Chapter 8
Equilibrium with Complete Markets

8.1. Time 0 versus sequential trading

This chapter describes competitive equilibria of a pure exchange infinite horizon economy with stochastic endowments. These are useful for studying risk sharing, asset pricing, and consumption. We describe two systems of markets: an *Arrow-Debreu* structure with complete markets in dated contingent claims all traded at time 0, and a sequential-trading structure with complete one-period *Arrow securities*. These two entail different assets and timings of trades, but have identical consumption allocations. Both are referred to as complete markets economies. They allow more comprehensive sharing of risks than do the incomplete markets economies to be studied in chapters 17 and 18, or the economies with imperfect enforcement or imperfect information, studied in chapters 20 and 21.

8.2. The physical setting: preferences and endowments

In each period $t \geq 0$, there is a realization of a stochastic event $s_t \in S$. Let the history of events up and until time t be denoted $s^t = [s_0, s_1, \ldots, s_t]$. The unconditional probability of observing a particular sequence of events s^t is given by a probability measure $\pi_t(s^t)$. For $t > \tau$, we write the probability of observing s^t conditional on the realization of s^τ as $\pi_t(s^t | s^\tau)$. In this chapter, we shall assume that trading occurs after observing s_0, which we capture by setting $\pi_0(s_0) = 1$ for the initially given value of s_0.[1]

In section 8.9 we shall follow much of the literatures in macroeconomics and econometrics and assume that $\pi_t(s^t)$ is induced by a Markov process. We wait to impose that special assumption until section 8.9 because some important findings do not require making that assumption.

[1] Most of our formulas carry over to the case where trading occurs before s_0 has been realized; just postulate a nondegenerate probability distribution $\pi_0(s_0)$ over the initial state.

There are I agents named $i = 1, \ldots, I$. Agent i owns a stochastic endowment of one good $y_t^i(s^t)$ that depends on the history s^t. The history s^t is publicly observable. Household i purchases a history-dependent consumption plan $c^i = \{c_t^i(s^t)\}_{t=0}^\infty$ and orders these consumption streams by[2]

$$U\left(c^i\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t^i\left(s^t\right)\right] \pi_t\left(s^t\right), \tag{8.2.1}$$

where $0 < \beta < 1$. The right side is equal to $E_0 \sum_{t=0}^\infty \beta^t u(c_t^i)$, where E_0 is the mathematical expectation operator, conditioned on s_0. Here $u(c)$ is an increasing, twice continuously differentiable, strictly concave function of consumption $c \geq 0$ of one good. The utility function satisfies the Inada condition[3]

$$\lim_{c \downarrow 0} u'(c) = +\infty.$$

Notice that in assuming (8.2.1), we are imposing identical preference orderings across all individuals i that can be represented in terms of discounted expected utility with common β, common utility function $u(\cdot)$, and common probability distributions $\pi_t(s^t)$. As we proceed through this chapter, watch for results that would evaporate if we were instead to allow $\beta, u(\cdot)$, or $\pi_t(s^t)$ to depend on i.

A *feasible allocation* satisfies

$$\sum_i c_t^i\left(s^t\right) \leq \sum_i y_t^i\left(s^t\right) \tag{8.2.2}$$

for all t and for all s^t.

[2] Exercises 8.13 - 8.17 consider examples in which we replace (8.2.1) with

$$U^i\left(c^i\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t^i\left(s^t\right)\right] \pi_t^i\left(s^t\right),$$

where $\pi^i(s^t)$ is a personal probability distribution specific to agent i. Blume and Easley (2006) study such settings, focusing particularly on which agents' beliefs ultimately influence the tails of allocations and prices. Throughout most of this chapter, we adopt the assumption, routinely employed in much of macroeconomics, that all agents share probabilities.

[3] One role of this Inada condition is to make the consumption of each agent strictly positive in every date-history pair. A related role is to deliver a state-by-state borrowing limit to impose in economies with sequential trading of Arrow securities.

8.3. Alternative trading arrangements

For a two-event stochastic process $s_t \in S = \{0, 1\}$, the trees in Figures 8.3.1 and 8.3.2 give two portraits of how histories s^t unfold. From the perspective of time 0 given $s_0 = 0$, Figure 8.3.1 portrays all prospective histories possible up to time 3. Figure 8.3.2 portrays a *particular* history that it is known the economy has indeed followed up to time 2, together with the two possible one-period continuations into period 3 that can occur after that history.

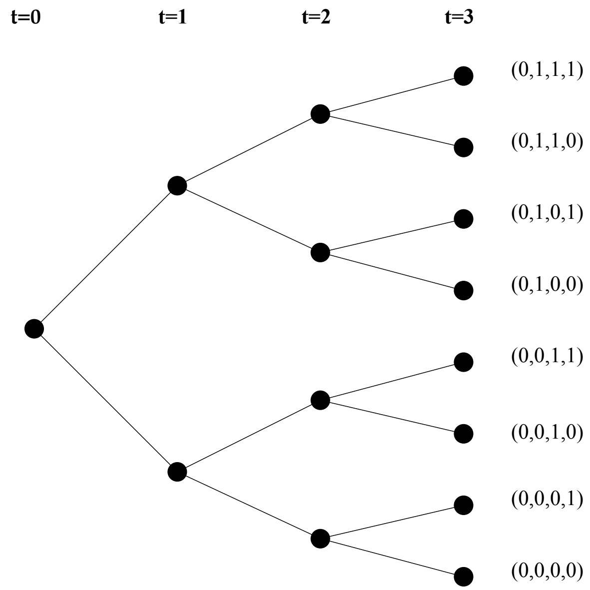

Figure 8.3.1: The Arrow-Debreu commodity space for a two-state Markov chain. At time 0, there are trades in time $t = 3$ goods for each of the eight nodes that signify histories that can possibly be reached starting from the node at time 0.

In this chapter we shall study two distinct trading arrangements that correspond, respectively, to the two views of the economy in Figures 8.3.1 and 8.3.2. One is what we shall call the Arrow-Debreu structure. Here markets meet at time 0 to trade claims to consumption at all times $t > 0$ and that are contingent on all possible histories up to t, s^t. In that economy, at time 0 and

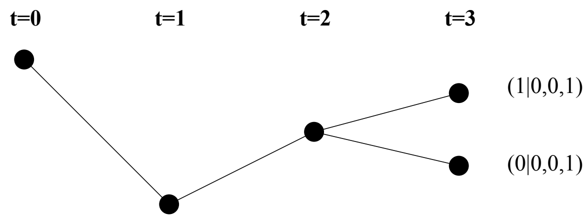

Figure 8.3.2: The commodity space with Arrow securities. At date $t=2$, there are trades in time 3 goods for only those time $t=3$ nodes that can be reached from the realized time $t=2$ history $(0,0,1)$.

for all $t \geq 1$, households trade claims on the time t consumption good *at all nodes* s^t. After time 0, no further trades occur. The other economy has *sequential* trading of only one-period-ahead state-contingent claims. Here trades of one-period ahead state-contingent claims occur at each date $t \geq 0$. Trades for history s^{t+1}–contingent date $t+1$ goods occur only at the *particular* date t history s^t that has been reached at t, as in Figure 8.3.2. It turns out that these two trading arrangements support identical equilibrium allocations. Those allocations share the notable property of being functions only of the *aggregate* endowment realization $\sum_{i=1}^{I} y_t^i(s^t)$ and time-invariant parameters describing the initial distribution of wealth.

8.3.1. History dependence

Before trading, the situation of household i at time t depends on the history s^t. A natural measure of household i's luck in life is $\{y_0^i(s_0), y_1^i(s^1), \ldots,\ y_t^i(s^t)\}$, which evidently in general depends on the history s^t. A question that will occupy us in this chapter and in chapters 18 and 20 is whether, after trading, the household's consumption allocation at time t is also history dependent. Remarkably, in the complete markets models of this chapter, the consumption allocation at time t depends only on the aggregate endowment realization at time t and some time-invariant parameters that describe the time 0 *initial* distribution of wealth. The market incompleteness of chapter 18 and the information and enforcement frictions of chapter 20 will break that result and put history dependence into equilibrium allocations.

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this discussion

8.4. Pareto problem

As a benchmark against which to measure allocations attained by a market economy, we seek efficient allocations. An allocation is said to be efficient if it is Pareto optimal: it has the property that any reallocation that makes one household strictly better off also makes one or more other households worse off. We can find efficient allocations by posing a Pareto problem for a fictitious social planner. The planner attaches nonnegative *Pareto weights* $\lambda_i, i = 1, \ldots, I$ to the consumers' utilities and chooses allocations $c^i, i = 1, \ldots, I$ to maximize

$$W = \sum_{i=1}^{I} \lambda_i U\left(c^i\right) \tag{8.4.1}$$

subject to (8.2.2). We call an allocation *efficient* if it solves this problem for some set of nonnegative λ_i's. Let $\theta_t(s^t)$ be a nonnegative Lagrange multiplier on the feasibility constraint (8.2.2) for time t and history s^t, and form the Lagrangian

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^{I} \lambda_i \beta^t u\left(c_t^i\left(s^t\right)\right) \pi_t\left(s^t\right) + \theta_t\left(s^t\right) \sum_{i=1}^{I} \left[y_t^i\left(s^t\right) - c_t^i\left(s^t\right)\right] \right\}.$$

The first-order condition for maximizing L with respect to $c_t^i(s^t)$ is

$$\beta^t u'\left(c_t^i\left(s^t\right)\right) \pi_t\left(s^t\right) = \lambda_i^{-1} \theta_t\left(s^t\right) \tag{8.4.2}$$

for each i, t, s^t. Taking the ratio of (8.4.2) for consumers i and 1, respectively, gives

$$\frac{u'\left(c_t^i\left(s^t\right)\right)}{u'\left(c_t^1\left(s^t\right)\right)} = \frac{\lambda_1}{\lambda_i}$$

which implies

$$c_t^i\left(s^t\right) = u'^{-1}\left(\lambda_i^{-1} \lambda_1 u'\left(c_t^1\left(s^t\right)\right)\right). \tag{8.4.3}$$

Substituting (8.4.3) into feasibility condition (8.2.2) at equality gives

$$\sum_i u'^{-1}\left(\lambda_i^{-1} \lambda_1 u'\left(c_t^1\left(s^t\right)\right)\right) = \sum_i y_t^i\left(s^t\right). \tag{8.4.4}$$

Equation (8.4.4) is one equation in the one unknown $c_t^1(s^t)$. The right side of (8.4.4) is the realized aggregate endowment, so the left side is a function only of the aggregate endowment. Thus, given $\{\lambda_i\}_{i=1}^I$, $c_t^1(s^t)$ depends only on the current realization of the aggregate endowment and not separately either on the date t or on the specific history s^t leading up to that aggregate endowment or the cross-section distribution of individual endowments realized at t. Equation (8.4.3) then implies that for all i, $c_t^i(s^t)$ depends only on the aggregate endowment realization. We thus have:

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PROPOSITION 1: An efficient allocation is a function of the realized aggregate endowment and does not depend separately on either the specific history s^t leading up to that aggregate endowment or on the cross-section distribution of individual endowments realized at t: $c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau)$ for s^t and $\tilde{s}^\tau$ such that $\sum_j y_t^j(s^t) = \sum_j y_\tau^j(\tilde{s}^\tau)$.

To compute the optimal allocation, first solve (8.4.4) for $c_t^1(s^t)$, then solve (8.4.3) for $c_t^i(s^t)$. Note from (8.4.3) that only the ratios of the Pareto weights matter, so that we are free to normalize the weights, e.g., to impose $\sum_i \lambda_i = 1$.

8.4.1. Time invariance of Pareto weights

Through equations (8.4.3) and (8.4.4), the allocation $c_t^i(s^t)$ assigned to consumer i depends in a time-invariant way on the aggregate endowment $\sum_j y_t^j(s^t)$. Consumer i's share of the aggregate endowment varies directly with his Pareto weight λ_i. In chapter 20, we shall see that the constancy through time of the Pareto weights $\{\lambda_j\}_{j=1}^I$ is a telltale sign that there are no enforcement- or information-related incentive problems in this economy. In chapter 20, when we inject those imperfections into the environment, the time invariance of the Pareto weights evaporates.

8.5. Time 0 trading: Arrow-Debreu securities

We now describe how an optimal allocation can be attained by a competitive equilibrium with the Arrow-Debreu timing. Households trade dated history-contingent claims to consumption. There is a complete set of securities. Trades occur at time 0, after s_0 has been realized. At $t=0$, households can exchange claims on time t consumption, contingent on history s^t at price $q_t^0(s^t)$, measured in some unit of account. The superscript 0 refers to the date at which trades occur, while the subscript t refers to the date that deliveries are to be made. The household's budget constraint is

why not numeraire?

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) c_t^i\left(s^t\right) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) y_t^i\left(s^t\right). \tag{8.5.1}$$

The household's problem is to choose c^i to maximize expression $(8.2.1)$ subject to inequality $(8.5.1)$.

Underlying the *single* budget constraint $(8.5.1)$ is the fact that multilateral trades are possible through a clearing operation that keeps track of net claims.[4] All trades occur at time 0. After time 0, trades that were agreed to at time 0 are executed, but no more trades occur.

Attach a Lagrange multiplier μ_i to each household's budget constraint $(8.5.1)$. We obtain the first-order conditions for the household's problem:

$$\frac{\partial U\left(c^i\right)}{\partial c_t^i\left(s^t\right)} = \mu_i q_t^0\left(s^t\right), \tag{8.5.2}$$

for all i, t, s^t. The left side is the derivative of total utility with respect to the time t, history s^t component of consumption. Each household has its own Lagrange multiplier μ_i that is independent of time. With specification $(8.2.1)$ of the utility functional, we have

$$\frac{\partial U\left(c^i\right)}{\partial c_t^i\left(s^t\right)} = \beta^t u'\left[c_t^i\left(s^t\right)\right]\pi_t\left(s^t\right). \tag{8.5.3}$$

This expression implies that equation $(8.5.2)$ can be written

$$\beta^t u'\left[c_t^i\left(s^t\right)\right]\pi_t\left(s^t\right) = \mu_i q_t^0\left(s^t\right). \tag{8.5.4}$$

[4] In the language of modern payments systems, this is a system with net settlements, not gross settlements, of trades.

We use the following definitions:

DEFINITIONS: A *price system* is a sequence of functions $\{q_t^0(s^t)\}_{t=0}^{\infty}$. An *allocation* is a list of sequences of functions $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$, one for each i.

DEFINITION: A *competitive equilibrium* is a feasible allocation and a price system such that, given the price system, the allocation solves each household's problem.

Notice that equation (8.5.4) implies

$$\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_t^j\left(s^t\right)\right]} = \frac{\mu_i}{\mu_j} \tag{8.5.5}$$

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for all pairs (i,j). Thus, ratios of marginal utilities between pairs of agents are constant across all histories and dates.

An equilibrium allocation solves equations (8.2.2), (8.5.1), and (8.5.5). Note that equation (8.5.5) implies that

$$c_t^i\left(s^t\right) = u'^{-1}\left\{u'\left[c_t^1\left(s^t\right)\right]\frac{\mu_i}{\mu_1}\right\}. \tag{8.5.6}$$

Substituting this into equation (8.2.2) at equality gives

$$\sum_i u'^{-1}\left\{u'\left[c_t^1\left(s^t\right)\right]\frac{\mu_i}{\mu_1}\right\} = \sum_i y_t^i\left(s^t\right). \tag{8.5.7}$$

The right side of equation (8.5.7) is the current realization of the aggregate endowment. Therefore, the left side, and so $c_t^1(s^t)$, must also depend only on the current aggregate endowment, as well as on the ratio $\frac{\mu_i}{\mu_1}$. It follows from equation (8.5.6) that the equilibrium allocation $c_t^i(s^t)$ for each i depends only on the economy's aggregate endowment as well as on $\frac{\mu_i}{\mu_1}$. We summarize this analysis in the following proposition:

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PROPOSITION 2: The competitive equilibrium allocation is a function of the realized aggregate endowment and does not depend on time t or the specific history or on the cross section distribution of endowments: $c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau)$ for all histories s^t and $\tilde{s}^\tau$ such that $\sum_j y_t^j(s^t) = \sum_j y_\tau^j(\tilde{s}^\tau)$.

8.5.1. Equilibrium pricing function

Suppose that c^i, $i = 1, \ldots, I$ is an equilibrium allocation. Then the marginal condition (8.5.2) or (8.5.4) can be regarded as determining the price system $q_t^0(s^t)$ as a function of the equilibrium allocation assigned to household i, for any i. But to exploit this fact in computation, we need a way first to compute an equilibrium allocation without simultaneously computing prices. As we shall see soon, solving the planning problem provides a convenient way to do that.

Because the units of the price system are arbitrary, one of the prices can be normalized at any positive value. We shall set $q_0^0(s_0) = 1$, putting the price system in units of time 0 goods. This choice implies that $\mu_i = u'[c_0^i(s_0)]$ for all i.

8.5.2. Optimality of equilibrium allocation

A competitive equilibrium allocation is a particular Pareto optimal allocation, one that sets the Pareto weights $\lambda_i = \mu_i^{-1}$. These weights are unique up to multiplication by a positive scalar. Furthermore, at a competitive equilibrium allocation, the *shadow prices* $\theta_t(s^t)$ for the associated planning problem equal the prices $q_t^0(s^t)$ for goods to be delivered at date t contingent on history s^t associated with the Arrow-Debreu competitive equilibrium. That allocations for the planning problem and the competitive equilibrium are identical reflects the two fundamental theorems of welfare economics (see Mas-Colell, Whinston, and Green (1995)). The first welfare theorem states that a competitive equilibrium allocation is efficient. The second welfare theorem states that any efficient allocation can be supported by a competitive equilibrium with an appropriate initial distribution of wealth.

