity of demand ε and is proportional to aggregate output Y_t.

1.6.2 Intermediate Producers

A typical intermediate producer produces output according to a CRS technology in labor, with a common productivity shock A_t

$$Y_t(j) = A_t N_t(j) \tag{30}$$

Intermediate producers face a common wage. They are not freely able to adjust price so as to maximize profit each period, but will always act to minimize cost. The cost minimization problem is to minimize total cost subject to the constraint of producing enough to satisfy demand

$$\begin{aligned} \min_{N_t(j)} \quad & W_t N_t(j) \\ s.t. \quad & A_t N_t(j) \geq Y_t(j) \\ & Y_t(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} Y_t \end{aligned}$$

The lagrange multiplier on the constraint will have the interpretation of marginal cost: how much will costs change if you are forced to produce an extra unit of output.

Taking the FOC one finds $\varphi_t = \frac{W_t}{A_t}$, so that the marginal cost φ_t is common to all intermediate goods firms.

1.6.3 Price-setting: Calvo (1983)

Prices are set according to the mechanism in Calvo (1983). A firm is only allowed to reset its price with probability $1-\theta$ in any given period. As a result, in each period only a subset of measure $1-\theta$ of all firms change their prices and the average price remains fixed for $\frac{1}{1-\theta}$ periods.

The domestic firm's problem is to maximize its discounted profit stream

$$
\begin{aligned}
\max \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} Q_{0,t}[P_{Ht}(j)Y_t(j) - W_t N_t(j)] \\
s.t. \quad & Y_t(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} Y_t \\
& Y_t(j) = A_t N_t(j)
\end{aligned}
$$

which can be rewritten as

$$
\max_{P^*_{Ht}(j)} \quad \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[P^*_{Ht}(j) - \frac{W_{t+k}}{A_{t+k}}\right] \left[\frac{P^*_{Ht}(j)}{P_{Ht+k}}\right]^{-\varepsilon} Y_{t+k}
$$

Taking the FOC

$$
\begin{aligned}
\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[\frac{P^*_{Ht}(j)}{P_{Ht+k}}\right]^{-\varepsilon} Y_{t+k} + \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[P^*_{Ht}(j) - \frac{W_{t+k}}{A_{t+k}}\right](-\varepsilon) \left[\frac{P^*_{Ht}(j)}{P_{Ht+k}}\right]^{-\varepsilon-1} \frac{1}{P_{Ht+k}} Y_{t+k} &= 0 \\
\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[\frac{P^*_{Ht}(j)}{P_{Ht+k}}\right]^{-\varepsilon} Y_{t+k} \left\{1 - \varepsilon \left[P^*_{Ht}(j) - \frac{W_{t+k}}{A_{t+k}}\right] \frac{1}{P^*_{Ht}(j)}\right\} &= 0 \\
\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[\frac{P^*_{Ht}(j)}{P_{Ht+k}}\right]^{-\varepsilon} Y_{t+k} \left[1 - \varepsilon + \varepsilon \frac{W_{t+k}}{A_{t+k} P^*_{Ht}(j)}\right] &= 0 \\
\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[\frac{P^*_{Ht}(j)}{P_{Ht+k}}\right]^{-\varepsilon} Y_{t+k} \left[P^*_{Ht}(j) - \mathcal{M} \frac{W_{t+k}}{A_{t+k}}\right] &= 0
\end{aligned}
$$

where in the last step we multiplied both sides by $\frac{P^*_{Ht}(j)}{1-\varepsilon}$, and we set $\mathcal{M} = \frac{\varepsilon}{\varepsilon-1}$. Separating both sides

$$
\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} P^*_{Ht}(j)^{1-\varepsilon} P^{\varepsilon}_{Ht+k} Y_{t+k} = \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left[\frac{P^*_{Ht}(j)}{P_{Ht+k}}\right]^{-\varepsilon} Y_{t+k} \mathcal{M} \frac{W_{t+k}}{A_{t+k}}
$$

In equilibrium, all firms that are allowed to optimize behave identically. This means that they all charge the same price and produce the same output. Inserting $Q_{t,t+k}$ and solving for P^*_{Ht}

$$\text{LHS: } \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}} (P_{Ht}^*)^{1-\varepsilon} P_{Ht+k}^{\varepsilon} Y_{t+k} = \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}} P_{Ht+k}^{\varepsilon} Y_{t+k} (P_{Ht}^*)^{1-\varepsilon}$$

$$\text{RHS: } \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}} \left[\frac{P_{Ht}^*}{P_{Ht+k}}\right]^{-\varepsilon} Y_{t+k} \mathcal{M} \frac{W_{t+k}}{A_{t+k}} = \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}} (P_{Ht+k})^{\varepsilon} Y_{t+k} \mathcal{M} \frac{W_{t+k}}{A_{t+k}} ($$

$$\text{LHS=RHS: } \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}} P_{Ht+k}^{\varepsilon} Y_{t+k} (P_{Ht}^*)^{1-\varepsilon} =$$

$$= \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}} (P_{Ht+k})^{\varepsilon} Y_{t+k} \mathcal{M} \frac{W_{t+k}}{A_{t+k}} (P_{Ht}^*)^{-\varepsilon}$$

$$C_t^{\sigma} P_t (P_{Ht}^*)^{1-\varepsilon} \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} \frac{1}{P_{t+k}} P_{Ht+k}^{\varepsilon} Y_{t+k} = \mathcal{M} C_t^{\sigma} P_t (P_{Ht}^*)^{-\varepsilon} \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} \frac{1}{P_{t+k}} P_{Ht+k}^{\varepsilon} Y_{t+k} \frac{W_{t+k}}{A_{t+k}}$$

$$P_{Ht}^* \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} \frac{1}{P_{t+k}} P_{Ht+k}^{\varepsilon} Y_{t+k} = \mathcal{M} \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} P_{Ht+k}^{\varepsilon} Y_{t+k} \underbrace{\frac{W_{t+k}}{A_{t+k}} \frac{1}{P_{t+k}}}_{MC_{t+k}}$$

$$P_{Ht}^* = \mathcal{M} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} P_{Ht+k}^{\varepsilon} Y_{t+k} MC_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} \frac{1}{P_{t+k}} P_{Ht+k}^{\varepsilon} Y_{t+k}} \tag{31}$$

with flexible prices ($\theta = 0$), the previous expression collapses to $P_{Ht}^* = \mathcal{M} MC_t P_t = \mathcal{M} \frac{W_t}{A_t}$, which gives the frictionless mark-up.