8.5.3. Interpretation of trading arrangement

In the competitive equilibrium, all trades occur at $t = 0$ in one market. Deliveries occur after $t = 0$, but no more trades. A vast clearing or credit system operates at $t = 0$. It ensures that condition (8.5.1) holds for each household i. A symptom of the once-and-for-all and net-clearing trading arrangement is that each household faces one budget constraint that accounts for trades across all dates and histories.

In section 8.8, we describe another trading arrangement with more trading dates but fewer securities at each date.

8.5.4. Equilibrium computation

To compute an equilibrium, we have somehow to determine ratios of the Lagrange multipliers, μ_i/μ_1, $i = 1, \ldots, I$, that appear in equations (8.5.6) and (8.5.7). The following *Negishi algorithm* accomplishes this.[5]

1. Fix a positive value for one μ_i, say μ_1, throughout the algorithm. Guess some positive values for the remaining μ_i's. Then solve equations (8.5.6) and (8.5.7) for a candidate consumption allocation $c^i, i = 1, \ldots, I$.

2. Use (8.5.4) for any household i to solve for the price system $q_t^0(s^t)$.

3. For $i = 1, \ldots, I$, check the budget constraint (8.5.1). For those i's for which the cost of consumption exceeds the value of their endowment, raise μ_i, while for those i's for which the reverse inequality holds, lower μ_i.

4. Iterate to convergence on steps 1-3.

Multiplying all of the μ_i's by a positive scalar simply changes the units of the price system. That is why we are free to normalize as we have in step 1. (μ_1)

In general, the equilibrium price system and distribution of wealth are mutually determined. Along with the equilibrium allocation, they solve a vast system of simultaneous equations. The Negishi algorithm provides one way to solve those equations. In applications, it can be complicated to implement. Therefore, in order to simplify things, most of the examples and exercises in this chapter specialize preferences in a way that eliminates the dependence of equilibrium prices on the distribution of wealth.

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5 See Negishi (1960).

8.6. Simpler computational algorithm

The preference specification in the following example enables us to avoid iterating on Pareto weights as in the Negishi algorithm.

8.6.1. Example 1: risk sharing

Suppose that the one-period utility function is of the constant relative risk-aversion (CRRA) form

$$u(c) = (1-\gamma)^{-1} c^{1-\gamma}, \ \gamma > 0.$$

Then equation (8.5.5) implies

$$\left[c_t^i\left(s^t\right)\right]^{-\gamma} = \left[c_t^j\left(s^t\right)\right]^{-\gamma} \frac{\mu_i}{\mu_j}$$

or

$$c_t^i\left(s^t\right) = c_t^j\left(s^t\right) \left(\frac{\mu_i}{\mu_j}\right)^{-\frac{1}{\gamma}}. \tag{8.6.1}$$

Equation (8.6.1) states that time t elements of consumption allocations to distinct agents are constant fractions of one another. With a power utility function, it says that individual consumption is perfectly correlated with the aggregate endowment or aggregate consumption.[6]

The fractions of the aggregate endowment assigned to each individual are independent of the realization of s^t. Thus, there is extensive cross-history and cross-time consumption sharing. The constant-fractions-of-consumption characterization comes from two aspects of the theory: (1) complete markets and (2) a homothetic one-period utility function.

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6 Equation (8.6.1) implies that conditional on the history s^t, time t consumption $c_t^i(s^t)$ is independent of the household's individual endowment at t, s^t, $y_t^i(s^t)$. Mace (1991), Cochrane (1991), and Townsend (1994) have tested and rejected versions of this conditional independence hypothesis. In chapter 20, we study how particular impediments to trade explain these rejections.

8.6.2. Implications for equilibrium computation

Equation (8.6.1) and the pricing formula (8.5.4) imply that an equilibrium price vector satisfies

$$q_t^0\left(s^t\right) = \mu_i^{-1}\alpha_i^{-\gamma}\beta^t\left(\overline{y}_t\left(s^t\right)\right)^{-\gamma}\pi_t\left(s^t\right), \tag{8.6.2}$$

where $c_t^i(s_t) = \alpha_i \overline{y}_t(s^t)$, $\overline{y}_t(s^t) = \sum_i y_t^i(s^t)$, and α_i is consumer i's fixed consumption share of the aggregate endowment. We are free to normalize the price system by setting $\mu_i \alpha_i^{-\gamma}$ for one consumer to an arbitrary positive number.

The homothetic CRRA preference specification that leads to equation (8.6.2) allows us to compute an equilibrium using the following steps:

1. Use (8.6.2) to compute an equilibrium price system.
2. Use this price system and consumer i's budget constraint to compute

$$\alpha_i = \frac{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) y_t^i\left(s^t\right)}{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) \overline{y}_t\left(s^t\right)}.$$

Thus, consumer i's fixed consumption share α_i equals its share of aggregate wealth evaluated at the competitive equilibrium price vector.

8.6.3. Example 2: no aggregate uncertainty

In this example, the endowment structure is sufficiently simple that we can compute an equilibrium without assuming a homothetic one-period utility function. Let the stochastic event s_t take values on the unit interval $[0,1]$. There are two households, with $y_t^1(s^t) = s_t$ and $y_t^2(s^t) = 1 - s_t$. Note that the aggregate endowment is constant, $\sum_i y_t^i(s^t) = 1$. Then equation (8.5.7) implies that $c_t^1(s^t)$ is constant over time and across histories, and equation (8.5.6) implies that $c_t^2(s^t)$ is also constant. Thus, the equilibrium allocation satisfies $c_t^i(s^t) = \overline{c}^i$ for all t and s^t, for $i = 1,2$. Then from equation (8.5.4),

$$q_t^0\left(s^t\right) = \beta^t \pi_t\left(s^t\right)\frac{u'\left(\overline{c}^i\right)}{\mu_i}, \tag{8.6.3}$$

for all t and s^t, for $i = 1,2$. Household i's budget constraint implies

$$\frac{u'\left(\overline{c}^i\right)}{\mu_i}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\left[\overline{c}^i - y_t^i\left(s^t\right)\right] = 0.$$

Solving this equation for $\bar{c}^i$ gives

$$\bar{c}^i = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)y_t^i\left(s^t\right). \tag{8.6.4}$$

Summing equation (8.6.4) verifies that $\bar{c}^1 + \bar{c}^2 = 1$.[7]

8.6.4. Example 3: periodic endowment processes

Consider the special case of the previous example in which s_t is deterministic and alternates between the values 1 and 0; $s_0 = 1$, $s_t = 0$ for t odd, and $s_t = 1$ for t even. Thus, the endowment processes are perfectly predictable sequences $(1,0,1,\ldots)$ for the first agent and $(0,1,0,\ldots)$ for the second agent. Let $\tilde{s}^t$ be the history of $(1,0,1,\ldots)$ up to t. Evidently, $\pi_t(\tilde{s}^t) = 1$, and the probability assigned to all other histories up to t is zero. The equilibrium price system is then

$$q_t^0\left(s^t\right) = \begin{cases}\beta^t, & \text{if } s^t = \tilde{s}^t;\\ 0, & \text{otherwise;}\end{cases}$$

when using the time 0 good as numeraire, $q_0^0(\tilde{s}_0) = 1$. From equation (8.6.4), we have

$$\bar{c}^1 = (1-\beta)\sum_{j=0}^{\infty}\beta^{2j} = \frac{1}{1+\beta}, \tag{8.6.5a}$$

$$\bar{c}^2 = (1-\beta)\,\beta\sum_{j=0}^{\infty}\beta^{2j} = \frac{\beta}{1+\beta}. \tag{8.6.5b}$$

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[7] If we let $\beta^{-1} = 1+r$, where r is interpreted as the risk-free rate of interest, then note that (8.6.4) can be expressed as

$$\bar{c}^i = \left(\frac{r}{1+r}\right)E_0\sum_{t=0}^{\infty}(1+r)^{-t}\,y_t^i\left(s^t\right).$$

Hence, equation (8.6.4) is a version of Friedman's permanent income model, which asserts that a household with zero financial assets consumes the annuity value of its human wealth defined as the expected discounted value of its labor income (which for present purposes we take to be $y_t^i(s^t)$). In the present example, the household completely smooths its consumption across time and histories, something that the household in Friedman's model typically cannot do. See chapter 17.

Consumer 1 consumes more every period because he is richer by virtue of receiving his endowment earlier.

8.6.5. Example 4

In this example, we assume that the one-period utility function is $\frac{c^{1-\gamma}}{1-\gamma}$. There are two consumers named $i = 1,2$. Their endowments are $y_t^1 = y_t^2 = .5$ for $t = 0,1$ and $y_t^1 = s_t$ and $y_t^2 = 1 - s_t$ for $t \geq 2$. The state space $s_t = \{0,1\}$ and s_t is governed by a Markov chain with probability $\pi(s_0 = 1) = 1$ for the initial state and time-varying transition probabilities $\pi_1(s_1 = 1|s_0 = 1) = 1, \pi_2(s_2 = 1|s_1 = 1) = \pi_2(s_2 = 0|s_1 = 1) = .5, \pi_t(s_t = 1|s_{t-1} = 1) = 1, \pi_t(s_t = 0|s_{t-1} = 0) = 1$ for $t > 2$. This specification implies that $\pi_t(1,1,\ldots,1,1,1) = .5$ and $\pi_t(0,0,\ldots,0,1,1) = .5$ for all $t > 2$.

We can apply the method of subsection 8.6.2 to compute an equilibrium. The aggregate endowment is $\overline{y}_t(s^t) = 1$ for all t and all s^t. Therefore, an equilibrium price vector is $q_1^0(1,1) = \beta, q_2^0(0,1,1) = q_2^0(1,1,1) = .5\beta^2$ and $q_t^0(1,1,\ldots,1,1) = q_t^0(0,0,\ldots,1,1) = .5\beta^t$ for $t > 2$. Use these prices to compute the value of agent i's endowment: $\sum_t \sum_{s^t} q_t^0(s^t) y_t^i(s^t) = \sum_t \beta^t .5[.5 + .5 + 0 + \ldots + 0] + \sum_t \beta^t .5[.5 + .5 + 1 + \ldots + 1] = 2 \sum_t \beta^t .5[.5 + .5 + \ldots + .5] = .5 \sum_t \beta^t = \frac{.5}{1-\beta}$. Consumer i's budget constraint is satisfied when he consumes a constant consumption of .5 each period in each state: $c_t^i(s^t) = .5$ for all t for all s^t.

In subsection 8.9.4, we shall use the equilibrium allocation from the Arrow-Debreu economy in this example to synthesize an equilibrium in an economy with sequential trading.

8.7. Primer on asset pricing

Many asset-pricing models assume complete markets and price an asset by breaking it into a sequence of history-contingent claims, evaluating each component of that sequence with the relevant "state price deflator" $q_t^0(s^t)$, then adding up those values. The asset is *redundant*, in the sense that it offers a bundle of history-contingent dated claims, each component of which has already been priced by the market. While we shall devote chapters 13 and 14 entirely to asset-pricing theories, it is useful to give some pricing formulas at this point because they help illustrate the complete market competitive structure. ← why?

8.7.1. Pricing redundant assets

Let $\{d_t(s^t)\}_{t=0}^{\infty}$ be a stream of claims on time t, history s^t consumption, where $d_t(s^t)$ is a measurable function of s^t. The price of an asset entitling the owner to this stream must be

$$p_0^0\left(s_0\right) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0\left(s^t\right) d_t\left(s^t\right). \tag{8.7.1}$$

If this equation did not hold, someone could make unbounded profits by synthesizing this asset through purchases or sales of history-contingent dated commodities and then either buying or selling the asset. We shall elaborate this arbitrage argument below and later in chapter 13 on asset pricing.

8.7.2. Riskless consol

As an example, consider the price of a *riskless consol*, that is, an asset offering to pay one unit of consumption for sure each period. Then $d_t(s^t) = 1$ for all t and s^t, and the price of this asset is

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0\left(s^t\right). \tag{8.7.2}$$

8.7.3. Riskless strips

As another example, consider a sequence of *strips* of payoffs on the riskless consol. The time t strip is just the payoff process $d_\tau = 1$ if $\tau = t \geq 0$, and 0 otherwise. Thus, the owner of the strip is entitled to the time t coupon only. The value of the time t strip at time 0 is evidently

$$\sum_{s^t} q_t^0\left(s^t\right).$$

Compare this to the price of the consol (8.7.2). We can think of the t-period riskless strip as a t-period zero-coupon bond. See appendix B of chapter 14 for an account of a closely related model of yields on such bonds.

8.7.4. Tail assets

Return to the stream of dividends $\{d_t(s^t)\}_{t\geq 0}$ generated by the asset priced in equation (8.7.1). For $\tau \geq 1$, suppose that we strip off the first $\tau - 1$ periods of the dividend and want the time 0 value of the remaining dividend stream $\{d_t(s^t)\}_{t\geq\tau}$. Specifically, we seek the value of this asset for a particular possible realization of s^τ. Let $p_\tau^0(s^\tau)$ be the time 0 price of an asset that entitles the owner to dividend stream $\{d_t(s^t)\}_{t\geq\tau}$ if history s^τ is realized,

$$p_\tau^0\left(s^\tau\right) = \sum_{t\geq\tau}\sum_{s^t|s^\tau} q_t^0\left(s^t\right) d_t\left(s^t\right), \tag{8.7.3}$$

where the summation over $s^t|s^\tau$ means that we sum over all possible subsequent histories $\tilde{s}^t$ such that $\tilde{s}^\tau = s^\tau$. When the units of the price are time 0, state s_0 goods, the normalization is $q_0^0(s_0) = 1$. To convert the price into units of time τ, history s^τ consumption goods, divide by $q_\tau^0(s^\tau)$ to get

$$p_\tau^\tau\left(s^\tau\right) \equiv \frac{p_\tau^0\left(s^\tau\right)}{q_\tau^0\left(s^\tau\right)} = \sum_{t\geq\tau}\sum_{s^t|s^\tau} \frac{q_t^0\left(s^t\right)}{q_\tau^0\left(s^\tau\right)} d_t\left(s^t\right). \tag{8.7.4}$$

Notice that[8]

$$\begin{aligned} q_t^\tau\left(s^t\right) \equiv \frac{q_t^0\left(s^t\right)}{q_\tau^0\left(s^\tau\right)} &= \frac{\beta^t u'\left[c_t^i\left(s^t\right)\right]\pi_t\left(s^t\right)}{\beta^\tau u'\left[c_\tau^i\left(s^\tau\right)\right]\pi_\tau\left(s^\tau\right)} \\ &= \beta^{t-\tau}\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_\tau^i\left(s^\tau\right)\right]}\pi_t\left(s^t|s^\tau\right). \end{aligned} \tag{8.7.5}$$

[8] Because the marginal conditions hold for all consumers, this condition holds for all i.

Here $q_t^\tau(s^t)$ is the price of one unit of consumption delivered at time t, history s^t in terms of the date τ, history s^τ consumption good; $\pi_t(s^t|s^\tau)$ is the probability of history s^t conditional on history s^τ at date τ. Thus, the price at time τ, history s^τ for the "tail asset" is

$$p_\tau^\tau\left(s^\tau\right)=\sum_{t\geq\tau}\sum_{s^t|s^\tau}q_t^\tau\left(s^t\right)d_t\left(s^t\right). \tag{8.7.6}$$

When we want to create a time series of, say, equity prices, we use the "tail asset" pricing formula (8.7.6). An equity purchased at time τ entitles the owner to the dividends from time τ forward. Our formula (8.7.6) expresses the asset price in terms of prices with time τ, history s^τ good as numeraire.

8.7.5. One-period returns

The one-period version of equation (8.7.5) is

$$q_{\tau+1}^\tau\left(s^{\tau+1}\right)=\beta\frac{u'\left[c_{\tau+1}^i\left(s^{\tau+1}\right)\right]}{u'\left[c_\tau^i\left(s^\tau\right)\right]}\pi_{\tau+1}\left(s^{\tau+1}|s^\tau\right).$$

The right side is the one-period *pricing kernel* at time τ. If we want to find the price at time τ at history s^τ of a claim to a random payoff $\omega(s_{\tau+1})$, we use

$$p_\tau^\tau\left(s^\tau\right)=\sum_{s_{\tau+1}}q_{\tau+1}^\tau\left(s^{\tau+1}\right)\omega\left(s_{\tau+1}\right)$$

or

$$p_\tau^\tau\left(s^\tau\right)=E_\tau\left[\beta\frac{u'\left(c_{\tau+1}\right)}{u'\left(c_\tau\right)}\omega\left(s_{\tau+1}\right)\right], \tag{8.7.7}$$

where E_τ is the conditional expectation operator. We have deleted the i superscripts on consumption, with the understanding that equation (8.7.7) is true for any consumer i; we have also suppressed the dependence of c_τ on s^τ, which is implicit.

Let $R_{\tau+1}\equiv\omega(s_{\tau+1})/p_\tau^\tau(s^\tau)$ be the one-period gross *return* on the asset. Then for any asset, equation (8.7.7) implies

$$1=E_\tau\left[\beta\frac{u'\left(c_{\tau+1}\right)}{u'\left(c_\tau\right)}R_{\tau+1}\right]\equiv E_\tau\left[m_{\tau+1}R_{\tau+1}\right]. \tag{8.7.8}$$

The term $m_{\tau+1} \equiv \beta u'(c_{\tau+1})/u'(c_\tau)$ functions as a *stochastic discount factor.* Like $R_{\tau+1}$, it is a random variable measurable with respect to $s_{\tau+1}$, given s^τ. Equation (8.7.8) is a restriction on the conditional moments of returns and m_{t+1}. Applying the law of iterated expectations to equation (8.7.8) gives the unconditional moments restriction

$$1 = E\left[\beta \frac{u'(c_{\tau+1})}{u'(c_\tau)} R_{\tau+1}\right] \equiv E\left[m_{\tau+1} R_{\tau+1}\right]. \tag{8.7.9}$$

In chapters 13 and 14 we shall many more instances of this equation.

In the next section, we display another market structure in which the one-period pricing kernel $q^t_{t+1}(s^{t+1})$ also plays a decisive role. This structure uses the celebrated one-period "Arrow securities," the sequential trading of which substitutes perfectly for the comprehensive trading of long horizon claims at time 0.

8.8. Sequential trading: Arrow securities

This section describes an alternative market structure that preserves both the equilibrium allocation and the key one-period asset-pricing formula (8.7.7).

8.8.1. Arrow securities

We build on an insight of Arrow (1964) that one-period securities are enough to implement complete markets, provided that new one-period markets are re-opened for trading each period and provided that time t, history s^t wealth is properly assigned to each agent. Thus, at each date $t \geq 0$, but only at the history s^t actually realized, trades occur in a set of claims to one-period-ahead state-contingent consumption. We describe a competitive equilibrium of this sequential-trading economy. With a full array of these one-period-ahead claims, the sequential-trading arrangement attains the same allocation as the competitive equilibrium that we described earlier.

8.8.2. Financial wealth as an endogenous state variable

A key step in constructing a sequential-trading arrangement is to identify a variable to serve as the state in a value function for the household at date t. We find this state by taking an equilibrium allocation and price system for the (Arrow-Debreu) time 0 trading structure and applying a guess-and-verify method. We begin by asking the following question. In the competitive equilibrium where all trading takes place at time 0, what is the implied continuation wealth of household i at time t after history s^t, but before adding in its time t, history s^t endowment $y_t^i(s^t)$? To answer this question, in period t, conditional on history s^t, we sum up the value of the household's purchased claims to current and future goods net of its outstanding liabilities. Since history s^t has been realized, we discard all claims and liabilities contingent on time t histories $\tilde{s}^t \neq s^t$ that were not realized. Household i's net claim to delivery of goods in a future period $\tau \geq t$ contingent on history $\tilde{s}^\tau$ whose time t partial history $\tilde{s}^t = s^t$ is $[c_\tau^i(\tilde{s}^\tau) - y_\tau^i(\tilde{s}^\tau)]$. Thus, the household's financial wealth, or the value of all its current and future net claims, expressed in terms of the date t, history s^t consumption good is

$$\Upsilon_t^i\left(s^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t\left(s^\tau\right)\left[c_\tau^i\left(s^\tau\right) - y_\tau^i\left(s^\tau\right)\right]. \tag{8.8.1}$$

Notice that feasibility constraint $(8.2.2)$ at equality implies that

$$\sum_{i=1}^{I} \Upsilon_t^i\left(s^t\right) = 0, \qquad \forall t, s^t.$$

8.8.3. Financial and non-financial wealth

Define $\Upsilon_t^i(s^t)$ as *financial wealth* and $\sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau)$ as *non-financial wealth*.[9] In terms of these concepts, (8.8.1) implies

$$\Upsilon_t^i\left(s^t\right) + \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^t\left(s^\tau\right) y_\tau^i\left(s^\tau\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q_\tau^t\left(s^\tau\right) c_\tau^i\left(s^\tau\right), \tag{8.8.2}$$

which states that at each time and each history, the sum of financial and non-financial wealth equals the present value of current and future consumption claims. At time 0, we have set $\Upsilon_t^i(s^0) = 0$ for all i. At $t > 0$, financial wealth $\Upsilon_t^i(s^t)$ typically differs from zero for individual i, but it sums to zero across i.

8.8.4. Reopening markets

Formula (8.7.5) takes the form of a pricing function for a complete markets economy with date- and history-contingent commodities whose markets can be regarded as having been reopened at date τ, history s^τ, starting from wealth levels implied by the tails of each household's endowment and consumption streams for a complete markets economy that originally convened at $t = 0$. We leave it as an exercise to the reader to prove the following proposition.

PROPOSITION 3: Start from the distribution of time t, history s^t wealth that is implicit in a time 0 Arrow-Debreu equilibrium. If markets are reopened at date t after history s^t, no trades occur. That is, given the price system (8.7.5), all households choose to continue the tails of their original consumption plans.

[9] In some applications, financial wealth is also called 'non-human wealth' and non-financial wealth is called 'human wealth'.

8.8.5. Debt limits

In moving from the Arrow-Debreu economy to one with sequential trading, we propose to match the time t, history s^t wealth of the household in the sequential economy with the equilibrium tail wealth $\Upsilon_t^i(s^t)$ from the Arrow-Debreu economy computed in equation (8.8.2). But first we have to say something about debt limits, a feature that was only implicit in the time 0 budget constraint (8.5.1) in the Arrow-Debreu economy. In moving to the sequential formulation, we restrict asset trades to prevent Ponzi schemes. We want the weakest possible restrictions. We synthesize restrictions that work by starting from the equilibrium allocation of the Arrow-Debreu economy (with time 0 markets), and find some state-by-state debt limits that support the equilibrium allocation that emerged from the Arrow-Debreu economy under a sequential trading arrangement. Often we'll refer to these weakest possible debt limits as the "natural debt limits." These limits come from the common sense requirement that it has to be *feasible* for the consumer to repay his state contingent debt in every possible state. Together with our assumption that $c_t^i(s^t)$ must be nonnegative, that feasibility requirement leads to the natural debt limits.

Let $q_\tau^t(s^\tau)$ be the Arrow-Debreu price, denominated in units of the date t, history s^t consumption good. Consider the value of the tail of agent i's endowment sequence at time t in history s^t:

$$A_t^i\left(s^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t\left(s^\tau\right) y_\tau^i\left(s^\tau\right). \tag{8.8.3}$$

We call $A_t^i(s^t)$ the *natural debt limit* at time t and history s^t. It is the maximal value that agent i can repay starting from that period, assuming that his consumption is zero always. With sequential trading, we shall require that household i at time $t-1$ and history s^{t-1} cannot promise to pay more than $A_t^i(s^t)$ conditional on the realization of s_t tomorrow, because it will not be feasible to repay more. Household i at time $t-1$ faces one such borrowing constraint for each possible realization of s_t tomorrow.

8.8.6. Sequential trading

There is a sequence of markets in one-period-ahead state-contingent claims. At each date $t \geq 0$, households trade claims to date $t+1$ consumption, whose payment is contingent on the realization of s_{t+1}. Let $\tilde{a}_t^i(s^t)$ denote the claims to time t consumption, other than its time t endowment $y_t^i(s^t)$, that household i brings into time t in history s^t. Suppose that $\tilde{Q}_t(s_{t+1}|s^t)$ is a *pricing kernel* to be interpreted as follows: $\tilde{Q}_t(s_{t+1}|s^t)$ is the price of one unit of time $t+1$ consumption, contingent on the realization s_{t+1} at $t+1$, when the history at t is s^t. The household faces a sequence of budget constraints for $t \geq 0$, where the time t, history s^t budget constraint is

$$\tilde{c}_t^i\left(s^t\right) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i\left(s_{t+1}, s^t\right) \tilde{Q}_t\left(s_{t+1}|s^t\right) \leq y_t^i\left(s^t\right) + \tilde{a}_t^i\left(s^t\right). \tag{8.8.4}$$

At time t, a household chooses $\tilde{c}_t^i(s^t)$ and $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$, where $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$ is a vector of claims on time $t+1$ consumption, there being one element of the vector for each value of the time $t+1$ realization of s_{t+1}. To rule out Ponzi schemes, we impose the state-by-state borrowing constraints

$$-\tilde{a}_{t+1}^i\left(s^{t+1}\right) \leq A_{t+1}^i\left(s^{t+1}\right), \tag{8.8.5}$$

where $A_{t+1}^i(s^{t+1})$ is computed in equation (8.8.3).

Let $\eta_t^i(s^t)$ and $\nu_t^i(s^t; s_{t+1})$ be nonnegative Lagrange multipliers on the budget constraint (8.8.4) and the borrowing constraint (8.8.5), respectively, for time t and history s^t. Form the Lagrangian

$$\begin{aligned} L^i = \sum_{t=0}^{\infty} \sum_{s^t} \Big\{ & \beta^t u(\tilde{c}_t^i(s^t)) \pi_t(s^t) \\ & + \eta_t^i(s^t) \Big[y_t^i(s^t) + \tilde{a}_t^i(s^t) - \tilde{c}_t^i(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1}|s^t) \Big] \\ & + \sum_{s_{t+1}} \nu_t^i(s^t; s_{t+1}) \left[A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1}) \right] \Big\}, \end{aligned}$$

for a given initial wealth $\tilde{a}_0^i(s_0)$. The first-order conditions for maximizing L^i with respect to $\tilde{c}_t^i(s^t)$ and $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}$ are

$$\beta^t u'(\tilde{c}_t^i(s^t)) \pi_t(s^t) - \eta_t^i(s^t) = 0, \tag{8.8.6a}$$

$$-\eta_t^i(s^t) \tilde{Q}_t(s_{t+1}|s^t) + \nu_t^i(s^t; s_{t+1}) + \eta_{t+1}^i(s_{t+1}, s^t) = 0, \tag{8.8.6b}$$

for all s_{t+1}, t, s^t. In the optimal solution to this problem, the natural debt limit (8.8.5) will not be binding, and hence the Lagrange multipliers $\nu_t^i(s^t; s_{t+1})$ all equal zero for the following reason: if there were any history s^{t+1} leading to a binding natural debt limit, the household would from then on have to set consumption equal to zero in order to honor its debt. Because the household's utility function satisfies the Inada condition $\lim_{c \downarrow 0} u'(c) = +\infty$, that would mean that all future marginal utilities would be infinite. Thus, it would be easy to find alternative affordable allocations that yield higher expected utility by postponing earlier consumption to periods after such a binding constraint.

After setting $\nu_t^i(s^t; s_{t+1}) = 0$ in equation (8.8.6b), the first-order conditions imply the following restrictions on the optimally chosen consumption allocation,

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u'(\tilde{c}_{t+1}^i(s^{t+1}))}{u'(\tilde{c}_t^i(s^t))} \pi_t(s^{t+1}|s^t), \tag{8.8.7}$$

for all s_{t+1}, t, s^t.

Definition: A *distribution of wealth* is a vector $\vec{\tilde{a}}_t(s^t) = \{\tilde{a}_t^i(s^t)\}_{i=1}^I$ satisfying $\sum_i \tilde{a}_t^i(s^t) = 0$.

Definition: A *competitive equilibrium with sequential trading of one-period Arrow securities* is an initial distribution of wealth $\vec{\tilde{a}}_0(s_0)$, a collection of borrowing limits $\{A_t^i(s^t)\}$ satisfying (8.8.3) for all i, for all t, and for all s^t, a feasible allocation $\{\tilde{c}^i\}_{i=1}^I$, and pricing kernels $\tilde{Q}_t(s_{t+1}|s^t)$ such that
(a) for all i, given $\tilde{a}_0^i(s_0)$, the borrowing limits $\{A_t^i(s^t)$, and the pricing kernels, the consumption allocation $\tilde{c}^i$ solves the household's problem for all i;
(b) for all realizations of $\{s^t\}_{t=0}^\infty$, the households' consumption allocations and implied portfolios $\{\tilde{c}_t^i(s^t), \{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}\}_i$ satisfy $\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t)$ and $\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$ for all s_{t+1}.

This definition leaves open the initial distribution of wealth. We'll say more about the initial distribution of wealth soon.

8.8.7. Equivalence of allocations

By making an appropriate guess about the form of the pricing kernels, it is easy to show that a competitive equilibrium allocation of the complete markets model with time 0 trading is also an allocation for a competitive equilibrium with sequential trading of one-period Arrow securities, one with a particular initial distribution of wealth. Thus, take $q_t^0(s^t)$ as given from the Arrow-Debreu equilibrium and suppose that the pricing kernel $\tilde{Q}_t(s_{t+1}|s^t)$ makes the following recursion true:

$$q_{t+1}^0(s^{t+1}) = \tilde{Q}_t(s_{t+1}|s^t) q_t^0(s^t),$$

or

$$\tilde{Q}_t(s_{t+1}|s^t) = q_{t+1}^t(s^{t+1}), \tag{8.8.8}$$

where recall that $q_{t+1}^t(s^{t+1}) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$.

Let $\{c_t^i(s^t)\}$ be a competitive equilibrium allocation in the Arrow-Debreu economy. If equation (8.8.8) is satisfied, that allocation is also a sequential-trading competitive equilibrium allocation. To show this fact, take the household's first-order conditions (8.5.4) for the Arrow-Debreu economy from two successive periods and divide one by the other to get

$$\frac{\beta u'[c_{t+1}^i(s^{t+1})]\pi(s^{t+1}|s^t)}{u'[c_t^i(s^t)]} = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \tilde{Q}_t(s_{t+1}|s^t). \tag{8.8.9}$$

If the pricing kernel satisfies equation (8.8.8), this equation is equivalent with the first-order condition (8.8.7) for the sequential-trading competitive equilibrium economy. It remains for us to choose the initial wealth of the sequential-trading equilibrium so that the sequential-trading competitive equilibrium duplicates the Arrow-Debreu competitive equilibrium allocation.

We conjecture that the initial wealth vector $\vec{a}_0(s_0)$ of the sequential-trading economy should be chosen to be the zero vector. This is a natural conjecture, because it means that each household must rely on its own endowment stream to finance consumption, in the same way that households are constrained to finance their history-contingent purchases for the infinite future at time 0 in the Arrow-Debreu economy. To prove that the conjecture is correct, we must show that the zero initial wealth vector enables household i to finance $\{c_t^i(s^t)\}$ and leaves no room to increase consumption in any period after any history.

The proof proceeds by guessing that, at time $t \geq 0$ and history s^t, household i chooses a portfolio given by $\tilde{a}^i_{t+1}(s_{t+1}, s^t) = \Upsilon^i_{t+1}(s^{t+1})$ for all s_{t+1}. The value of this portfolio expressed in terms of the date t, history s^t consumption good is

$$\begin{aligned}\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) &= \sum_{s^{t+1}|s^t} \Upsilon^i_{t+1}(s^{t+1}) q^t_{t+1}(s^{t+1}) \\ &= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} q^t_\tau(s^\tau)\left[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)\right], \quad (8.8.10)\end{aligned}$$

where we have invoked expressions (8.8.2) and (8.8.8).[10] To demonstrate that household i can afford this portfolio strategy, we now use budget constraint (8.8.4) to compute the implied consumption plan $\{\tilde{c}^i_\tau(s^\tau)\}$. First, in the initial period $t = 0$ with $\tilde{a}^i_0(s_0) = 0$, the substitution of equation (8.8.10) into budget constraint (8.8.4) at equality yields

$$\tilde{c}^i_0(s_0) + \sum_{t=1}^{\infty} \sum_{s^t} q^0_t(s^t)\left[c^i_t(s^t) - y^i_t(s^t)\right] = y^i_t(s_0) + 0\,.$$

This expression together with budget constraint (8.5.1) at equality imply $\tilde{c}^i_0(s_0) = c^i_0(s_0)$. In other words, the proposed portfolio is affordable in period 0 and the associated consumption plan is the same as in the competitive equilibrium of the Arrow-Debreu economy. In all consecutive future periods $t > 0$ and histories s^t, we replace $\tilde{a}^i_t(s^t)$ in constraint (8.8.4) by $\Upsilon^i_t(s^t)$, and after noticing that the value of the asset portfolio in (8.8.10) can be written as

$$\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) = \Upsilon^i_t(s^t) - \left[c^i_t(s^t) - y^i_t(s^t)\right], \qquad (8.8.11)$$

it follows immediately from (8.8.4) that $\tilde{c}^i_t(s^t) = c^i_t(s^t)$ for all periods and histories.

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10 We have also used the following identities,

$$q^{t+1}_\tau(s^\tau) q^t_{t+1}(s^{t+1}) = \frac{q^0_\tau(s^\tau)}{q^0_{t+1}(s^{t+1})} \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = q^t_\tau(s^\tau) \text{ for } \tau > t.$$

We have shown that the proposed portfolio strategy attains the same consumption plan as in the competitive equilibrium of the Arrow-Debreu economy, but what precludes household i from further increasing current consumption by reducing some component of the asset portfolio? The answer lies in the debt limit restrictions to which the household must adhere. In particular, if the household wants to ensure that consumption plan $\{c^i_\tau(s^\tau)\}$ can be attained starting next period in all possible future states, the household should subtract the value of this commitment to future consumption from the natural debt limit in (8.8.3). Thus, the household is facing a state-by-state borrowing constraint that is more restrictive than restriction (8.8.5): for any s^{t+1},

$$-\tilde{a}^i_{t+1}(s^{t+1}) \leq A^i_{t+1}(s^{t+1}) - \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}_\tau(s^\tau) c^i_\tau(s^\tau) = -\Upsilon^i_{t+1}(s^{t+1}),$$

or

$$\tilde{a}^i_{t+1}(s^{t+1}) \geq \Upsilon^i_{t+1}(s^{t+1}).$$

Hence, household i does not want to increase consumption at time t by reducing next period's wealth below $\Upsilon^i_{t+1}(s^{t+1})$ because that would jeopardize attaining the preferred consumption plan that satisfies first-order conditions (8.8.7) for all future periods and histories.

8.9. Recursive competitive equilibrium

We have established that equilibrium allocations are the same in the Arrow-Debreu economy with complete markets in dated contingent claims all traded at time 0 and in a sequential-trading economy with a complete set of one-period Arrow securities. This finding holds for arbitrary individual endowment processes $\{y^i_t(s^t)\}_i$ that are measurable functions of the history of events s^t, which in turn are governed by some arbitrary probability measure $\pi_t(s^t)$. At this level of generality, the pricing kernels $\tilde{Q}_t(s_{t+1}|s^t)$ and the wealth distributions $\vec{\tilde{a}}_t(s^t)$ in the sequential-trading economy both depend on the history s^t, so both are time-varying functions of all past events $\{s_\tau\}^t_{\tau=0}$. This can make it difficult to formulate an economic model that can be used to confront empirical observations. We want a framework in which economic outcomes are functions of a limited number of "state variables" that summarize the effects of past events and current information. This leads us to make the following specialization of the exogenous forcing processes that facilitates a recursive formulation of the sequential-trading equilibrium.