Divide (31) by P_{Ht-1}

$$\frac{P_{Ht}^*}{P_{Ht-1}} = \mathcal{M} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} P_{Ht+k}^{\varepsilon} Y_{t+k} MC_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} \frac{1}{P_{t+k}} P_{Ht+k}^{\varepsilon} Y_{t+k}} \frac{1}{P_{Ht-1}} \implies$$

$$\implies \frac{P_{Ht}^*}{P_{Ht-1}} \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} \frac{1}{P_{t+k}} P_{Ht+k}^{\varepsilon} Y_{t+k} = \mathbb{E}_t \sum_{k=0}^{\infty} (\theta\beta)^k C_{t+k}^{-\sigma} P_{Ht+k}^{\varepsilon} Y_{t+k} MC_{t+k} \frac{1}{P_{Ht-1}}$$

Log-linearizing the firm's problem around a symmetric steady state where $P_{Ht} = P_t = P$, $\Pi_t = \frac{P_t}{P_{t-1}} = 1$.

$$\text{LHS: } \frac{P^*_{Ht}}{P_{Ht-1}}\mathbb{E}_t\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}_{t+k}\frac{1}{P_{t+k}}P^{\varepsilon}_{Ht+k}Y_{t+k} \approx \sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}Y +$$

$$\frac{1}{P}\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}Y(P^*_{Ht}-P) - \frac{1}{P}\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}Y(P_{Ht-1}-P)-$$

$$\sigma\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma-1}P^{\varepsilon-1}Y(C_{t+k}-C) - \sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-2}Y(P_{t+k}-P)+$$

$$\varepsilon\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-2}Y(P_{Ht+k}-P) + \sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}(Y_{t+k}-Y) =$$

$$= \sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}Y\left\{1 + p^*_{Ht} - p_{Ht-1} - \sigma c_{t+k} + p_{t+k} + \varepsilon p_{Ht+k} + y_{t+k}\right\}$$

$$\text{RHS: } \mathbb{E}_t\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}_{t+k}P^{\varepsilon}_{Ht+k}Y_{t+k}MC_{t+k}\frac{1}{P_{Ht-1}} \approx \sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}YMC-$$

$$-\sigma\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma-1}P^{\varepsilon-1}YMC(C_{t+k}-C) + \varepsilon\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-2}YMC(P_{Ht+k}-P)+$$

$$+\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}MC(Y_{t+k}-Y) + \sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}Y(MC_{t+k}-MC)-$$

$$-\sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-2}YMC(P_{Ht-1}-P)$$

$$= \sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma}P^{\varepsilon-1}YMC\left\{1 - \sigma c_{t+k} + \varepsilon p_{Ht+k} + y_{t+k} + \log MC_{t+k} - \log MC - p_{Ht-1}\right\}$$

$$\text{LHS=RHS: } \mathbb{E}_t\sum_{t=0}^{\infty}(\theta\beta)^k[p^*_{Ht}-p_{t+k}] = \mathbb{E}_t\sum_{t=0}^{\infty}(\theta\beta)^k[\log MC_{t+k} - \log MC]$$

$$\frac{p^*_{Ht}}{1-\theta\beta} - \mathbb{E}_t\sum_{t=0}^{\infty}(\theta\beta)^k p_{t+k} = -\frac{\log MC}{1-\theta\beta} + \mathbb{E}_t\sum_{t=0}^{\infty}(\theta\beta)^k\log MC_{t+k}$$

$$p^*_{Ht} = -\log MC + (1-\theta\beta)\mathbb{E}_t\sum_{t=0}^{\infty}(\theta\beta)^k[p_{t+k} - \log MC_{t+k}]$$

we know that $MC = \frac{1}{\mathcal{M}}$, hence $\log MC = -\log\mathcal{M} = -\mu$

$$p^*_{Ht} = \mu + (1-\theta\beta)\mathbb{E}_t\sum_{t=0}^{\infty}(\theta\beta)^k[p_{t+k} - \log MC_{t+k}] \tag{32}$$

1.7 Equilibrium

1.7.1 Aggregate Demand and Output

Market clearing for good j in the home economy implies

$$Y_t(j) = C_{Ht}(j) + \int_0^1 C_{Ht}^i(j)di \tag{33}$$

The supply of domestically produced good j is denoted $Y_t(j)$, the domestic demand is denoted $C_{Ht}(j)$ and country i's demand for good j produced in the home economy is denoted $C_{Ht}^i(j)$. Due to the nested structure, we can express demand in sub-markets in terms of total demand by combining all demand functions from each level. For instance, insert (7) into (2) and get

$$C_{Ht}(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} (1-\alpha) \left(\frac{P_{Ht}}{P_t}\right)^{-\eta} C_t \tag{34}$$