8.9.1. Endowments governed by a Markov process

Let $\pi(s'|s)$ be a Markov chain with given initial distribution $\pi_0(s)$ and state space $s \in S$. That is, $\text{Prob}(s_{t+1} = s'|s_t = s) = \pi(s'|s)$ and $\text{Prob}(s_0 = s) = \pi_0(s)$. As we saw in chapter 2, the chain induces a sequence of probability measures $\pi_t(s^t)$ on histories s^t via the recursions

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\ldots\pi(s_1|s_0)\pi_0(s_0). \tag{8.9.1}$$

In this chapter, we have assumed that trading occurs after s_0 has been observed, which we capture by setting $\pi_0(s_0) = 1$ for the initially given value of s_0.

Because of the Markov property, the conditional probability $\pi_t(s^t|s^\tau)$ for $t > \tau$ depends only on the state s_τ at time τ and does not depend on the history before τ,

$$\pi_t(s^t|s^\tau) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\ldots\pi(s_{\tau+1}|s_\tau). \tag{8.9.2}$$

Next, we assume that households' endowments in period t are time invariant measurable functions of s_t, $y_t^i(s^t) = y^i(s_t)$ for each i. Of course, all of our previous results continue to hold, but the Markov assumption for s_t imparts further structure to the equilibrium.

8.9.2. Equilibrium outcomes inherit the Markov property

Proposition 2 asserted a particular kind of history independence of the equilibrium allocation that prevails under any stochastic process for the endowments. In particular, each individual's consumption is a function only of the current realization of the aggregate endowment and does not depend on the specific history leading to that outcome.[11] Now, under our present assumption that $y_t^i(s^t) = y^i(s_t)$ for each i, it follows immediately that

$$c_t^i(s^t) = \bar{c}^i(s_t). \tag{8.9.3}$$

Substituting (8.9.2) and (8.9.3) into (8.8.7) shows that the pricing kernel in the sequential-trading equilibrium is a function only of the current state,

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u'(\bar{c}^i(s_{t+1}))}{u'(\bar{c}^i(s_t))} \pi(s_{t+1}|s_t) \equiv Q(s_{t+1}|s_t). \tag{8.9.4}$$

After similar substitutions with respect to equation (8.7.5), we can also establish history independence of the relative prices in the Arrow-Debreu economy:

Proposition 4: If time t endowments are a function of a Markov state s_t, the Arrow-Debreu equilibrium price of date-$t \geq 0$, history s^t consumption goods expressed in terms of date τ $(0 \leq \tau \leq t)$, history s^τ consumption goods is not history dependent: $q_t^\tau(s^t) = q_k^j(\tilde{s}^k)$ for $j, k \geq 0$ such that $t - \tau = k - j$ and $[s_\tau, s_{\tau+1}, \ldots, s_t] = [\tilde{s}_j, \tilde{s}_{j+1}, \ldots, \tilde{s}_k]$.

Using this proposition, we can verify that both the natural debt limits (8.8.3) and households' wealth levels (8.8.2) exhibit history independence,

$$A_t^i(s^t) = \bar{A}^i(s_t)\,, \tag{8.9.5}$$

$$\Upsilon_t^i(s^t) = \bar{\Upsilon}^i(s_t)\,. \tag{8.9.6}$$

The finding concerning wealth levels (8.9.6) conveys a useful insight into how the sequential-trading competitive equilibrium attains the first-best outcome in which no idiosyncratic risk is borne by individual households. In particular, each household enters every period with a wealth level that is independent of past realizations of his endowment. That is, his past trades have fully insured

[11] Of course, the equilibrium allocation also depends on the distribution of $\{y_t^i(s^t)\}$ *processes* across agents i, as reflected in the relative values of the Lagrange multipliers μ_i.

him against the idiosyncratic outcomes of his endowment. And from that very same insurance motive, the household now enters the present period with a wealth level that is a function of the current state s_t. It is a state-contingent wealth level that was chosen by the household in the previous period $t-1$, and this wealth will be just sufficient to continue a trading strategy previously designed to insure against future idiosyncratic risks. The optimal holding of wealth is a function of s_t alone because the current state s_t determines the current endowment and the current pricing kernel and contains all information relevant for predicting future realizations of the household's endowment process as well as future prices. It can be shown that a household especially wants higher wealth levels for those states next period that either make his next period endowment low or more generally signal poor future prospects for its endowment into the more distant future. Of course, individuals' desires are tempered by differences in the economy's aggregate endowment across states (as reflected in equilibrium asset prices). Aggregate shocks cannot be diversified away but must be borne somehow by all of the households. The pricing kernel $Q(s_t|s_{t-1})$ and the assumed clearing of all markets set into action an "invisible hand" that coordinates households' transactions at time $t-1$ in such a way that only aggregate risk and no idiosyncratic risk is borne by the households.

8.9.3. Recursive formulation of optimization and equilibrium

The fact that the pricing kernel $Q(s'|s)$ and the endowment $y^i(s)$ are functions of a Markov process s motivates us to seek a recursive formulation of the household's optimization problem. Household i's state at time t is its wealth a_t^i and the current realization s_t. We seek a pair of optimal policy functions $h^i(a,s)$, $g^i(a,s,s')$ such that the household's optimal decisions are

$$c_t^i = h^i(a_t^i, s_t), \tag{8.9.7a}$$

$$a_{t+1}^i(s_{t+1}) = g^i(a_t^i, s_t, s_{t+1}). \tag{8.9.7b}$$

Let $v^i(a,s)$ be the optimal value of household i's problem starting from state (a,s); $v^i(a,s)$ is the maximum expected discounted utility household that household i with current wealth a can attain in state s. The Bellman equation for the household's problem is

$$v^i(a,s) = \max_{c,\hat{a}(s')} \left\{ u(c) + \beta \sum_{s'} v^i[\hat{a}(s'), s']\pi(s'|s) \right\} \tag{8.9.8}$$

where the maximization is subject to the following version of constraint (8.8.4):

$$c + \sum_{s'} \hat{a}(s')Q(s'|s) \leq y^i(s) + a \tag{8.9.9}$$

and also

$$c \geq 0, \tag{8.9.10a}$$

$$-\hat{a}(s') \leq \bar{A}^i(s'), \qquad \forall s'. \tag{8.9.10b}$$

Let the optimum decision rules be

$$c = h^i(a,s), \tag{8.9.11a}$$

$$\hat{a}(s') = g^i(a,s,s'). \tag{8.9.11b}$$

Note that the solution of the Bellman equation implicitly depends on $Q(\cdot|\cdot)$ because it appears in the constraint (8.9.9). In particular, use the first-order conditions for the problem on the right of equation (8.9.8) and the Benveniste-Scheinkman formula and rearrange to get

$$Q(s_{t+1}|s_t) = \frac{\beta u'(c^i_{t+1})\pi(s_{t+1}|s_t)}{u'(c^i_t)}, \tag{8.9.12}$$

where it is understood that $c^i_t = h^i(a^i_t, s_t)$ and $c^i_{t+1} = h^i(a^i_{t+1}(s_{t+1}), s_{t+1}) = h^i(g^i(a^i_t, s_t, s_{t+1}), s_{t+1})$.

DEFINITION: A *recursive competitive equilibrium* is an initial distribution of wealth $\vec{a}_0$, a set of borrowing limits $\{\bar{A}^i(s)\}_{i=1}^I$, a pricing kernel $Q(s'|s)$, sets of value functions $\{v^i(a,s)\}_{i=1}^I$, and decision rules $\{h^i(a,s), g^i(a,s,s')\}_{i=1}^I$ such that

(a) The state-by-state borrowing constraints satisfy the recursion

$$\bar{A}^i(s) = y^i(s) + \sum_{s'} Q(s'|s)\bar{A}^i(s'|s). \tag{8.9.13}$$

(b) For all i, given a_0^i, $\bar{A}^i(s)$, and the pricing kernel, the value functions and decision rules solve the household's problem;

(c) For all realizations of $\{s_t\}_{t=0}^\infty$, the consumption and asset portfolios $\{\{c_t^i, \{\hat{a}_{t+1}^i(s')\}_{s'}\}_i\}_t$ implied by the decision rules satisfy $\sum_i c_t^i = \sum_i y^i(s_t)$ and $\sum_i \hat{a}_{t+1}^i(s') = 0$ for all t and s'.

We shall use the recursive competitive equilibrium concept extensively in our discussion of asset pricing in chapter 13.

8.9.4. Computing an equilibrium with sequential trading of Arrow-securities

We use example 4 from subsection 8.6.5 to illustrate the following algorithm for computing an equilibrium in an economy with sequential trading of a complete set of Arrow securities:

1. Compute an equilibrium of the Arrow-Debreu economy with time 0 trading.
2. Set the equilibrium allocation for the sequential trading economy to the equilibrium allocation from Arrow-Debreu time 0 trading economy.
3. Compute equilibrium prices from formula (8.9.12) for a Markov economy or the corresponding formula (8.8.9) for a non-Markov economy.
4. Compute debt limits from (8.9.13).
5. Compute portfolios of one-period Arrow securities by first computing implied time t, history s^t wealth $\Upsilon_t^i(s^t)$ from (8.8.2) evaluated at the Arrow-Debreu equilibrium prices, then set $a_t^i(s_t) = \Upsilon_t^i(s^t)$.

Applying this procedure to example 4 from section 8.6.5 gives us the price system $Q_0(s_1 = 1|s_0 = 1) = \beta, Q_0(s_1 = 0|s_0 = 1) = 0, Q_1(s_2 = 1|s_1 = 1) = .5\beta, Q_1(s_2 = 0|s_1 = 0) = .5\beta$ and $Q_t(s_{t+1} = 1|s_t = 1) = Q_t(s_{t+1} = 0|s_t = 0) = \beta$ for $t \geq 2$. Also, $\Upsilon_t^i(s^t) = 0$ for $i = 1, 2$ and $t = 0, 1$. For $t \geq 2$, $\Upsilon_t^1(s_t = 1) = \sum_{\tau \geq t} \beta^{\tau-t}[.5 - 1] = \frac{-.5}{1-\beta}$ and $\Upsilon_t^2(s_t = 1) = \sum_{\tau \geq t} \beta^{\tau-t}[.5 - 0] = \frac{.5}{1-\beta}$. Therefore, in period 1, the first consumer trades Arrow securities in amounts $a_2^1(s_2 = 1) = \frac{-.5}{1-\beta}, a_2^1(s_2 = 0) = \frac{.5}{1-\beta}$, while the second consumer trades Arrow securities in amounts $a_2^1(s_2 = 1) = \frac{.5}{1-\beta}, a_2^1(s_2 = 0) = \frac{-.5}{1-\beta}$ After period 2, the consumers perpetually roll over their debts or assets of either $\frac{.5}{1-\beta}$ or $\frac{-.5}{1-\beta}$.

8.10. j-step pricing kernel

We are sometimes interested in the price at time t of a claim to one unit of consumption at date $\tau > t$ contingent on the time τ state being s_τ, *regardless* of the particular history by which s_τ is reached at τ. We let $Q_j(s'|s)$ denote the j-step pricing kernel to be interpreted as follows: $Q_j(s'|s)$ gives the price of one unit of consumption j periods ahead, contingent on the state in that future period being s', given that the current state is s. For example, $j = 1$ corresponds to the one-step pricing kernel $Q(s'|s)$.

With markets in all possible j-step-ahead contingent claims, the counterpart to constraint (8.8.4), the household's budget constraint at time t, is

$$c_t^i + \sum_{j=1}^{\infty} \sum_{s_{t+j}} Q_j(s_{t+j}|s_t) z_{t,j}^i(s_{t+j}) \leq y^i(s_t) + a_t^i. \tag{8.10.1}$$

Here $z_{t,j}^i(s_{t+j})$ is household i's holdings at the end of period t of contingent claims that pay one unit of the consumption good j periods ahead at date $t+j$, contingent on the state at date $t+j$ being s_{t+j}. The household's wealth in the next period depends on the chosen asset portfolio and the realization of s_{t+1},

$$a_{t+1}^i(s_{t+1}) = z_{t,1}^i(s_{t+1}) + \sum_{j=2}^{\infty} \sum_{s_{t+j}} Q_{j-1}(s_{t+j}|s_{t+1}) z_{t,j}^i(s_{t+j}).$$

The realization of s_{t+1} determines which element of the vector of one-period-ahead claims $\{z_{t,1}^i(s_{t+1})\}$ pays off at time $t+1$, and also the capital gains and losses inflicted on the holdings of longer horizon claims implied by equilibrium prices $Q_j(s_{t+j+1}|s_{t+1})$.

With respect to $z_{t,j}^i(s_{t+j})$ for $j > 1$, use the first-order condition for the problem on the right of (8.9.8) and the Benveniste-Scheinkman formula and rearrange to get

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} \frac{\beta u'[c_{t+1}^i(s_{t+1})]\pi(s_{t+1}|s_t)}{u'(c_t^i)} Q_{j-1}(s_{t+j}|s_{t+1}). \tag{8.10.2}$$

This expression, evaluated at the competitive equilibrium consumption allocation, characterizes two adjacent pricing kernels.[12] Together with first-order

[12] According to expression (8.9.3), the equilibrium consumption allocation is not history dependent, so that $(c_t^i, \{c_{t+1}^i(s_{t+1})\}_{s_{t+1}}) = (\bar{c}^i(s_t), \{\bar{c}^i(s_{t+1})\}_{s_{t+1}})$. Because marginal conditions hold for all households, the characterization of pricing kernels in (8.10.2) holds for any i.

condition (8.9.12), formula (8.10.2) implies that the kernels $Q_j, j = 2, 3, \ldots$, can be computed recursively:

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} Q_1(s_{t+1}|s_t) Q_{j-1}(s_{t+j}|s_{t+1}). \tag{8.10.3}$$

8.10.1. Arbitrage-free pricing

It is useful briefly to describe how arbitrage free pricing theory deduces restrictions on asset prices by manipulating budget sets with redundant assets. We now present an arbitrage argument as an alternative way of deriving restriction (8.10.3) that was established above by using households' first-order conditions evaluated at the equilibrium consumption allocation. In addition to j-step-ahead contingent claims, we illustrate the arbitrage pricing theory by augmenting the trading opportunities in our Arrow securities economy by letting the consumer also trade an ex-dividend Lucas tree. Because markets are already complete, these additional assets are redundant. They have to be priced in a way that leaves the budget set unaltered.[13]

Assume that at time t, in addition to purchasing a quantity $z_{t,j}(s_{t+j})$ of j-step-ahead claims paying one unit of consumption at time $t+j$ if the state takes value s_{t+j} at time $t+j$, the consumer also purchases N_t units of a stock or Lucas tree. Let the ex-dividend price of the tree at time t be $p(s_t)$. Next period, the tree pays a dividend $d(s_{t+1})$ depending on the state s_{t+1}. Ownership of the N_t units of the tree at the beginning of $t+1$ entitles the consumer to a claim on $N_t[p(s_{t+1}) + d(s_{t+1})]$ units of time $t+1$ consumption.[14] As before, let a_t be the wealth of the consumer, apart from his endowment, $y(s_t)$. In this setting, the augmented version of constraint (8.10.1), the consumer's budget constraint, is

$$c_t + \sum_{j=1}^{\infty} \sum_{s_{t+j}} Q_j(s_{t+j}|s_t) z_{t,j}(s_{t+j}) + p(s_t) N_t \leq a_t + y(s_t) \tag{8.10.4a}$$

[13] That the additional assets are redundant follows from the fact that trading Arrow securities is sufficient to complete markets.

[14] We calculate the price of this asset using a different method in chapter 13.

and

$$a_{t+1}(s_{t+1}) = z_{t,1}(s_{t+1}) + [p(s_{t+1}) + d(s_{t+1})]\, N_t + \sum_{j=2}^{\infty} \sum_{s_{t+j}} Q_{j-1}(s_{t+j}|s_{t+1}) z_{t,j}(s_{t+j}). \tag{8.10.4b}$$

Multiply equation $(8.10.4b)$ by $Q_1(s_{t+1}|s_t)$, sum over s_{t+1}, solve for $\sum_{s_{t+1}} Q_1(s_{t+1}|s_t) z_{t,1}(s_t)$, and substitute this expression in $(8.10.4a)$ to get

$$\begin{aligned} c_t + &\left\{ p(s_t) - \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)[p(s_{t+1}) + d(s_{t+1})] \right\} N_t \\ &+ \sum_{j=2}^{\infty} \sum_{s_{t+j}} \left\{ Q_j(s_{t+j}|s_t) - \sum_{s_{t+1}} Q_{j-1}(s_{t+j}|s_{t+1}) Q_1(s_{t+1}|s_t) \right\} z_{t,j}(s_{t+j}) \\ &+ \sum_{s_{t+1}} Q_1(s_{t+1}|s_t) a_{t+1}(s_{t+1}) \leq a_t + y(s_t). \end{aligned} \tag{8.10.5}$$

If the two terms in braces are not zero, the consumer can attain unbounded consumption and future wealth by purchasing or selling either the stock (if the first term in braces is not zero) or a state-contingent claim (if any of the terms in the second set of braces is not zero). Therefore, so long as the utility function has no satiation point, in any equilibrium, the terms in the braces must be zero. Thus, we have the arbitrage pricing formulas

$$p(s_t) = \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)[p(s_{t+1}) + d(s_{t+1})], \tag{8.10.6a}$$

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} Q_{j-1}(s_{t+j}|s_{t+1}) Q_1(s_{t+1}|s_t). \tag{8.10.6b}$$

These are called *arbitrage pricing formulas* because if they were violated, there would exist an *arbitrage*. An arbitrage is defined as a risk-free transaction that earns positive profits.

8.11. Recursive version of Pareto problem

At the very outset of this chapter, we characterized Pareto optimal allocations. This section considers how to formulate a Pareto problem recursively, which will give a preview of things to come in chapters 20 and 23. For this purpose, we consider a special case of the earlier section 8.6.3 example 2 of an economy with a constant aggregate endowment and two types of household with $y_t^1 = s_t, y_t^2 = 1 - s_t$. We now assume that the s_t process is i.i.d., so that $\pi_t(s^t) = \pi(s_t)\pi(s_{t-1})\cdots\pi(s_0)$. Also, let's assume that s_t has a discrete distribution so that $s_t \in [\overline{s}_1, \ldots, \overline{s}_S]$ with probabilities $\Pi_i = \text{Prob}(s_t = \overline{s}_i)$ where $\overline{s}_{i+1} > \overline{s}_i$ and $\overline{s}_1 \geq 0$ and $\overline{s}_S \leq 1$.

In our recursive formulation, each period a planner delivers a pair of previously promised discounted utility streams by assigning a state-contingent consumption allocation today and a pair of state-contingent promised discounted utility streams starting tomorrow. Both the state-contingent consumption today and the promised discounted utility tomorrow are functions of the initial promised discounted utility levels.

Define v as the expected discounted utility of a type 1 person and $P(v)$ as the maximal expected discounted utility that can be offered to a type 2 person, given that a type 1 person is offered at least v. Each of these expected values is to be evaluated before the realization of the state at the initial date.

The Pareto problem is to choose stochastic processes $\{c_t^1(s^t), c_t^2(s^t)\}_{t=0}^{\infty}$ to maximize $P(v)$ subject to the utility constraint $\sum_{t=0}^{\infty}\sum_{s^t}\beta^t u(c_t^1(s^t))\pi_t(s^t) \geq v$ and $c_t^1(s^t) + c_t^2(s^t) = 1$. In terms of the competitive equilibrium allocation calculated for the section 8.6.3 example 2 economy above, let $\overline{c} = \overline{c}^1$ be the constant consumption allocated to a type 1 person and $1 - \overline{c} = \overline{c}^2$ be the constant consumption allocated to a type 2 person. Since we have shown that the competitive equilibrium allocation is a Pareto optimal allocation, we already know one point on the Pareto frontier $P(v)$. In particular, when a type 1 person is promised $v = u(\overline{c})/(1-\beta)$, a type 2 person attains life-time utility $P(v) = u(1-\overline{c})/(1-\beta)$.

We can express the discounted values v and $P(v)$ recursively[15] as

$$v = \sum_{i=1}^{S} [u(c_i) + \beta w_i] \, \Pi_i$$

and

$$P(v) = \sum_{i=1}^{S} [u(1 - c_i) + \beta P(w_i)] \, \Pi_i,$$

where c_i is consumption of the type 1 person in state i, w_i is the continuation value assigned to the type 1 person in state i; and $1 - c_i$ and $P(w_i)$ are the consumption and the continuation value, respectively, assigned to a type 2 person in state i. Assume that the continuation values $w_i \in V$, where V is a set of admissible discounted values of utility. In this section, we assume that $V = [u(\epsilon)/(1-\beta), \, u(1)/(1-\beta)]$ where $\epsilon \in (0,1)$ is an arbitrarily small number.

In effect, before the realization of the current state, a Pareto optimal allocation offers the type 1 person a state-contingent vector of consumption c_i in state i and a state-contingent vector of continuation values w_i in state i, with each w_i itself being a present value of one-period future utilities. In terms of the pair of values $(v, P(v))$, we can express the Pareto problem recursively as

$$P(v) = \max_{\{c_i, w_i\}_{i=1}^S} \sum_{i=1}^{S} [u(1 - c_i) + \beta P(w_i)] \Pi_i \tag{8.11.1}$$

where the maximization is subject to

$$\sum_{i=1}^{S} [u(c_i) + \beta w_i] \Pi_i \geq v \tag{8.11.2}$$

where $c_i \in [0,1]$ and $w_i \in V$.

To solve the Pareto problem, form the Lagrangian

$$L = \sum_{i=1}^{S} \Pi_i [u(1 - c_i) + \beta P(w_i) + \theta(u(c_i) + \beta w_i)] - \theta v$$

---

15 This is our first example of a 'dynamic program squared'. We call it that because the state variable v that appears in the Bellman equation for $P(v)$ itself satisfies another Bellman equation.

where θ is a Lagrange multiplier on constraint (8.11.2). First-order conditions with respect to c_i and w_i, respectively, are

$$-u'(1-c_i) + \theta u'(c_i) = 0, \tag{8.11.3a}$$

$$P'(w_i) + \theta = 0. \tag{8.11.3b}$$

The envelope condition is $P'(v) = -\theta$. Thus, (8.11.3b) becomes $P'(w_i) = P'(v)$. But $P(v)$ happens to be strictly concave, so this equality implies $w_i = v$. Therefore, any solution of the Pareto problem leaves the continuation value w_i independent of the state i. Equation (8.11.3a) implies that

$$\frac{u'(1-c_i)}{u'(c_i)} = -P'(v). \tag{8.11.4}$$

Since the right side of (8.11.4) is independent of i, so is the left side, and therefore c is independent of i. And since v is constant over time (because $w_i = v$ for all i), it follows that c is constant over time.

Notice from (8.11.4) that $P'(v)$ serves as a relative Pareto weight on the type 1 person. The recursive formulation brings out that, because $P'(w_i) = P'(v)$, the relative Pareto weight remains constant over time and is independent of the realization of s_t. The planner imposes complete risk sharing.

In chapter 20, we shall encounter recursive formulations again. Impediments to risk sharing that occur in the form either of enforcement or of information constraints will impel the planner sometimes to make continuation values respond to the current realization of shocks to endowments or preferences.

8.12. Concluding remarks

The framework in this chapter serves much of macroeconomics either as foundation or straw man ("benchmark model" is a kinder phrase than "straw man"). It is the foundation of extensive literatures on asset pricing and risk sharing. We describe the literature on asset pricing in more detail in chapters 13 and 14. The model also serves as benchmark, or point of departure, for a variety of models designed to confront observations that seem inconsistent with complete markets. In particular, for models with exogenously imposed incomplete markets, see chapters 17 on precautionary saving and 18 on incomplete markets. For models with endogenous incomplete markets, see chapters 20 and 21 on enforcement and information problems. For models of money, see chapters 26 and 27. To take monetary theory as an example, complete markets models dispose of any need for money because they contain an efficient multilateral trading mechanism, with such extensive netting of claims that no medium of exchange is required to facilitate bilateral exchanges. Any modern model of money introduces frictions that impede complete markets. Some monetary models (e.g., the cash-in-advance model of Lucas, 1981) impose minimal impediments to complete markets, to preserve many of the asset-pricing implications of complete markets models while also allowing classical monetary doctrines like the quantity theory of money. The shopping time model of chapter 26 is constructed in a similar spirit. Other monetary models, such as the Townsend turnpike model of chapter 27 or the Kiyotaki-Wright search model of chapter 28, impose more extensive frictions on multilateral exchanges and leave the complete markets model farther behind. Before leaving the complete markets model, we'll put it to work in several of the following chapters.

A. Gaussian asset-pricing model

The theory of this chapter is readily adapted to a setting in which the state of the economy evolves according to a continuous-state Markov process. We use such a version in chapter 14. Here we give a taste of how such an adaptation can be made by describing an economy in which the state follows a linear stochastic difference equation driven by a Gaussian disturbance. If we supplement this with the specification that preferences are quadratic, we get a setting in which asset prices can be calculated swiftly.

Suppose that the state evolves according to the stochastic difference equation

$$s_{t+1} = As_t + Cw_{t+1} \tag{8.A.1}$$

where A is a matrix whose eigenvalues are bounded from above in modulus by $1/\sqrt{\beta}$ and w_{t+1} is a Gaussian martingale difference sequence adapted to the history of s_t. Assume that $Ew_{t+1}w_{t+1} = I$. The conditional density of s_{t+1} is Gaussian:

$$\pi(s_t|s_{t-1}) \sim \mathcal{N}(As_{t-1}, CC'). \tag{8.A.2}$$

More precisely,

$$\pi(s_t|s_{t-1}) = K \exp\left\{-.5(s_t - As_{t-1})(CC')^{-1}(s_t - As_{t-1})\right\}, \tag{8.A.3}$$

where $K = (2\pi)^{\frac{-k}{?2}} \det(CC')^{\frac{-1}{?2}}$ and s_t is $k \times 1$. We also assume that $\pi_0(s_0)$ is Gaussian.[16]

If $\{c_t^i(s_t)\}_{t=0}^{\infty}$ is the equilibrium allocation to agent i, and the agent has preferences represented by (8.2.1), the equilibrium pricing function satisfies

$$q_t^0(s^t) = \frac{\beta^t u'[c_t^i(s_t)]\pi(s^t)}{u'[c_0^i(s_0)]}. \tag{8.A.4}$$

Once again, let $\{d_t(s_t)\}_{t=0}^{\infty}$ be a stream of claims to consumption. The time 0 price of the asset with this dividend stream is

$$p_0 = \sum_{t=0}^{\infty} \int_{s^t} q_t^0(s^t) d_t(s_t) d\, s^t.$$

[16] If s_t is stationary, $\pi_0(s_0)$ can be specified to be the stationary distribution of the process.

Substituting equation (8.A.4) into the preceding equation gives

$$p_0 = \sum_t \int_{s^t} \beta^t \frac{u'[c_t^i(s_t)]}{u'[c_0^i(s_0)]} d_t(s_t) \pi(s^t) ds^t$$

or

$$p_0 = E \sum_{t=0}^{\infty} \beta^t \frac{u'[c_t(s_t)]}{u'[c_0(s_0)]} d_t(s_t). \tag{8.A.5}$$

This formula expresses the time 0 asset price as an inner product of a discounted marginal utility process and a dividend process.[17]

This formula becomes especially useful in the case that the one-period utility function $u(c)$ is quadratic, so that marginal utilities become linear, and the dividend process d_t is linear in s_t. In particular, assume that

$$u(c_t) = -.5(c_t - b)^2 \tag{8.A.6}$$

$$d_t = S_d s_t, \tag{8.A.7}$$

where $b > 0$ is a bliss level of consumption. Furthermore, assume that the equilibrium allocation to agent i is

$$c_t^i = S_{ci} s_t, \tag{8.A.8}$$

where S_{ci} is a vector conformable to s_t.

The utility function (8.A.6) implies that $u'(c_t^i) = b - c_t^i = b - S_{ci} s_t$. Suppose that unity is one element of the state space for s_t, so that we can express $b = S_b s_t$. Then $b - c_t = S_f s_t$, where $S_f = S_b - S_{ci}$, and the asset-pricing formula becomes

$$p_0 = \frac{E_0 \sum_{t=0}^{\infty} \beta^t s_t' S_f' S_d s_t}{S_f s_0}. \tag{8.A.9}$$

Thus, to price the asset, we have to evaluate the expectation of the sum of a discounted quadratic form in the state variable. This is easy to do by using results from chapter 2.

In chapter 2, we evaluated the conditional expectation of the geometric sum of the quadratic form

$$\alpha_0 = E_0 \sum_{t=0}^{\infty} \beta^t s_t' S_f' S_d s_t.$$

[17] For two scalar stochastic processes x, y, the inner product is defined as $< x, y >= E \sum_{t=0}^{\infty} \beta^t x_t y_t$.

We found that it could be written in the form

$$\alpha_0 = s_0' \mu s_0 + \sigma, \tag{8.A.10}$$

where μ is an $(n \times n)$ matrix and σ is a scalar that satisfy

$$\begin{aligned} \mu &= S_f' S_d + \beta A' \mu A \\ \sigma &= \beta \sigma + \beta \text{ trace } (\mu C C') \end{aligned} \tag{8.A.11}$$

The first equation of $(8.A.11)$ is a *discrete Lyapunov equation* in the square matrix μ, and can be solved by using one of several algorithms.[18] After μ has been computed, the second equation can be solved for the scalar σ.

B. The permanent income model revisited

This appendix is a variation on the theme that 'many single agent models can be reinterpreted as general equilibrium models'.

8.B.1. Reinterpreting the single-agent model

In this appendix, we cast the single-agent linear quadratic permanent income model of section 2.11 of chapter 2 as a competitive equilibrium with time 0 trading of a complete set of history-contingent securities. We begin by reformulating the model in that chapter as a planning problem. The planner has utility functional

$$E_0 \sum_{t=0}^{\infty} \beta^t u(\bar{c}_t) \tag{8.B.1}$$

where E_t is the mathematical expectation conditioned on the consumer's time t information, $\bar{c}_t$ is time t consumption, $u(c) = -.5(\gamma - \bar{c}_t)^2$, and $\beta \in (0,1)$ is a discount factor. The planner maximizes $(8.B.1)$ by choosing a consumption, borrowing plan $\{\bar{c}_t, b_{t+1}\}_{t=0}^{\infty}$ subject to the sequence of budget constraints

$$\bar{c}_t + b_t = R^{-1} b_{t+1} + y_t \tag{8.B.2}$$

[18] The Matlab control toolkit has a program called `dlyap.m`; also see a program called `doublej.m`.

where y_t is an exogenous stationary endowment process, R is a constant gross risk-free interest rate, $-R^{-1}b_t \equiv \bar{k}_t$ is the stock of an asset that bears a risk free one-period gross return of R, and b_0 is a given initial condition. We assume that $R^{-1} = \beta$ and that the endowment process has the state-space representation

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1} \tag{8.B.3a}$$

$$y_t = U_y z_t \tag{8.B.3b}$$

where w_{t+1} is an i.i.d. process with mean zero and identity contemporaneous covariance matrix, A_{22} is a stable matrix, its eigenvalues being strictly below unity in modulus, and U_y is a selection vector that identifies y with a particular linear combination of z_t. As shown in chapter 2, the solution of what we now interpret as a planning problem can be represented as the following versions of equations (2.11.9) and (2.11.20), respectively:

$$\bar{c}_t = (1-\beta)\left[U_y (I - \beta A_{22})^{-1} z_t - R\bar{k}_t\right] \tag{8.B.4}$$

$$\bar{k}_{t+1} = \bar{k}_t + R U_y (I - \beta A_{22})^{-1} (A_{22} - I) z_t. \tag{8.B.5}$$

We can represent the optimal consumption, capital accumulation path compactly as

$$\begin{bmatrix} \bar{k}_{t+1} \\ z_{t+1} \end{bmatrix} = A \begin{bmatrix} \bar{k}_t \\ z_t \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 \end{bmatrix} w_{t+1} \tag{8.B.6}$$

$$\bar{c}_t = S_c \begin{bmatrix} \bar{k}_t \\ z_t \end{bmatrix} \tag{8.B.7}$$

where the matrices A, S_c can readily be constructed from the solutions and specifications just mentioned. In addition, it is useful to have at our disposal the marginal utility of consumption process $p_t^0 \equiv (\gamma - \bar{c}_t)$, which can be represented as

$$p_t^0(z^t) = S_p \begin{bmatrix} \bar{k}_t \\ z_t \end{bmatrix} \tag{8.B.8}$$

and where S_p can be constructed easily from S_c. Solving equation (8.B.5) recursively shows that k_{t+1} is a function $k_{t+1}(z^t; k_0)$ of history z^t. In equation (8.B.8), $\bar{k}_t$ encodes the history dependence of $p_t^0(z^t)$.

Equations (8.B.6), (8.B.7), (8.B.8) together with the equation $r_t^0 = \alpha$ to be explained below turn out to be representations of the equilibrium price system in the competitive equilibrium to which we turn next.

8.B.2. Decentralization and scaled prices

Let $q_t^0(z^t)$ the time 0 price of a unit of time t consumption at history z^t. Let $\pi_t(z^t)$ the probability density of the history z^t induced by the state-space representation $(8.B.3)$. Define the adjusted Arrow-Debreu price scaled by discounting and probabilities as

$$p_t^0(z^t) = \frac{q_t^0(z^t)}{\beta^t \pi_t(z^t)}. \tag{8.B.9}$$

We find it convenient to express a representative consumer's problem and a representative firm's problem in terms of these scaled Arrow-Debreu prices.

Evidently, the present value of consumption, for example, can be represented as

$$\begin{aligned}\sum_{t=0}^{\infty}\sum_{z^t} q_t^0(z^t)c_t(z^t) &= \sum_{t=0}^{\infty}\sum_{z^t} \beta^t p_t^0(z^t)c_t(z^t)\pi_t(z^t) \\ &= E_0 \sum_{t=0}^{\infty} \beta^t p_t(z^t)c_t(z^t).\end{aligned}$$

Below, it will be convenient for us to represent present values as conditional expectations of discounted sums as is done in the second line.

We let $r_t^0(z^t)$ be the rental rate on capital, again scaled analogously to $(8.B.9)$. Both the consumer and the firm take these processes as given.

The consumer owns and operates the technology for accumulating capital. The consumer owns the endowment process $\{y_t\}_{t=0}^{\infty}$, which it sells to a firm that operates a production technology. The consumer rents capital to the firm. The firm uses the endowment and capital to produce output that it sells to the consumer at a competitive price. The consumer divides his time t purchases between consumption c_t and gross investment x_t.