Furthermore, the demand for domestically produced good j in country i is expressed by nesting up across different demand layers in country i. First, note that the consumption of domestically produced good j in country i is a function of country i's consumption of goods produced in the economy, given as in (2)

$$C_{Ht}^i(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} C_{Ht}^i \tag{35}$$

Second, note that country i's consumption of goods produced in the home economy is a function of country i's consumption of foreign goods, given as in (5)

$$C_{Ht}^i = \left(\frac{P_{Ht}}{\mathcal{E}_{it} P_{Ft}^i}\right)^{-\gamma} C_{Ft}^i \tag{36}$$

Third, note that consumption of imported goods in country i is a function of total consumption in that country, given as in (8)

$$C_{Ft}^i = \alpha \left(\frac{P_{Ft}^i}{P_t^i}\right)^{-\eta} C_t^i \tag{37}$$

Combining all these yields the demand for domestically produced good j in country i as a function of total consumption in that country

$$C_{Ht}^i(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} \left(\frac{P_{Ht}}{\mathcal{E}_{it} P_{Ft}^i}\right)^{-\gamma} \alpha \left(\frac{P_{Ft}^i}{P_t^i}\right)^{-\eta} C_t^i \tag{38}$$

Thus, we can insert (34) and (38) into (33)

$$
\begin{aligned}
Y_t(j) = \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} (1-\alpha) \left(\frac{P_{Ht}}{P_t}\right)^{-\eta} C_t + \int_0^1 \left\{ \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} \left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}}\right)^{-\gamma} \alpha \left(\frac{P^i_{Ft}}{P^i_t}\right)^{-\eta} C^i_t \right\} di = \\
= (1-\alpha) \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} \left(\frac{P_{Ht}}{P_t}\right)^{-\eta} C_t + \alpha \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} \int_0^1 \left\{ \left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}}\right)^{-\gamma} \left(\frac{P^i_{Ft}}{P^i_t}\right)^{-\eta} C^i_t \right\} di \\
= \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} \left[(1-\alpha) \left(\frac{P_{Ht}}{P_t}\right)^{-\eta} C_t + \alpha \int_0^1 \left\{ \left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}}\right)^{-\gamma} \left(\frac{P^i_{Ft}}{P^i_t}\right)^{-\eta} C^i_t \right\} di \right]
\end{aligned}
\tag{39}
$$

To aggregate, start with the definition of aggregate domestic output

$$
Y_t = \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{40}
$$

Insert (39) into (40)

$$
\begin{aligned}
Y_t &= \left(\int_0^1 \left\{ \left[\frac{P_{Ht}(j)}{P_{Ht}} \right]^{-\varepsilon} \left[(1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left\{ \left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}} \right)^{-\gamma} \left(\frac{P^i_{Ft}}{P^i_t} \right)^{-\eta} C^i_t \right\} di \right] \right\}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left(\int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}} \right]^{1-\varepsilon} \left[(1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left\{ \left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}} \right)^{-\gamma} \left(\frac{P^i_{Ft}}{P^i_t} \right)^{-\eta} C^i_t \right\} di \right]^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left(\left\{ \int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}} \right]^{1-\varepsilon} dj \right\} \left[(1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left\{ \left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}} \right)^{-\gamma} \left(\frac{P^i_{Ft}}{P^i_t} \right)^{-\eta} C^i_t \right\} di \right]^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left\{ \int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}} \right]^{1-\varepsilon} dj \right\}^{\frac{\varepsilon}{\varepsilon-1}} \left[(1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left\{ \left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}} \right)^{-\gamma} \left(\frac{P^i_{Ft}}{P^i_t} \right)^{-\eta} C^i_t \right\} di \right] \\
&= P^{\varepsilon}_{Ht} \underbrace{\left\{ \int_0^1 P_{Ht}(j)^{1-\varepsilon} dj \right\}^{\frac{\varepsilon}{\varepsilon-1}}}_{P^{-\varepsilon}_{Ht}} \left[(1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left\{ \left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}} \right)^{-\gamma} \left(\frac{P^i_{Ft}}{P^i_t} \right)^{-\eta} C^i_t \right\} di \right] \\
&= (1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 \left[\left(\frac{P_{Ht}}{\mathcal{E}_{it} P^i_{Ft}} \right)^{-\gamma} \left(\frac{P^i_{Ft}}{P^i_t} \right)^{-\eta} C^i_t \right] di \\
&= (1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 P^{-\gamma}_{Ht} \mathcal{E}^{\gamma}_{it} (P^i_{Ft})^{\gamma-\eta} (P^i_t)^{\eta} C^i_t di \\
&= (1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 P^{\eta-\gamma}_{Ht} P^{-\eta}_{Ht} \mathcal{E}^{\gamma-\eta}_{it} \mathcal{E}^{\eta}_{it} (P^i_{Ft})^{\gamma-\eta} (P^i_t)^{\eta} P^{\eta}_t P^{-\eta}_t C^i_t di \\
&= (1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \int_0^1 P^{-\eta}_{Ht} \left(\frac{\mathcal{E}_{it} P^i_{Ft}}{P_{Ht}} \right)^{\gamma-\eta} \left(\frac{\mathcal{E}_{it} P^i_t}{P_t} \right)^{\eta} P^{\eta}_t C^i_t di \\
&= (1-\alpha) \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + \alpha \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} \int_0^1 \left(\frac{\mathcal{E}_{it} P^i_{Ft}}{P_{Ht}} \right)^{\gamma-\eta} \mathcal{Q}^{\eta}_{it} C^i_t di \\
&= \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} \left[(1-\alpha) C_t + \alpha \int_0^1 \left(\frac{\mathcal{E}_{it} P^i_{Ft}}{P_{Ht}} \right)^{\gamma-\eta} \mathcal{Q}^{\eta}_{it} C^i_t di \right]
\end{aligned}
$$

Define the effective terms of trade for country i as $S^i_t = \frac{\mathcal{E}_{it} P^i_{Ft}}{P_{it}}$. Using it along with the bilateral terms of trade between the domestic economy and country i from (14) and (25)

$$
\begin{aligned}
Y_t &= \left(\frac{P_{Ht}}{P_t}\right)^{-\eta}\left[(1-\alpha)C_t + \alpha\int_0^1\left(\frac{\mathcal{E}_{it}P_{Ft}^i}{P_{Ht}}\right)^{\gamma-\eta}\mathcal{Q}_{it}^{\frac{\sigma\eta-1}{\sigma}}C_t di\right] \\
&= \left(\frac{P_{Ht}}{P_t}\right)^{-\eta}C_t\left[(1-\alpha) + \alpha\int_0^1\left(\frac{\mathcal{E}_{it}P_{Ft}^i}{P_{Ht}}\right)^{\gamma-\eta}\mathcal{Q}_{it}^{\frac{\sigma\eta-1}{\sigma}}di\right] \\
&= \left(\frac{P_{Ht}}{P_t}\right)^{-\eta}C_t\left[(1-\alpha) + \alpha\int_0^1\left(\frac{\mathcal{E}_{it}P_{Ft}^i}{P_{it}}\frac{P_{it}}{P_{Ht}}\right)^{\gamma-\eta}\mathcal{Q}_{it}^{\frac{\sigma\eta-1}{\sigma}}di\right] \\
&= \left(\frac{P_{Ht}}{P_t}\right)^{-\eta}C_t\left[(1-\alpha) + \alpha\int_0^1\left(S_t^iS_{it}\right)^{\gamma-\eta}\mathcal{Q}_{it}^{\frac{\sigma\eta-1}{\sigma}}di\right]
\end{aligned}
$$

Log-linearizing around a symmetric steady-state yields

$$
\begin{aligned}
Y_t \approx Y &- \eta\left(\frac{P_H}{P}\right)^{-\eta-1}\frac{1}{P}C\left[(1-\alpha) + \alpha\int_0^1\left(S^iS_i\right)^{\gamma-\eta}\mathcal{Q}_i^{\frac{\sigma\eta-1}{\sigma}}di\right](P_{Ht}-P_H)+ \\
&+ \eta\left(\frac{P_H}{P}\right)^{-\eta-1}\frac{1}{P}C\left[(1-\alpha) + \alpha\int_0^1\left(S^iS_i\right)^{\gamma-\eta}\mathcal{Q}_i^{\frac{\sigma\eta-1}{\sigma}}di\right](P_t-P)+ \\
&+ \left(\frac{P_H}{P}\right)^{-\eta}\left[(1-\alpha) + \alpha\int_0^1\left(S^iS_i\right)^{\gamma-\eta}\mathcal{Q}_i^{\frac{\sigma\eta-1}{\sigma}}di\right](C_t-C)+ \\
&+ \left(\frac{P_H}{P}\right)^{-\eta}C\alpha\int_0^1(\gamma-\eta)\left(S^iS_i\right)^{\gamma-\eta-1}S_i\mathcal{Q}_i^{\frac{\sigma\eta-1}{\sigma}}(S_t^i-S^i)di+ \\
&+ \left(\frac{P_H}{P}\right)^{-\eta}C\alpha\int_0^1(\gamma-\eta)\left(S^iS_i\right)^{\gamma-\eta-1}S^i\mathcal{Q}_i^{\frac{\sigma\eta-1}{\sigma}}(S_{it}-S_i)di+ \\
&+ \left(\frac{P_H}{P}\right)^{-\eta}C\alpha\int_0^1\left(S^iS_i\right)^{\gamma-\eta}\frac{\sigma\eta-1}{\sigma}\mathcal{Q}_i^{\frac{\sigma(\eta-1)}{\sigma}}(\mathcal{Q}_{it}-\mathcal{Q}_i)di = \\
= Y &- \eta C\frac{P_{Ht}-P_H}{P} + \eta C\frac{P_t-P}{P} + C\frac{C_t-C}{C} + C\alpha(\gamma-\eta)\underbrace{\int_0^1 s_t^i di}+ \\
&+ C\alpha(\gamma-\eta)\underbrace{\int_0^1 s_{it}di}_{s_t} + C\alpha\frac{\sigma\eta-1}{\sigma}\underbrace{\int_0^1 q_{it}di}_{q_t}
\end{aligned}
$$

$$
y_t = -\eta p_{Ht} + \eta p_t + c_t + \alpha(\gamma-\eta)s_t + \alpha\frac{\sigma\eta-1}{\sigma}q_t
$$

inserting (16) and (23)

$$
\begin{aligned}
y_t &= -\eta p_{Ht} + \eta(p_{Ht} + \alpha s_t) + c_t + \alpha(\gamma - \eta)s_t + \alpha\frac{\sigma\eta - 1}{\sigma}q_t \\
&= c_t + \alpha\gamma s_t + \alpha\frac{\sigma\eta - 1}{\sigma}q_t \\
&= c_t + \alpha\gamma s_t + \alpha\frac{\sigma\eta - 1}{\sigma}(1-\alpha)s_t \\
&= c_t + \frac{\alpha\omega}{\sigma}s_t
\end{aligned} \tag{41}
$$