8.B.2.1. The consumer

Let $\{p_t^0(z^t), r_t^0(z^t)\}_{t=0}^\infty$ be a price system, each component of which takes the form of a 'scaled Arrow-Debreu price' (attained by dividing a time-0 Arrow-Debreu price by a discount factor times a probability, as in the previous subsection). The representative consumer's problem is to choose processes $\{c_t, k_{t+1}\}_{t=0}^\infty$ to maximize

$$-.5 E_0 \sum_{t=0}^\infty \beta^t (\gamma - c_t)^2 \tag{8.B.10}$$

subject to

$$E_0 \sum_{t=0}^\infty \beta^t p_t^0(z^t) o_t(z^t) = E_0 \sum_{t=0}^\infty \beta^t \big(p_t^0(z^t) y_t + r_t^0(z^t) k_t(z^t)\big) \tag{8.B.11}$$

$$k_{t+1} = (1-\delta) k_t + x_t \tag{8.B.12}$$

$$o_t(z^t) = c_t(z^t) + x_t(z^t) \tag{8.B.13}$$

where k_0 is a given initial condition. Here x_t is gross investment and k_t is physical capital owned by the household and rented to firms. The consumer purchases output $o_t = c_t + x_t$ from competitive firms. The consumer sells its endowment y_t and rents its capital k_t to firms at prices $p_t^0(z^t)$ and $r_t^0(z^t)$. Equation $(8.B.12)$ is the law of motion for physical capital, where $\delta \in (0,1)$ is a depreciation rate.

8.B.2.2. The firm

A competitive representative firm chooses processes $\{k_t, c_t, x_t\}_{t=0}^\infty$ to maximize

$$E_0 \sum_{t=0}^\infty \beta^t \big\{p_t^0(z^t) o_t(z^t) - p_t^0(z^t) y_t - r_t^0(z^t) k_t\big\} \tag{8.B.14}$$

subject to the physical technology

$$o_t(z^t) = \alpha k_t + y_t(z_t), \tag{8.B.15}$$

where $\alpha > 0$. Since the marginal product of capital is α, a good guess is that

$$r_t^0(z^t) = \alpha. \tag{8.B.16}$$

8.B.3. Matching equilibrium and planning allocations

We impose the condition

$$\alpha + (1 - \delta) = R. \tag{8.B.17}$$

This makes the gross rates of return in investment identical in the planning and decentralized economies. In particular, if we substitute equation $(8.B.12)$ into equation $(8.B.15)$ and remember that $b_t \equiv Rk_t$, we obtain $(8.B.2)$.

It is straightforward to verify that the allocation $\{\bar{k}_{t+1}, \bar{c}_t\}_{t=0}^{\infty}$ that solves the planning problem is a competitive equilibrium allocation.

As in chapter 7, we have distinguished between the planning allocation $\{\bar{k}_{t+1}, \bar{c}_t\}_{t=0}^{\infty}$ that determines the equilibrium price functions defined in subsection 8.B.1 and the allocation chosen by the representative firm and the representative consumer who face those prices as price takers. This is yet another example of the 'big K, little k' device from chapter 7.

8.B.4. Interpretation

As we saw in section 2.11 of chapter 2 and also in representation $(8.B.4)$ $(8.B.5)$ here, what is now *equilibrium* consumption is a random walk. Why, despite his preference for a *smooth* consumption path, does the representative consumer accept fluctuations in his consumption? In the complete markets economy of this appendix, the consumer believes that it is possible for him completely to smooth consumption over time and across histories by purchasing and selling history contingent claims. But at the equilibrium prices facing him, the consumer prefers to tolerate fluctuations in consumption over time and across histories.

Exercises

Exercise 8.1 **Existence of representative consumer**

Suppose households 1 and 2 have one-period utility functions $u(c^1)$ and $w(c^2)$, respectively, where u and w are both increasing, strictly concave, twice differentiable functions of a scalar consumption rate. Consider the Pareto problem:

$$v_\theta(c) = \max_{\{c^1,c^2\}} \left[\theta u(c^1) + (1-\theta)w(c^2)\right]$$

subject to the constraint $c^1 + c^2 = c$. Show that the solution of this problem has the form of a concave utility function $v_\theta(c)$, which depends on the Pareto weight θ. Show that $v'_\theta(c) = \theta u'(c^1) = (1-\theta)w'(c^2)$.

The function $v_\theta(c)$ is the utility function of the *representative consumer.* Such a representative consumer always lurks within a complete markets competitive equilibrium even with heterogeneous preferences. At a competitive equilibrium, the marginal utilities of the representative agent and each and every agent are proportional.

Exercise 8.2 **Term structure of interest rates**

Consider an economy with a single consumer. There is one good in the economy, which arrives in the form of an exogenous endowment obeying[19]

$$y_{t+1} = \lambda_{t+1} y_t,$$

where y_t is the endowment at time t and $\{\lambda_{t+1}\}$ is governed by a two-state Markov chain with transition matrix

$$P = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix},$$

and initial distribution $\pi_\lambda = [\,\pi_0 \quad 1-\pi_0\,]$. The value of λ_t is given by $\bar\lambda_1 = .98$ in state 1 and $\bar\lambda_2 = 1.03$ in state 2. Assume that the history of y_s, λ_s up to t is observed at time t. The consumer has endowment process $\{y_t\}$ and has preferences over consumption streams that are ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

[19] Such a specification was made by Mehra and Prescott (1985).

where $\beta \in (0,1)$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma \geq 1$.

a. Define a competitive equilibrium, being careful to name all of the objects of which it consists.

b. Tell how to compute a competitive equilibrium.

For the remainder of this problem, suppose that $p_{11} = .8$, $p_{22} = .85$, $\pi_0 = .5$, $\beta = .96$, and $\gamma = 2$. Suppose that the economy begins with $\lambda_0 = .98$ and $y_0 = 1$.

c. Compute the (unconditional) average growth rate of consumption, computed before having observed λ_0.

d. Compute the time 0 prices of three risk-free discount bonds, in particular, those promising to pay one unit of time j consumption for $j = 0, 1, 2$, respectively.

e. Compute the time 0 prices of three bonds, in particular, ones promising to pay one unit of time j consumption contingent on $\lambda_j = \bar{\lambda}_1$ for $j = 0, 1, 2$, respectively.

f. Compute the time 0 prices of three bonds, in particular, ones promising to pay one unit of time j consumption contingent on $\lambda_j = \bar{\lambda}_2$ for $j = 0, 1, 2$, respectively.

g. Compare the prices that you computed in parts d, e, and f.

Exercise 8.3 An economy consists of two infinitely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer i consumes c_t^i at time t. Consumer i ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i),$$

where $\beta \in (0,1)$ and $u(c)$ is increasing, strictly concave, and twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_t^i = 1, 0, 0, 1, 0, 0, 1, \ldots$. Consumer 2 is endowed with a stream of the consumption good $0, 1, 1, 0, 1, 1, 0, \ldots$. Assume that there are complete markets with time 0 trading.

a. Define a competitive equilibrium.

b. Compute a competitive equilibrium.

c. Suppose that one of the consumers markets a derivative asset that promises to pay .05 units of consumption each period. What would the price of that asset be?

Exercise 8.4 Consider a pure endowment economy with a single representative consumer; $\{c_t, d_t\}_{t=0}^{\infty}$ are the consumption and endowment processes, respectively. Feasible allocations satisfy

$$c_t \leq d_t.$$

The endowment process is described by[20]

$$d_{t+1} = \lambda_{t+1} d_t.$$

The growth rate λ_{t+1} is described by a two-state Markov process with transition probabilities

$$P_{ij} = \text{Prob}(\lambda_{t+1} = \bar{\lambda}_j | \lambda_t = \bar{\lambda}_i).$$

Assume that

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix},$$

and that

$$\bar{\lambda} = \begin{bmatrix} .97 \\ 1.03 \end{bmatrix}.$$

In addition, $\lambda_0 = .97$ and $d_0 = 1$ are both known at date 0. The consumer has preferences over consumption ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

where E_0 is the mathematical expectation operator, conditioned on information known at time 0, $\gamma = 2, \beta = .95$.

Part I

At time 0, after d_0 and λ_0 are known, there are complete markets in date- and history-contingent claims. The market prices are denominated in units of time 0 consumption goods.

[20] See Mehra and Prescott (1985).

a. Define a competitive equilibrium, being careful to specify all the objects composing an equilibrium.

b. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, \ldots, \lambda_5) = (.97, .97, 1.03, .97, 1.03)$. Please give a numerical answer.

c. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, \ldots, \lambda_5) = (1.03, 1.03, 1.03, 1.03, .97)$.

d. Give a formula for the price at time 0 of a claim on the entire endowment sequence.

e. Give a formula for the price at time 0 of a claim on consumption in period 5, contingent on the growth rate λ_5 being .97 (regardless of the intervening growth rates).

Part II

Now assume a different market structure. Assume that at each date $t \geq 0$ there is a complete set of one-period forward Arrow securities.

f. Define a (recursive) competitive equilibrium with Arrow securities, being careful to define all of the objects that compose such an equilibrium.

g. For the representative consumer in this economy, for each state compute the "natural debt limits" that constrain state-contingent borrowing.

h. Compute a competitive equilibrium with Arrow securities. In particular, compute both the pricing kernel and the allocation.

i. An entrepreneur enters this economy and proposes to issue a new security each period, namely, a risk-free two-period bond. Such a bond issued in period t promises to pay one unit of consumption at time $t+1$ for sure. Find the price of this new security in period t, contingent on λ_t.

Exercise 8.5

An economy consists of two consumers, named $i = 1, 2$. The economy exists in discrete time for periods $t \geq 0$. There is one good in the economy, which is not storable and arrives in the form of an endowment stream owned by each consumer. The endowments to consumers $i = 1, 2$ are

$$
\begin{aligned}
y_t^1 &= s_t \\
y_t^2 &= 1
\end{aligned}
$$

where s_t is a random variable governed by a two-state Markov chain with values $s_t = \bar{s}_1 = 0$ or $s_t = \bar{s}_2 = 1$. The Markov chain has time invariant transition probabilities denoted by $\pi(s_{t+1} = s'|s_t = s) = \pi(s'|s)$, and the probability distribution over the initial state is $\pi_0(s)$. The *aggregate endowment* at t is $Y(s_t) = y_t^1 + y_t^2$.

Let c^i denote the stochastic process of consumption for agent i. Household i orders consumption streams according to

$$
U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)] \pi_t(s^t),
$$

where $\pi_t(s^t)$ is the probability of the history $s^t = (s_0, s_1, \ldots, s_t)$.

a. Give a formula for $\pi_t(s^t)$.

b. Let $\theta \in (0,1)$ be a Pareto weight on household 1. Consider the planning problem

$$
\max_{c^1, c^2} \left\{ \theta \ln(c^1) + (1-\theta) \ln(c^2) \right\}
$$

where the maximization is subject to

$$
c_t^1(s^t) + c_t^2(s^t) \leq Y(s_t).
$$

Solve the Pareto problem, taking θ as a parameter.

c. Define a *competitive equilibrium* with history-dependent Arrow-Debreu securities traded once and for all at time 0. Be careful to define all of the objects that compose a competitive equilibrium.

d. Compute the competitive equilibrium price system (i.e., find the prices of all of the Arrow-Debreu securities).

e. Tell the relationship between the solutions (indexed by θ) of the Pareto problem and the competitive equilibrium allocation. If you wish, refer to the two welfare theorems.

f. Briefly tell how you can compute the competitive equilibrium price system *before* you have figured out the competitive equilibrium allocation.

g. Now define a recursive competitive equilibrium with trading every period in one-period Arrow securities only. Describe all of the objects of which such an equilibrium is composed. (Please denominate the prices of one-period time $t+1$ state-contingent Arrow securities in units of time t consumption.) Define the "natural borrowing limits" for each consumer in each state. Tell how to compute these natural borrowing limits.

h. Tell how to compute the prices of one-period Arrow securities. How many prices are there (i.e., how many numbers do you have to compute)? Compute all of these prices in the special case that $\beta = .95$ and $\pi(s_j|s_i) = P_{ij}$ where $P = \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$.

i. Within the one-period Arrow securities economy, a new asset is introduced. One of the households decides to market a one-period-ahead riskless claim to one unit of consumption (a one-period real bill). Compute the equilibrium prices of this security when $s_t = 0$ and when $s_t = 1$. Justify your formula for these prices in terms of first principles.

j. Within the one-period Arrow securities equilibrium, a new asset is introduced. One of the households decides to market a two-period-ahead riskless claim to one unit of consumption (a two-period real bill). Compute the equilibrium prices of this security when $s_t = 0$ and when $s_t = 1$.

k. Within the one-period Arrow securities equilibrium, a new asset is introduced. One of the households decides at time t to market five-period-ahead claims to consumption at $t+5$ contingent on the value of s_{t+5}. Compute the equilibrium prices of these securities when $s_t = 0$ and $s_t = 1$ and $s_{t+5} = 0$ and $s_{t+5} = 1$.

Exercise 8.6 **Optimal taxation**

The government of a small country must finance an exogenous stream of government purchases $\{g_t\}_{t=0}^{\infty}$. Assume that g_t is described by a discrete-state Markov chain with transition matrix P and initial distribution π_0. Let $\pi_t(g^t)$ denote the probability of the history $g^t = g_t, g_{t-1}, \ldots, g_0$, conditioned on g_0. The state of the economy is completely described by the history g^t. There are complete markets in date-history claims to goods. At time 0, after g_0 has been realized, the government can purchase or sell claims to time t goods contingent on the history g^t at a price $p_t^0(g^t) = \beta^t \pi_t(g^t)$, where $\beta \in (0,1)$. The date-state prices are exogenous to the small country. The government finances its expenditures by raising history-contingent tax revenues of $R_t = R_t(g^t)$ at time t. The present value of its expenditures must not exceed the present value of its revenues.

Raising revenues by taxation is distorting. The government confronts a dead weight loss function $W(R_t)$ that measures the distortion at time t. Assume that W is an increasing, twice differentiable, strictly convex function that satisfies $W(0) = 0, W'(0) = 0, W'(R) > 0$ for $R > 0$ and $W''(R) > 0$ for $R \geq 0$. The government devises a state-contingent taxation and borrowing plan to minimize

$$E_0 \sum_{t=0}^{\infty} \beta^t W(R_t), \tag{1}$$

where E_0 is the mathematical expectation conditioned on g_0.

Suppose that g_t takes two possible values, $\bar{g}_1 = .2$ (peace) and $\bar{g}_2 = 1$ (war) and that $P = \begin{bmatrix} .8 & .2 \\ .5 & .5 \end{bmatrix}$. Suppose that $g_0 = .2$. Finally, suppose that $W(R) = .5R^2$.

a. Please write out (1) long hand, i.e., write out an explicit expression for the mathematical expectation E_0 in terms of a summation over the appropriate probability distribution.

b. Compute the optimal tax and borrowing plan. In particular, give analytic expressions for $R_t = R_t(g^t)$ for all t and all g^t.

c. There is an equivalent market setting in which the government can buy and sell one-period Arrow securities each period. Find the price of one-period Arrow securities at time t, denominated in units of the time t good.

d. Let $B_t(g_t)$ be the one-period Arrow securities at t that the government issued for state g_t at time $t-1$. For $t > 0$, compute $B_t(g_t)$ for $g_t = \bar{g}_1$ and $g_t = \bar{g}_2$.

e. Use your answers to parts b and d to describe the government's optimal policy for taxing and borrowing.

Exercise 8.7 **A competitive equilibrium**

An endowment economy consists of two type of consumers. Consumers of type 1 order consumption streams of the one good according to

$$\sum_{t=0}^{\infty} \beta^t c_t^1$$

and consumers of type 2 order consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t^2)$$

where $c_t^i \geq 0$ is the consumption of a type i consumer and $\beta \in (0,1)$ is a common discount factor. The consumption good is tradable but nonstorable. There are equal numbers of the two types of consumer. The consumer of type 1 is endowed with the consumption sequence

$$y_t^1 = \mu > 0 \quad \forall t \geq 0$$

where $\mu > 0$. The consumer of type 2 is endowed with the consumption sequence

$$y_t^2 = \begin{cases} 0 & \text{if } t \geq 0 \text{ is even} \\ \alpha & \text{if } t \geq 0 \text{ is odd} \end{cases}$$

where $\alpha = \mu(1 + \beta^{-1})$.

a. Define a competitive equilibrium with time 0 trading. Be careful to include definitions of all of the objects of which a competitive equilibrium is composed.

b. Compute a competitive equilibrium allocation with time 0 trading.

c. Compute the time 0 wealths of the two types of consumers using the competitive equilibrium prices.

d. Define a competitive equilibrium with sequential trading of Arrow securities.

e. Compute a competitive equilibrium with sequential trading of Arrow securities.

Exercise 8.8 **Corners**

A pure endowment economy consists of two type of consumers. Consumers of type 1 order consumption streams of the one good according to

$$\sum_{t=0}^{\infty} \beta^t c_t^1$$

and consumers of type 2 order consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t^2)$$

where $c_t^i \geq 0$ is the consumption of a type i consumer and $\beta \in (0,1)$ is a common discount factor. Please note the nonnegativity constraint on consumption of each person (the force of this is that c_t^i is *consumption*, not *production*). The consumption good is tradable but nonstorable. There are equal numbers of the two types of consumer. The consumer of type 1 is endowed with the consumption sequence

$$y_t^1 = \mu > 0 \quad \forall t \geq 0$$

where $\mu > 0$. The consumer of type 2 is endowed with the consumption sequence

$$y_t^2 = \begin{cases} 0 & \text{if } t \geq 0 \text{ is even} \\ \alpha & \text{if } t \geq 0 \text{ is odd} \end{cases}$$

where

$$\alpha = \mu(1 + \beta^{-1}). \tag{1}$$

a. Define a competitive equilibrium with time 0 trading. Be careful to include definitions of all of the objects of which a competitive equilibrium is composed.

b. Compute a competitive equilibrium allocation with time 0 trading. Compute the equilibrium price system. Please also compute the sequence of one-period gross interest rates. Do they differ between odd and even periods?

c. Compute the time 0 wealths of the two types of consumers using the competitive equilibrium prices.

d. Now consider an economy identical to the preceding one except in one respect. The endowment of consumer 1 continues to be 1 each period, but we assume that the endowment of consumer 2 is larger (though it continues to be zero in every even period). In particular, we alter the assumption about endowments in condition (1) to the new condition

$$\alpha > \mu(1+\beta^{-1}).$$

Compute the competitive equilibrium allocation and price system for this economy.

e. Compute the sequence of one-period interest rates implicit in the equilibrium price system that you computed in part d. Are interest rates higher or lower than those you computed in part b?