Note that $\omega = \sigma\gamma + (\sigma\eta - 1)(1-\alpha) > 0$ is reasonable because $\alpha \in (0, 1)$. A condition analogous to (41) will hold for all countries. Thus, for a generic country i it can be rewritten as $y_t^i = c_t^i + \frac{\alpha\omega}{\sigma}s_t^i$. By aggregating over all countries, a world market clearing condition can be derived

$$
y_t^* = \int_0^1 y_t^i di = \int_0^1 \left(c_t^i + \frac{\alpha\omega}{\sigma}s_t^i\right) di = \underbrace{\int_0^1 c_t^i di}_{c_t^*} + \frac{\alpha\omega}{\sigma}\underbrace{\int_0^1 s_t^i di}_{0} = c_t^* \tag{42}
$$

Inserting (27) and (42) into (41)

$$
y_t = c_t^* + \frac{1-\alpha}{\sigma} + \frac{\alpha\omega}{\sigma}s_t = y_t^* + \frac{1}{\varsigma}s_t \tag{43}
$$

where $\varsigma = \frac{\sigma}{1-\alpha(1-\omega)} > 0$. Finally, insert c_t from (41) into the Euler equation (12) to get the IS equation

$$
\begin{aligned}
y_t - \frac{\alpha\omega}{\sigma}s_t &= \mathbb{E}_t\left(y_{t+1} - \frac{\alpha\omega}{\sigma}s_{t+1}\right) - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1} - \rho) \\
y_t &= \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1} - \rho) - \frac{\alpha\omega}{\sigma}\mathbb{E}_t\Delta s_{t+1}
\end{aligned} \tag{44}
$$

The IS equation is similar to the one in a closed economy except that now there is an additional term linking domestic output to the international environment.

An alternative representation including domestic inflation is found by inserting (17) into (44)

$$
\begin{aligned}
y_t &= \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}[i_t - \mathbb{E}_t(\pi_{Ht+1} + \alpha\Delta s_{t+1}) - \rho] - \frac{\alpha\omega}{\sigma}\mathbb{E}_t\Delta s_{t+1} \\
&= \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{Ht+1} - \rho) - \frac{\alpha\psi}{\sigma}\mathbb{E}_t\Delta s_{t+1}
\end{aligned} \tag{45}
$$

Note that $\psi = \omega - 1 > 0$ if γ and η are sufficiently high.

Yet another representation is found by inserting for s_t from (43)

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{Ht+1} - \rho) - \frac{\alpha\psi}{\sigma}\varsigma \mathbb{E}_t[(y_{t+1} - y_t) - (y^*_{t+1} - y^*_t)] \\ &= \mathbb{E}_t y_{t+1} - \frac{1}{\sigma - \alpha}(i_t - \mathbb{E}_t \pi_{Ht+1} - \rho) - \frac{\alpha\psi}{\sigma}\varsigma \mathbb{E}_t[(y_{t+1} - y_t) - (y^*_{t+1} - y^*_t)] \\ &= \mathbb{E}_t y_{t+1} - \frac{1}{\sigma - \alpha\psi\varsigma}(i_t - \mathbb{E}_t \pi_{Ht+1} - \rho) + \frac{\alpha\psi\varsigma}{\sigma - \alpha\psi\varsigma}\mathbb{E}_t \Delta y^*_{t+1} \\ &= \mathbb{E}_t y_{t+1} - \frac{1}{\varsigma}(i_t - \mathbb{E}_t \pi_{Ht+1} - \rho) + \alpha\psi \mathbb{E}_t \Delta y^*_{t+1} \end{aligned} \tag{46}$$

The last term $\mathbb{E}_t \Delta y^*_{t+1}$ is exogenous to domestic allocations. Note that, in general, the degree of openness α influences the sensitivity of output to any given change in the domestic real rate $r_{Ht} = i_t - \mathbb{E}_t \pi_{Ht+1}$. Also note from (45) that if $\psi = \omega - 1 > 0$ (i.e., if η and γ are sufficiently high), we have that $\varsigma < \sigma$, and output is more sensitive to real rate changes than in the closed economy case. The reason is the direct negative effect of an increase in the real rate on aggregate demand and output is amplified by the induced real appreciation and the consequent switch of expenditure towards foreign goods. This will be partly offset by any increase in CPI inflation relative to domestic inflation induced by the expected real depreciation, which would dampen the change in the consumption based real rate, $r_t = i_t - \mathbb{E}_t \pi_{t+1}$, which is the one ultimately relevant for aggregate demand.

1.7.2 The Trade Balance

We define net exports nx_t as the difference between total domestic production and total domestic consumption, relative to steady-state output

$$nx_t = \frac{Y_t - \frac{P_t}{P_{Ht}} C_t}{Y} \tag{47}$$

A first-order approximation around a symmetric steady-state with price level $P_t = P_{Ht} = P$ and output level $Y_t = C_t = Y$, i.e., zero net exports, yields

$$\begin{aligned} nx_t &\approx nx + \frac{1}{Y}(Y_t - Y) - \frac{C}{PY}(P_t - P) + \frac{C}{PY}(P_{Ht} - P) - \frac{P}{PY}(C_t - C) \\ &= y_t - p_t + p_{Ht} - c_t = y_t - c_t - \alpha s_t \end{aligned}$$

Inserting (41)

$$nx_t = \frac{\alpha\omega}{\sigma} s_t - \alpha s_t = \frac{\alpha(\omega - \sigma)}{\sigma} s_t \tag{48}$$

1.7.3 The Supply Side: Marginal Cost and Inflation Dynamics

We could rewrite (32) using $\widehat{mc}_{t+k} = \log MC_{t+k} - \log MC$.

$$p^*_{Ht} = (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t[p_{t+k} + \widehat{mc}_{t+k}] = (1-\theta\beta)(p_t + \widehat{mc}_t) + (1-\theta\beta) \underbrace{\sum_{k=1}^{\infty}(\theta\beta)^k \mathbb{E}_t[p_{t+k} + \widehat{mc}_{t+k}]}_{\theta\beta\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t[p_{t+k+1}+\widehat{mc}_{t+k+1}]}$$

Iterating one period forward

$$p^*_{Ht+1} = (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_{t+1}[p_{t+k+1} + \widehat{mc}_{t+k+1}]$$

Using time-t expectations and the law of iterated expectations ($\mathbb{E}_t(X) = \mathbb{E}_t[\mathbb{E}_{t+1}(X)]$),

$$\mathbb{E}_t p^*_{Ht+1} = (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t[\mathbb{E}_{t+1}[p_{t+k+1} + \widehat{mc}_{t+k+1}]]$$

Hence, we can write

$$p^*_{Ht} = (1-\theta\beta)(p_t + \widehat{mc}_t) + \theta\beta\mathbb{E}_t p^*_{Ht+1} \tag{49}$$

Let us now turn to inflation dynamics. Let Ω_t denote the subset of firms not reoptimizing at time t

$$\begin{aligned} P_{Ht} &= \left(\int_0^1 P_{Ht}(j)^{1-\varepsilon}dj\right)^{\frac{1}{1-\varepsilon}} = \left[\int_{\Omega_t} P_{Ht-1}(j)^{1-\varepsilon}dj + \int_{\Omega_t^C}(P^*_{Ht})^{1-\varepsilon}dj\right]^{\frac{1}{1-\varepsilon}} = \\ &= \left[\theta\int_0^1 P_{Ht-1}(j)^{1-\varepsilon}dj + (1-\theta)\int_0^1 (P^*_{Ht})^{1-\varepsilon}dj\right]^{\frac{1}{1-\varepsilon}} \\ &= \left[\theta P^{1-\varepsilon}_{Ht-1} + (1-\theta)(P^*_{Ht})^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \end{aligned}$$

$$\implies \left(\frac{P_{Ht}}{P_{Ht-1}}\right)^{1-\varepsilon} = \theta\left(\frac{P_{Ht-1}}{P_{Ht-1}}\right)^{1-\varepsilon} + (1-\theta)\left(\frac{P^*_{Ht}}{P_{Ht-1}}\right)^{1-\varepsilon} \implies (1+\pi_{Ht})^{1-\varepsilon} = \theta + (1-\theta)\left(\frac{P^*_{Ht}}{P_{Ht-1}}\right)^{1-\varepsilon}$$

Log-linearizing

$$(1-\varepsilon)\log(1+\pi_{Ht}) = \log\left[\theta + (1-\theta)\left(\frac{P^*_{Ht}}{P_{Ht-1}}\right)^{1-\varepsilon}\right]$$

$$(1-\varepsilon)\frac{1}{1+\pi_H}(\pi_{Ht} - \pi_H) \approx \frac{(1-\theta)(1-\varepsilon)\left(\frac{P}{P}\right)^{-\varepsilon}\frac{1}{P}}{\theta + (1-\theta)\left(\frac{P}{P}\right)^{1-\varepsilon}}(P^*_{Ht} - P) - \frac{(1-\theta)(1-\varepsilon)\left(\frac{P}{P}\right)^{-\varepsilon}\frac{1}{P}}{\theta + (1-\theta)\left(\frac{P}{P}\right)^{1-\varepsilon}}(P_{Ht-1} - P)$$

$$\pi_{Ht} = (1-\theta)(p^*_{Ht} - p_{Ht-1})$$

Plugging it into (49), and introducing (16)

$$
\begin{aligned}
\frac{\pi_{Ht}}{1-\theta} + p_{Ht-1} &= (1-\theta\beta)(p_t + \widehat{mc}_t) + \theta\beta\mathbb{E}_t\left[\frac{\pi_{Ht}}{1-\theta} + p_{Ht-1}\right] \\
\frac{\pi_{Ht}}{1-\theta} + p_{Ht-1} &= (1-\theta\beta)(p_{Ht} + \alpha s_t + \widehat{mc}_t) + \theta\beta\mathbb{E}_t\left[\frac{\pi_{Ht}}{1-\theta} + p_{Ht-1}\right] \\
\pi_{Ht} &= \lambda(\alpha s_t + \widehat{mc}_t) + \beta\mathbb{E}_t\pi_{Ht+1}
\end{aligned}
$$

where $\lambda = \frac{(1-\theta\beta)(1-\theta)}{\theta}$ and the last expression could be written as

$$
\pi_{Ht} = \lambda\widehat{mc}_{Ht} + \beta\mathbb{E}_t\pi_{Ht+1} \tag{50}
$$

since $\widehat{mc}_{Ht} = w_t - a_t - p_{Ht} - \log MC = \widehat{mc}_t + \alpha s_t$.
Let us now move to the labor market clearing condition

$$
N_t = \int_0^1 N_t(j)dj \tag{51}
$$

Using the production function (30) and plugging in (29)

$$
N_t = \int_0^1 \frac{Y_t(j)}{A_t}dj = \int_0^1 \frac{\left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} Y_t}{A_t}dj = \frac{Y_t}{A_t}\int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} dj
$$

Take logs

$$
n_t = y_t - a_t + \log\left\{\int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} dj\right\} \tag{52}
$$

Recall the consumer price index (3). Rearranging gives

$$
1 = \left\{\int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{1-\varepsilon} dj\right\}^{\frac{1}{1-\varepsilon}} = \left[\int_0^1 e^{(1-\varepsilon)[p_{Ht}(j)-p_{Ht}]}dj\right]^{\frac{1}{1-\varepsilon}} \implies 1 = \int_0^1 e^{(1-\varepsilon)[p_{Ht}(j)-p_{Ht}]}dj
$$