Exercise 8.9 **Equivalent martingale measure**

Let $\{d_t(s_t)\}_{t=0}^{\infty}$ be a stream of payouts. Suppose that there are complete markets. From (8.5.4) and (8.7.1), the price at time 0 of a claim on this stream of dividends is

$$a_0 = \sum_{t=0} \sum_{s^t} \beta^t \frac{u'(c_t^i(s^t))}{\mu_i} \pi_t(s^t) d_t(s_t).$$

Show that this a_0 can also be represented as

$$\begin{aligned} a_0 &= \sum_t b_t \sum_{s^t} d_t(s_t) \tilde{\pi}_t(s^t) \\ &= \tilde{E}_0 \sum_{t=0}^{\infty} b_t d_t(s_t) \end{aligned} \tag{1}$$

where $\tilde{E}$ is the mathematical expectation with respect to the twisted measure $\tilde{\pi}_t(s^t)$ defined by

$$\begin{aligned} \tilde{\pi}_t(s^t) &= b_t^{-1} \beta^t \frac{u'(c_t^i(s^t))}{\mu_i} \pi_t(s^t) \\ b_t &= \sum_{s^t} \beta^t \frac{u'(c_t^i(s^t))}{\mu_i} \pi_t(s^t). \end{aligned}$$

Prove that $\tilde{\pi}_t(s^t)$ is a probability measure. Interpret b_t itself as a price of particular asset. Note: $\tilde{\pi}_t(s^t)$ is called an *equivalent martingale measure*. See chapters 13 and 14.

Exercise 8.10 **Harrison-Kreps prices**

Show that the asset price in (1) of the previous exercise can also be represented as

$$\begin{aligned} a_0 &= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t p_t^0(s^t) d_t(s^t) \pi_t(s^t) \\ &= E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 d_t \end{aligned}$$

where $p_t^0(s^t) = q_t^0(s^t)/[\beta^t \pi_t(s^t)]$.

Exercise 8.11 **Early resolution of uncertainty**

An economy consists of two households named $i = 1, 2$. Each household evaluates streams of a single consumption good according to $\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t)$. Here $u(c)$ is an increasing, twice continuously differentiable, strictly concave function of consumption c of one good. The utility function satisfies the Inada condition $\lim_{c \downarrow 0} u'(c) = +\infty$. A feasible allocation satisfies $\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t)$. The households' endowments of the one nonstorable good are both functions of a state variable $s_t \in \mathbf{S} = \{0, 1, 2\}$; s_t is described by a time invariant Markov chain with initial distribution $\pi_0 = [\,0 \quad 1 \quad 0\,]'$ and transition density defined by the stochastic matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .5 & 0 & .5 \\ 0 & 0 & 1 \end{bmatrix}.$$

The endowments of the two households are

$$\begin{aligned} y_t^1 &= s_t/2 \\ y_t^2 &= 1 - s_t/2. \end{aligned}$$

a. Define a competitive equilibrium with Arrow securities.

b. Compute a competitive equilibrium with Arrow securities.

c. By hand, simulate the economy. In particular, for every possible realization of the histories s^t, describe time series of c_t^1, c_t^2 and the wealth levels a_t^i of the households. (Note: Usually this would be an impossible task by hand, but this problem has been set up to make the task manageable.)

Exercise 8.12 donated by Pierre-Olivier Weill

An economy is populated by a continuum of infinitely lived consumers of types $j \in \{0,1\}$, with a measure one of each. There is one nonstorable consumption good arriving in the form of an endowment stream owned by each consumer. Specifically, the endowments are

$$y_t^0(s_t) = (1-s_t)\bar{y}^0$$
$$y_t^1(s_t) = s_t\bar{y}^1,$$

where s_t is a two-state time-invariant Markov chain valued in $\{0,1\}$ and $\bar{y}^0 < \bar{y}^1$. The initial state is $s_0 = 1$. Transition probabilities are denoted $\pi(s'|s)$ for $(s,s') \in \{0,1\}^2$, where $'$ denotes a next period value. The aggregate endowment is $y_t(s_t) \equiv (1-s_t)\bar{y}^0 + s_t\bar{y}^1$. Thus, this economy fluctuates stochastically between recessions $y_t(0) = \bar{y}^0$ and booms $y_t(1) = \bar{y}^0$. In a recession, the aggregate endowment is owned by type 0 consumers, while in a boom it is owned by a type 1 consumers. A consumer orders consumption streams according to:

$$U(c^j) = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t|s_0)\frac{c_t^j(s^t)^{1-\gamma}}{1-\gamma},$$

where $s^t = (s_t, s_{t-1}, \ldots, s_0)$ is the history of the state up to time t, $\beta \in (0,1)$ is the discount factor, and $\gamma > 0$ is the coefficient of relative risk aversion.

a. Define a competitive equilibrium with time 0 trading. Compute the price system $\{q_t^0(s^t)\}_{t=0}^{\infty}$ and the equilibrium allocation $\{c^j(s^t)\}_{t=0}^{\infty}$, for $j \in \{0,1\}$.

b. Find a utility function $\bar{U}(c) = E_0\left(\sum_{t=0}^{\infty}\beta^t u(c_t)\right)$ such that the price system $q_t^0(s^t)$ and the aggregate endowment $y_t(s_t)$ is an equilibrium allocation of the single-agent economy $\left(\bar{U}, \{y_t(s_t)\}_{t=0}^{\infty}\right)$. How does your answer depend on the initial distribution of endowments $y_t^j(s_t)$ among the two types $j \in \{0,1\}$? How would you defend the representative agent assumption in this economy?

c. Describe the equilibrium allocation under the following three market structures: (i) at each node s^t, agents can trade only claims on their entire endowment streams; (ii) at each node s^t, there is a complete set of one-period ahead Arrow securities; and (iii) at each node s^t, agents can only trade two risk-free assets, namely, a one-period zero-coupon bond that pays one unit of consumption for sure at $t+1$ and a two-period zero-coupon bond that pays one unit of the consumption good for sure at $t+2$. How would you modify your answer in the absence of aggregate uncertainty?

d. Assume that $\pi(1|0) = 1$, $\pi(0|1) = 1$, and as before $s_0 = 1$. Compute the allocation in an equilibrium with time 0 trading. Does the type $j = 1$ agent always consume the largest share of the aggregate endowment? How does it depend on parameter values? Provide economic intuition for your results.

e. Assume that $\pi(1|0) = 1$ and $\pi(0|1) = 1$. Remember that $s_0 = 1$. Assume that at $t = 1$ agent $j = 0$ is given the option to default on her financial obligation. For example, in the time 0 trading economy, these obligations are deliveries of goods. Upon default, it is assumed that the agent is excluded from the market and has to consume her endowment forever. Will the agent ever exercise her option to default?

Exercise 8.13 **Diverse beliefs, I**

A pure endowment economy is populated by two consumers. Consumer i has preferences over history-contingent consumption sequences $\{c_t^i(s^t)\}$ that are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t^i(s^t),$$

where $u(c) = \ln(c)$ and where $\pi_t^i(s^t)$ is a density that consumer i assigns to history s^t. The state space is time invariant. In particular, $s_t \in S = \{0, .5, 1\}$ for all $t \geq 0$. Only two histories are possible for $t = 0, 1, 2, \ldots$:

$$\begin{aligned} &\text{history } 1 : .5, 1, 1, 1, 1, \ldots \\ &\text{history } 2 : .5, 0, 0, 0, 0, \ldots \end{aligned}$$

Consumer 1 assigns probability 1/3 to history 1 and probability 2/3 to history 2, while consumer 2 assigns probability 2/3 to history 1 and probability 1/3 to history 2. Nature assigns equal probabilities to the two histories. The endowments of the two consumers are:

$$\begin{aligned} y_t^1 &= s_t \\ y_t^2 &= 1 - s_t. \end{aligned}$$

a. Define a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities.

b. Compute a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities.

c. Is the equilibrium allocation Pareto optimal?

Exercise 8.14 **Diverse beliefs, II**

Consider the following I person pure endowment economy. There is a state variable $s_t \in S$ for all $t \geq 0$. Let s^t denote a history of s from 0 to t. The time t aggregate endowment is a function of the history, so $Y_t = Y_t(s^t)$. Agent i attaches a personal probability of $\pi_t^i(s^t)$ to history s^t. The history s^t is observed by all I people at time t. Assume that for all i, $\pi_t^i(s_t) > 0$ if and only if $\pi_t^1(s_t) > 0$ (so the consumers agree about which histories have positive probability). Consumer i ranks consumption plans $c_t^i(s^t)$ that are measurable functions of histories via the expected utility functional

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln(c_t^i(s^t)) \pi_t^i(s^t) \tag{1}$$

The ownership structure of the economy is not yet determined.

A planner puts positive Pareto weights $\lambda_i > 0$ on consumers $i = 1, \ldots, I$ and solves a time 0 Pareto problem that respects each consumer's preferences as represented by (1).

a. Show how to solve for a Pareto optimal allocation. Display an expression for $c_t^i(s^t)$ as a function of $Y_t(s^t)$ and other pertinent variables.

b. Under what circumstances does the Pareto plan imply complete risk-sharing among the I consumers?

c. Under what circumstances does the Pareto plan imply an allocation that is not history dependent? By 'not history dependent', we mean that $Y_t(s^t) = Y_t(\tilde{s}^t)$ would imply the same allocation at time t?

d. For a given set of Pareto weights, find an associated equilibrium price vector and an initial distribution of wealth among the I consumers that makes the Pareto allocation be the allocation associated with a competitive equilibrium with time 0 trading of history-contingent claims on consumption.

e. Find a formula for the equilibrium price vector in terms of equilibrium quantities and the beliefs of consumers.

f. Suppose that $I = 2$. Show that as $\lambda_2/\lambda_1 \to +\infty$, the planner would distribute initial wealth in a way that makes consumer 2's beliefs more and more influential in determining equilibrium prices.

Exercise 8.15 **Diverse beliefs, III**

An economy consists of two consumers named $i = 1, 2$. Each consumer evaluates streams of a single nonstorable consumption good according to

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)] \pi_t^i(s^t).$$

Here $\pi_t^i(s^t)$ is consumer i's subjective probability over history s^t. A feasible allocation satisfies $\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t)$ for all $t \geq 0$ and for all s^t. The consumers' endowments of the one good are functions of a state variable $s_t \in \mathbf{S} = \{0, 1, 2\}$. In truth, s_t is described by a time invariant Markov chain with initial distribution $\pi_0 = [\,0 \quad 1 \quad 0\,]'$ and transition density defined by the stochastic matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .5 & 0 & .5 \\ 0 & 0 & 1 \end{bmatrix}$$

where $P_{ij} = \text{Prob}[s_{t+1} = j-1 | s_t = i-1]$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. The endowments of the two consumers are

$$y_t^1 = s_t/2$$
$$y_t^2 = 1 - s_t/2.$$

In part I, both consumers know the true probabilities over histories s^t (i.e., they know both π_0 and P). In part II, the two consumers have different subjective probabilities.

Part I:

Assume that both consumers know (π_0, P), so that $\pi_t^1(s^t) = \pi_t^2(s^t)$ for all $t \geq 0$ for all s^t.

a. Show how to deduce $\pi_t^i(s^t)$ from (π_0, P).

b. Define a competitive equilibrium with sequential trading of Arrow securities.

c. Compute a competitive equilibrium with sequential trading of Arrow securities.

d. By hand, simulate the economy. In particular, for every possible realization of the histories s^t, describe time series of c_t^1, c_t^2 and the wealth levels for the two consumers.

Part II:

Now assume that while consumer 1 knows (π_0, P), consumer 2 knows π_0 but thinks that P is

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 \\ .4 & 0 & .6 \\ 0 & 0 & 1 \end{bmatrix}.$$

e. Deduce $\pi_t^2(s^t)$ from $(\pi_0, \hat{P})$ for all $t \geq 0$ for all s^t.

f. Formulate and solve a Pareto problem for this economy.

g. Define an equilibrium with time 0 trading of a complete set of Arrow-Debreu history-contingent securities.

h. Compute an equilibrium with time 0 trading of a complete set of Arrow-Debreu history-contingent securities.

i. Compute an equilibrium with sequential trading of Arrow securities. For every possible realization of s^t for all $t \geq 0$, please describe time series of c_t^1, c_t^2 and the wealth levels for the two consumers.

Exercise 8.16 **Diverse beliefs, IV**

A pure exchange economy is populated by two consumers. Consumer i has preferences over history-contingent consumption sequences $\{c_t^i(s^t)\}$ that are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t^i(s^t),$$

where $u(c) = \ln(c)$, $\beta \in (0,1)$, and $\pi_t^i(s^t)$ is a density that consumer i assigns to history s^t. The state space is time invariant. In particular, $s_t \in S = \{0, .5, 1\}$ for all $t \geq 0$. Only two histories are possible for $t = 0, 1, 2, \ldots$:

$$\text{history } 1 : .5, 1, 1, 1, 1, \ldots$$

$$\text{history } 2 : .5, 0, 0, 0, 0, \ldots$$

Consumer 1 assigns probability 1 to history 1 and probability 0 to history 2, while consumer 2 assigns probability 0 to history 1 and probability 1 to history

2. Nature assigns equal probabilities to the two histories. The endowments of the two consumers are:

$$\begin{aligned} y_t^1 &= s_t \\ y_t^2 &= 1 - s_t. \end{aligned}$$

a. Formulate and solve a Pareto problem for this economy.

b. Define a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities.

c. Does a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities exist for this economy? If it does, compute it. If it does not, explain why.

Exercise 8.17 **Diverse beliefs, V**

A pure exchange economy is populated by two consumers. Consumer i has preferences over history-contingent consumption sequences $\{c_t^i(s^t)\}$ that are ordered by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t^i(s^t),$$

where $u(c) = \ln(c)$, $\beta \in (0,1)$, and $\pi_t^i(s^t)$ is a density that consumer i assigns to history s^t. The state space is time invariant. In particular, $s_t \in S = \{0, .5, 1\}$ for all $t \geq 0$. Only two histories are possible for $t = 0, 1, 2, \ldots$:

$$\begin{aligned} &\text{history } 1 : .5, 1, 1, 1, 1, \ldots \\ &\text{history } 2 : .5, 0, 0, 0, 0, \ldots \end{aligned}$$

Consumer 1 assigns probability 1 to history 1 and probability 0 to history 2, while consumer 2 assigns probability 0 to history 1 and probability 1 to history 2. Nature assigns equal probabilities to the two histories. The endowments of the two consumers are:

$$\begin{aligned} y_t^1 &= 1 - s_t \\ y_t^2 &= s_t. \end{aligned}$$

a. Formulate and solve a Pareto problem for this economy.

b. Define a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities.

c. Does a competitive equilibrium with sequential trading of a complete set of one-period Arrow securities exist for this economy? If it does, compute it. If it does not, explain why.

Exercise 8.18 **Risk-free bonds**

An economy consists of a single representative consumer who ranks streams of a single nonstorable consumption good according to $\sum_{t=0}^{\infty}\sum_{s^t}\beta^t \ln[c_t(s^t)]\pi_t(s^t)$. Here $\pi_t(s^t)$ is the subjective probability that the consumer attaches to a history s^t of a Markov state s_t, where $s_t \in \{1,2\}$. Assume that the subjective probability $\pi_t(s^t)$ equals the objective probability. Feasibility for this pure endowment economy is expressed by the condition $c_t \leq y_t$, where y_t is the endowment at time t. The endowment is exogenous and governed by

$$y_{t+1} = \lambda_{t+1}\lambda_t \cdots \lambda_1 y_0$$

for $t \geq 0$ where $y_0 > 0$. Here λ_t is a function of the Markov state s_t. Assume that $\lambda_t = 1$ when $s_t = 1$ and $\lambda_t = 1+\zeta$ when $s_t = 2$, where $\zeta > 0$. States $s^t = [s_t, \ldots, s_0]$ are known at time t, but future states are not. The state s_t is described by a time invariant Markov chain with initial probability distribution $\pi_0 = [1 \quad 0]'$ and transition density defined by the stochastic matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{ij} = \text{Prob}[s_{t+1} = j | s_t = i]$ for $i = 1,2$ and $j = 1,2$. Assume that $P_{ij} \geq 0$ for all pairs (i,j).

a. Show how to deduce $\pi_t(s^t)$ from (π_0, P).

b. Define a competitive equilibrium with sequential trading of Arrow securities.

c. Compute a competitive equilibrium with sequential trading of Arrow securities.

d. Let p_t^b be the time t price of a risk-free claim to one unit of consumption at time $t+1$. Define a competitive equilibrium with sequential trading of risk-free claims to consumption one period ahead.

e. Let $R_t = (p_t^b)^{-1}$ be the one-period risk-free gross interest rate. Give a formula for R_t and tell how it depends on the history s^t.

f. Suppose that $\beta = .95, \zeta = .02$ and $P = \begin{bmatrix} 1 & 0 \\ .5 & .5 \end{bmatrix}$. Please compute R_t when $s_t = 1$. Then compute R_t when $s_t = 2$.

g. What parts of your answers depend on assuming that the subjective probability $\pi_t(s^t)$ equals the objective probability?

COMPETITIVE EQUILIBRIUM ANALYSIS

1. COMPLETE MARKET ECONOMY WITH INFINITELY-LIVED AGENTS

Environment:

- Agents $i = 1, 2, \dots, I$.
- Discrete time $t = 0, 1, 2, \dots$
- In each period $t \geq 0$, a stochastic event $s_t \in S$ is realized
- History of stochastic events $s^t = [s_0 \; s_1 \; s_2 \cdots s_t]$
- Unconditional probability of observing history s^t, $\pi_t(s^t)$
- Conditional probability of observing history s^t, given earlier history s^τ, $\pi_t(s^t | s^\tau)$

Two alternatives concerning $t=0$:

a) agents can trade before observing s_0

b) economy starts-off after realization of s_0, i.e., $\pi_0(s_0) = 1$, for some s_0.