A second order approximation yields

$$
\begin{aligned}
1 &\approx \int_0^1 \left\{e^0 + (1-\varepsilon)e^0[p_{Ht}(j) - p_{Ht}] + \frac{(1-\varepsilon)^2}{2}e^0[p_{Ht}(j)-p_{Ht}]^2\right\}dj = \\
&= 1 - (1-\varepsilon)p_{Ht} + (1-\varepsilon)\int_0^1 p_{Ht}(j)dj + \frac{(1-\varepsilon)^2}{2}\int_0^1 [p_{Ht}(j)-p_{Ht}]^2 dj \implies
\end{aligned}
$$

$$
\implies p_{Ht} = \int_0^1 p_{Ht}(j)dj + \frac{1-\varepsilon}{2}\int_0^1 [p_{Ht}(j)-p_{Ht}]^2 dj \tag{53}
$$

Hence, $p_{Ht} = \int_0^1 p_{Ht}(j)dj$ up to a first-order approximation.

Now lets do a second-order approximation of $\int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} dj$

$$\begin{aligned}
\int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} dj &= \int_0^1 e^{-\varepsilon[p_{Ht}(j)-p_{Ht}]} dj \approx \\
&\approx \int_0^1 \left\{e^0 - \varepsilon e^0 [p_{Ht}(j) - p_{Ht}] dj + \frac{\varepsilon^2}{2} e^0 [p_{Ht}(j) - p_{Ht}]^2 \right\} dj \\
&= 1 + \varepsilon p_{Ht} - \varepsilon \int_0^1 p_{Ht}(j) dj + \frac{\varepsilon^2}{2} [p_{Ht}(j) - p_{Ht}]^2 dj
\end{aligned}$$

Inserting (53)

$$\begin{aligned}
\int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} dj &\approx 1 + \varepsilon \left\{\int_0^1 p_{Ht}(j) dj + \frac{1-\varepsilon}{2} \int_0^1 [p_{Ht}(j) - p_{Ht}]^2 dj \right\} - \varepsilon \int_0^1 p_{Ht}(j) dj + \frac{\varepsilon^2}{2} \int_0^1 [p_{Ht}(j) - p \\
&= 1 + \frac{\varepsilon}{2} \int_0^1 [p_{Ht}(j) - p_{Ht}]^2 dj
\end{aligned}$$

Hence, up to a first-order approximation, $\int_0^1 \left[\frac{P_{Ht}(j)}{P_{Ht}}\right]^{-\varepsilon} dj = 1$ and

$$n_t = y_t - a_t \implies y_t = a_t + n_t \tag{54}$$

Going back to (50) we can rewrite mc_{Ht} plugging in (27), (43) and the labor supply condition

$$\begin{aligned}
mc_{Ht} &= w_t - a_t - p_{Ht} = \\
&= w_t - a_t - p_t - \alpha s_t = \\
&= \sigma c_t + \phi n_t - a_t - \alpha s_t = \\
&= \sigma \left[c_t^* + \frac{1-\alpha}{\sigma} \varsigma (y_t - y_t^*)\right] + \phi(y_t - a_t) - a_t - \alpha\varsigma(y_t - y_t^*) = \\
&= (\sigma - \varsigma) y_t^* + (\phi + \varsigma) y_t - (1 + \phi) a_t
\end{aligned} \tag{55}$$

Domestic output affects marginal costs through its impact on employment (captured by ϕ) and the terms of trade (captured by ς, which is a function of the degree of openness α and the substitutability between domestic and foreign goods). World output, on the other hand, affects marginal costs through its effect on consumption (and hence, the real wage as captured by σ) and the terms of trade (captured by ς). The effect of world output on marginal costs is positive if $\sigma > \varsigma$). This is because with sufficiently high η and γ, the size of the real appreciation needed to absorb the change in relative supplies is small, with its negative effect on marginal costs more than offset by the positive effect from a higher real wage.

What about the natural level of output y_t^n, i.e. the output when prices are flexible? We know from earlier that in this case, $mc_H = mc = -\mu$. Thus, the flexible price version of (55) is simply:

$$mc_H = -\mu = (\sigma - \varsigma)y_t^* + (\phi + \varsigma)y_t^n - (1+\phi)a_t \tag{56}$$

Solving the last expression for natural output,

$$\begin{aligned} y_t^n &= -\frac{\mu}{\phi+\varsigma} - \frac{\sigma-\varsigma}{\phi+\varsigma}y_t^* + \frac{1+\phi}{\phi+\varsigma}a_t \\ &= \Gamma_0 + \Gamma_* y_t^* + \Gamma_a a_t \end{aligned} \tag{57}$$

where $\Gamma_0 = -\frac{\mu}{\phi+\varsigma}$, $\Gamma_* = -\frac{\sigma-\varsigma}{\phi+\varsigma}$ and $\Gamma_a = \frac{1+\phi}{\phi+\varsigma}$. Again the effect of world output on natural output is ambiguous, depending on the effect of world output on domestic marginal costs, which in turn depends on the relative importance of the terms of trade effect discussed above.

1.8 The New Keynesian Phillips Curve and the Dynamic IS Equation

Denote $\tilde{y}_t = y_t - y_t^n$ as the domestic output gap from flexible price output. Obtaining $\widehat{mc}_{Ht}$,

$$\begin{aligned} \widehat{mc}_{Ht} &= mc_{Ht} - mc_H = \\ &= [(\sigma-\varsigma)y_t^* + (\phi+\varsigma)y_t - (1+\phi)a_t] - [(\sigma-\varsigma)y_t^* + (\phi+\varsigma)y_t^n - (1+\phi)a_t] = \\ &= (\phi+\varsigma)\tilde{y}_t \end{aligned} \tag{58}$$

Insert (58) into (50)

$$\begin{aligned} \pi_{Ht} &= \lambda(\phi+\varsigma)\tilde{y}_t + \beta \mathbb{E}_t \pi_{Ht+1} \\ &= \varphi \tilde{y}_t + \beta \mathbb{E}_t \pi_{Ht+1} \end{aligned} \tag{59}$$

Note that (59) nests the special case of a closed economy because $\alpha = 0$ implies that $\sigma = \varsigma$ and then (59) becomes identical to the closed economy equivalent. In general, the relation between the degree of openness parameter α, an increase in the output gap, and domestic inflation, depends on the sign on ψ. If $\psi > 0$ (i.e. if η and γ are sufficiently high), an increase in the openness will make domestic inflation less responsive to a change in the output gap. On the other hand, if $\psi < 0$, then more openness will make domestic inflation more responsive to output gap changes.

To derive the open economy DIS equation, define the real interest rate as

$$r_{Ht} = i_t - \mathbb{E}_t \pi_{Ht+1}$$

(46) can be rewritten as

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\varsigma}(r_{Ht} - \rho) + \alpha\psi \mathbb{E}_t \Delta y_{t+1}^*$$

The natural output is given as a function of the natural real interest rate

$$y_t^n = \mathbb{E}_t y_{t+1}^n - \frac{1}{\varsigma}(r_{Ht}^n - \rho) + \alpha\psi \mathbb{E}_t \Delta y_{t+1}^* \tag{60}$$

Substracting (60) from (46) yields the DIS equation

$$\begin{aligned}
\tilde{y}_t &= y_t - y_t^n = \\
&= [\mathbb{E}_t y_{t+1} - \frac{1}{\varsigma}(i_t - \mathbb{E}_t \pi_{Ht+1} - \rho) + \alpha\psi \mathbb{E}_t \Delta y_{t+1}^*] - [\mathbb{E}_t y_{t+1}^n - \frac{1}{\varsigma}(r_{Ht}^n - \rho) + \alpha\psi \mathbb{E}_t \Delta y_{t+1}^*] \\
&= \mathbb{E}_t \tilde{y}_{t+1} - \frac{1}{\varsigma}(i_t - \mathbb{E}_t \pi_{Ht+1} - r_{Ht}^n) \qquad (61)
\end{aligned}$$

Equations (59) and (61), together with an equilibrium process for the natural real rate r_{Ht}^n, constitute the non-policy block of the small open economy version of the New Keynesian model.

The domestic natural real rate can be extracted from (61), but first one should note that (46) implies that

$$\mathbb{E}_t \Delta y_{t+1} = \frac{1}{\varsigma}(i_t - \mathbb{E}_t \pi_{Ht+1} - \rho) - \alpha\psi \mathbb{E}_t \Delta y_{t+1}^*$$

Second, (57) implies that

$$\mathbb{E}_t \Delta y_{t+1}^n = \Gamma_* \mathbb{E}_t \Delta y_{t+1}^* + \Gamma_a \mathbb{E}_t \Delta a_{t+1}$$

Third, (59) implies

$$\mathbb{E}_t \Delta \tilde{y}_{t+1} = \frac{1}{\varsigma}(i_t - \mathbb{E}_t \pi_{Ht+1} - r_{Ht}^n)$$

Using these results, one can solve for r_{Ht}^n

$$\begin{aligned}
r_{Ht}^n &= i_t - \mathbb{E}_t \pi_{Ht+1} - \varsigma \mathbb{E}_t \Delta \tilde{y}_{t+1} \\
&= i_t - \mathbb{E}_t \pi_{Ht+1} - \varsigma \left[\mathbb{E}_t (\Delta y_{t+1} - \Delta y_{t+1}^n)\right] \\
&= i_t - \mathbb{E}_t \pi_{Ht+1} - \varsigma \left[\frac{1}{\varsigma}(i_t - \mathbb{E}_t \pi_{Ht+1} - \rho) - \alpha\psi \mathbb{E}_t \Delta y_{t+1}^* - \Gamma_* \mathbb{E}_t \Delta y_{t+1}^* - \Gamma_a \mathbb{E}_t \Delta a_{t+1}\right] \\
&= \rho + \varsigma(\alpha\psi + \Gamma_*)\mathbb{E}_t \Delta y_{t+1}^* + \varsigma \Gamma_a \mathbb{E}_t \Delta a_{t+1} \\
&= \rho + \varsigma(\alpha\psi + \Gamma_*)\mathbb{E}_t \Delta y_{t+1}^* + \varsigma \Gamma_a (\rho_a - 1) a_t
\end{aligned}$$

where we have made use of $a_{t+1} = \rho_a a_t + \epsilon_{t+1}$. Thus, we see that the New Keynesian Phillips curve and the DIS equation in the small open economy equilibrium is similar to the counterparts in the closed economy. A couple of differences appear however. First, the degree of openness influences the sensitivity of the output gap to interest rate changes. Second, openness generally makes the natural real interest rate depend on expected world output growth, in addition to domestic productivity. Finally, it is convenient to define the real rate gap as:

$$\hat{r}^n_{Ht} = r^n_{Ht} - \rho = \varsigma(\alpha\psi + \Gamma_*)\mathbb{E}_t\Delta y^*_{t+1} + \varsigma\Gamma_a(\rho_a - 1)a_t \quad (62)$$

As in the closed economy case the real rate converges to the discount rate once technology shocks and world output growth is turned off. Note however, that the real rate will typically by higher than the discount rate because the world experiences a positive growth on average.

1.9 Monetary Authority

The economy starts out of a Currency Union, with a Central Bank following a country-specific Taylor Rule

$$i_t = \rho + \phi_\pi\pi_{Ht} + \phi_y\tilde{y}_t + \nu_t$$