Preferences: agent i

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta_i^t \, u_i\left[c_t^i(s^t)\right] \pi_t(s^t)$$

$c_t^i(s^t) \equiv$ agent i's consumption at time t, history s^t

"regular condition": non-satiation : $u_i'(c) > 0$

Inada condition : $u'(c) \xrightarrow[c \to 0]{} \infty$

Technology: no storage of the good.

Endowment: agent i receives at time t, after history s^t, $y_t^i(s^t)$

I. Time-0 trading (Arrow-Debreu securities)

II. Sequential trading (Arrow securities)

Social Planner problem (Pareto problem)

max welfare

s.t. feasibility

(budget constraints, no markets, no incentives)

$$\max \ \lambda_1 \sum_{t=0}^{\infty} \sum_{s^t} \beta_1^t \pi_t(s^t) u_1\left[c_t^1(s^t)\right] + \dots + \lambda_I \sum_{t=0}^{\infty} \sum_{s^t} \beta_I^t \pi_t(s^t) u_I\left[c_t^I(s^t)\right]$$

$$\text{s.t.} \quad \sum_{i=1}^{I} c_t^i(s^t) \leq \sum_{i=1}^{I} y_t^i(s^t) \quad \forall t, \forall s^t \qquad (1)$$

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^{I} \lambda_i \beta_i^t \pi_t(s^t) u_i\left[c_t^i(s^t)\right] + \theta_t(s^t) \sum_{i=1}^{I} \left[y_t^i(s^t) - c_t^i(s^t) \right] \right\}$$

FOC

$$c_t^i(s^t): \quad \lambda_i \beta_i^t \pi_t(s^t) u_i'\left[c_t^i(s^t)\right] - \theta_t(s^t) = 0$$

Use expression for individual i and 1

$$\lambda_i \beta_i^t \cancel{\pi_t(s^t)} u_i'\left[c_t^i(s^t)\right] = \lambda_1 \beta_1^t \cancel{\pi_t(s^t)} u_1'\left[c_t^1(s^t)\right] \Longrightarrow$$

$$\Rightarrow \left(\frac{\beta_i}{\beta_1}\right)^t \frac{u_i'\left[c_t^i(s^t)\right]}{u_1'\left[c_t^1(s^t)\right]} = \frac{\lambda_1}{\lambda_i}$$

$$\beta_1 > \beta_i \quad \forall i \neq 1, \quad \begin{array}{l} u_i' \longrightarrow \infty \\ u_1' \longrightarrow 0 \end{array}$$

from now on, assume $\beta_i = \beta \ \forall i$

$$\boxed{\frac{u_i'\left[c_t^i(s^t)\right]}{u_1'\left[c_t^1(s^t)\right]} = \frac{\lambda_1}{\lambda_i}}$$

Solve for $c_t^i(s^t)$

$$c_t^i(s^t) = (u_i')^{-1}\left[\frac{\lambda_1}{\lambda_i} u_1'\left[c_t^1(s^t)\right]\right] \qquad (2)$$

Substitute (2) into (1)

$$\sum_{i=1}^{I} (u_i')^{-1}\left[\frac{\lambda_1}{\lambda_i} u_1'\left[c_t^1(s^t)\right]\right] = \sum_{i=1}^{I} y_t^i(s^t)$$

(one equation and one unknown $c_t^1(s^t)$)

insured!!!

Prop. An efficient allocation is a function of the realized <u>aggregate</u> endowment and depend neither on the specific history leading to that outcome nor on the realizations of individual endowments.

I. Time-0 Trading

At $t=0$, all trade takes place at prices $q_t^0(s^t)$.

$q_t^0(s^t)$ is the price of one good derived at time t, contingent on history s^t, being realized.

$c_t^i(s^t)$ is agent i's purchase of such claims

Budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, y_t^i(s^t)$$

Agent i's optimization problem

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \beta^t \pi_t(s^t)\, u_i\left[c_t^i(s^t)\right] \right\} + \mu_i \left\{ \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \left[y_t^i(s^t) - c_t^i(s^t) \right] \right\}$$

FOC

$$c_t^i(s^t) : \beta^t \pi_t(s^t)\, u_i'\left[c_t^i(s^t)\right] - \mu_i\, q_t^0(s^t) = 0$$

use FOC for agents i and 1

$$\frac{\cancel{\beta^t \pi_t(s^t)}\, u_i'\left[c_t^i(s^t)\right]}{\mu_i} = \frac{\cancel{\beta^t \pi_t(s^t)}\, u_1'\left[c_t^1(s^t)\right]}{\mu_1}$$

$$\frac{u_i'\left[c_t^i(s^t)\right]}{u_1'\left[c_t^1(s^t)\right]} = \frac{\mu_i}{\mu_1}$$

prefer to have low lagrange multiplier!
If the resource constraint is relaxed (one extra apple) Bill Gates doesn't care ... but I would! And Bill is better.

Prop 2: the analogue

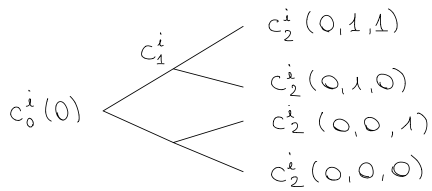
Def: price system $\{q_t^0(s^t)\}_{\forall t, \forall s^t}$

allocation $\{c_t^i(s^t)\}_{\forall i, t, s^t}$

A competitive equilibrium is a feasible allocation and a price system such that

1. given the price system, the allocation solves each agent's problem.
2. market clearing: $\sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} y_t^i(s^t) \quad \forall t, s^t$

Suppose $S = \{0, 1\}$ and $s_0 = 0$



Budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, y_t^i(s^t)$$

many social planner probs (λ's)

Suppose you have found the consumption allocation from the corresponding social planner problem

Then, read off equilibrium price from agents' FOCs. More specifically, pick any agent

$$\beta^t \pi_t(s^t)\, u_i'\left[c_t^i(s^t)\right] - \mu_i\, q_t^0(s^t) = 0$$

Pick a numeraire: $q_0^0(s^0) = 1$

Evaluate FOC at time 0

$$\beta^0 \pi_0(s_0)\, u_i'\left[c_0^i(s_0)\right] - \mu_i\, q_0^0(s_0) = 0 \implies u_i'\left[c_0^i(s_0)\right] = \mu_i$$

Solve for any price

$$q_t^0(s^t) = \beta^t \pi_t(s^t)\, \frac{u_i'\left[c_t^i(s^t)\right]}{u_i'\left[c_0^i(s_0)\right]}$$

II. Sequential Trading

In each period $t \geq 0$, agents trade in one-period-ahead state-contingent claims.

At time t after history s^t,

- $\tilde{Q}_t(s_{t+1} | s^t)$ is the price of one good delivered at time $t+1$, contingent on state s_{t+1} being realized.
- $\tilde{a}_{t+1}^i(s_{t+1}, s^t)$ agent i's purchases of such claim
- Budget constraint at time t, history s^t

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\, \tilde{Q}_t(s_{t+1} | s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s_t, s^{t-1})$$

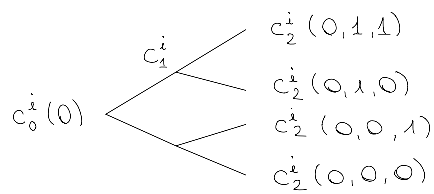
Suppose $S = \{0, 1\}$ and $s_0 = 0$

Trade $t = 0$:

$\tilde{c}_0^i(0)$

$\tilde{a}_1^i(0, [0])$

$\tilde{a}_1^i(1, [0])$



Suppose $s_1 = 0$

$\tilde{c}_1^i(0)$

$\tilde{a}_2^i(0, [0,0])$

$\tilde{a}_2^i(1, [0,0])$

$$\tilde{c}_1^i([0,0]) + \tilde{a}_2^i(1,[0,0])\, \tilde{Q}_1(1 \mid [0,0]) + \tilde{a}_2^i(0,[0,0])\, \tilde{Q}_1(0 \mid [0,0]) \leq y_1^i([0,0]) + a_1^i(0,[0])$$

Before we go to sequential trading equilibrium, suppose time 0 trading equilibrium has reached period t, history s^t. Convert the price

$$\left\{ q_{t+j}^0(s^{t+j}) \right\}_{j=0,\ \forall s^{t+j}}^{\infty}$$

into one expressed in units of time t history s^t goods.

$$q_\tau^0(s^\tau)$$

$$q_\tau^t(s^\tau) = \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} = \frac{\beta^\tau \pi_\tau(s^\tau) \frac{u_i'[c_\tau^i(s^\tau)]}{u_i'[c_0^i(s^0)]}}{\beta^t \pi_t(s^t) \frac{u_i'[c_t^i(s^t)]}{u_i'[c_0^i(s^0)]}} = \beta^{\tau-t} \pi_\tau(s^\tau \mid s^t) \frac{u_i'[c_\tau^i(s^\tau)]}{u_i'[c_t^i(s^t)]}$$

Compute agent i's wealth in period t, history s^t, expressed in terms of time t history s^t prices.

$$W_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau \mid s^t} q_\tau^t(s^\tau)\, c_\tau^i(s^\tau)$$

Compute agent i's original endowments

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau \mid s^t} q_\tau^t(s^\tau)\, y_\tau^i(s^\tau)$$

Compute agent i's wealth minus endowments

$$\Upsilon_t^i(s^t) = W_t^i(s^t) - A_t^i(s^t)$$

$$\sum_{i=1}^{I} \Upsilon_t^i(s^t) = 0$$

Agent i's optimization problem

$$\max_{\{\tilde{c}_t^i(s^t),\ \{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}\}_{\forall t, \forall s^t}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)\, u_i\left[\tilde{c}_t^i(s^t)\right]$$

s.t.

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\, \tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s_t, s^{t-1})$$

$\forall t, \forall s^t$

given $\tilde{a}_0^i(s_0)$

$c_t^i(s^t) \geq 0$

non-Ponzi constraint : $-\tilde{a}_{t+1}^i(s_{t+1}, s^t) \leq$ "natural debt limit"

value of the maximal amount that agent i can repay starting from that period, when not eating anything from there on. $\equiv A_{t+1}^i(s^{t+1})$

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \Big\{ \beta^t \pi_t(s^t)\, u_i\left[\tilde{c}_t^i(s^t)\right] + \eta_t^i(s^t)\Big[y_t^i(s^t) + \tilde{a}_t^i(s^t) - \tilde{c}_t^i(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t)\Big] +$$

$$+ \sum_{s_{t+1}} \nu_t^i(s^t, s_{t+1})\left[A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1})\right]\Big\}$$

FOC

$$\tilde{c}_t^i(s^t): \quad \beta^t \pi_t(s^t)\, u_i'\left[c_t^i(s^t)\right] - \eta_t^i(s^t) = 0$$

$$\tilde{a}_{t+1}^i(s_{t+1}, s^t): \quad -\eta_t^i(s^t)\tilde{Q}_t(s_{t+1}|s^t) + \nu_t^i(s^t, s_{t+1}) + \eta_{t+1}^i(s_{t+1}, s^t) = 0$$

Because of the Inada condition on consumption

$$\nu_t^i(s^t, s_{t+1}) = 0$$ (constraint not binding! Won't borrow up to the limit)

Hence,

$$\tilde{Q}_t(s_{t+1}|s^t) = \frac{\eta_{t+1}^i(s^{t+1})}{\eta_t^i(s^t)} = \frac{\beta^{t+1}\pi_{t+1}(s^{t+1})\, u_i'\left[\tilde{c}_{t+1}^i(s^{t+1})\right]}{\beta^t \pi_t(s^t)\, u_i'\left[\tilde{c}_t^i(s^t)\right]} = \beta\, \pi_{t+1}(s^{t+1}|s^t)\, \frac{u_i'\left[\tilde{c}_{t+1}^i(s^{t+1})\right]}{u_i'\left[\tilde{c}_t^i(s^t)\right]}$$

<u>Def</u>: A sequential-trading competitive equilibrium is an initial distribution of wealth $\{a_0^i(s_0)\}_{\forall i}$, a consumption allocation $\{\tilde{c}_t^i(s^t)\}_{\forall i, t, s^t}$ and a price system $\{\tilde{Q}_t(s_{t+1}|s^t)\}_{\forall t, s^t, s_{t+1}}$ such that

(a) given initial wealth distribution and price system, the allocation solves each agent's problem

(b) for all realization of $\{s^t\}$ the consumption allocation and implied asset portfolio satisfy

$$\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t) \qquad \forall t, s^t$$

$$\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0 \qquad \forall t+1, s^{t+1}, s^t$$

<u>Equivalence of allocations</u>

Guess that $\quad a_0^i(s_0) = 0$

$$\tilde{Q}_t(s_{t+1}|s^t) = q_{t+1}^t(s^{t+1})$$

verify that a competitive equilibrium allocation of the complete market model with time 0 trading is also a sequential-trading competitive equilibrium allocation

$$\tilde{a}_{t+1}^i(s_{t+1}, s^t) = \Upsilon_{t+1}^i(s^{t+1})$$

<u>Time - 0 trading equilibrium</u>

Agent i's wealth in period t, after history s^t, expressed in terms of time-t history-s^t goods

$$W_t^i(s^t) = \sum_{t=\tau}^{\infty} \sum_{s^\tau|s^t} q_\tau^t(s^\tau)\, c_\tau^i(s^\tau)$$

Value of agent i's original endowment process, ...

$$A_t^i(s^t) = \sum_{t=\tau}^{\infty} \sum_{s^\tau|s^t} q_\tau^t(s^\tau)\, y_\tau^i(s^\tau)$$

Define $\Upsilon_t^i(s^t) \equiv W_t^i(s^t) - A_t^i(s^t)$

<u>Sequential trading equilibrium</u>

No-Ponzi restrictions

$$-\tilde{a}_{t+1}^i(s_{t+1}, s^t) \leq A_{t+1}^i(s^{t+1})$$

Equilibrium asset portfolio satisfy

$$\tilde{a}_{t+1}^i(s_{t+1}, s^t) = \Upsilon_{t+1}^i(s^{t+1})$$

When assuming arbitrary probability measures $\pi_t(s^t)$, then the equilibrium prices $\tilde{Q}_t(s_{t+1}|s^t)$ and wealth distribution $\tilde{a}_t^i(s^t)$ depend on the entire history s^t.

<u>Specialization</u> Endowment governed by a Markov process $\Longrightarrow$ a recursive formulation of the sequential trading equilibrium

i) Markov chain for stochastic events

$$\pi(s'|s) \equiv \text{Prob}(s_{t+1} = s' \mid s_t = s)$$

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\,\pi(s_{t-1}|s_{t-2}) \cdots \pi(s_0)$$

$$\pi_t(s^t|s^\tau) = \pi(s_t|s_{t-1})\,\pi(s_{t-1}|s_{t-2}) \cdots \pi(s_{\tau+1}|s_\tau)$$

ii) $y_t^i(s^t) = y^i(s^t) \qquad$ Hence, endowments follow a markov process

By Prop. 2, $(ii) \Longrightarrow c_t^i(s^t) = \bar{c}^i(s^t)$

$$\tilde{Q}_t(s_{t+1}|s^t) = q_{t+1}^t(s^{t+1}) = \beta\,\pi_{t+1}(s^{t+1}|s^t)\,\frac{u_i'[\tilde{c}_{t+1}^i(s^{t+1})]}{u_i'[\tilde{c}_t^i(s^t)]}$$

involving (i) - (ii)

$$\tilde{Q}_t(s_{t+1}|s^t) = q_{t+1}^t(s^{t+1}) = \beta\,\pi_{t+1}(s^{t+1}|s^t)\,\frac{u_i'[\bar{c}^i(s^{t+1})]}{u_i'[\bar{c}^i(s^t)]}$$

$$A_t^i(s^t) = \bar{A}^i(s^t)$$

$$\Upsilon_t^i(s^t) = \bar{\Upsilon}^i(s^t)$$

How to compute a time-0 trading equilibrium

$$\max \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\,\pi_t(s^t)\,u_i[c_t^i(s^t)] \qquad i = 1, 2, \dots, I$$

$$\text{s.t.} \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\,c_t^i(s^t) \le \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\,y_t^i(s^t) \tag{1}$$

$$\sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} y_t^i(s^t) \qquad \forall t, s^t \tag{2}$$

From optimization and equilibrium expressions

$$\beta^t\,\pi_t(s^t)\,u_i'[c_t^i(s^t)] - \mu_i\,q_t^0(s^t) = 0 \tag{3}$$

$$\frac{u_i'[c_t^i(s^t)]}{u_1'[c_t^1(s^t)]} = \frac{\mu_i}{\mu_1} \tag{4}$$

$$\sum_{i=1}^{I}(u_i')^{-1}\left(\frac{\mu_i}{\mu_1}\,u_1'[c_t^1(s^t)]\right) = \sum_{i=1}^{I} y_t^i(s^t) \tag{5}$$

Solution method: Guess and verify

Approach I: guess $\{c_t^i(s^t)\}_{i,t,s^t}$, $\{q_t^0(s^t)\}_{t,s^t}$

Approach II: guess $\{\mu_i\}_i$

Negishi algorithm:

1. Given a guess $\{\mu_i\}_i$, use (4)-(5) to compute the allocation $\{c_t^i(s^t)\}_{i,t,s^t}$

2. Use FOC (3) [for some agent i] to solve for the system $\{q_t^0(s^t)\}_{t,s^t}$

3. For all agents $i = 1, \dots, I$ check their budget constraints

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \left[y_t^i(s^t) - c_t^i(s^t) \right] \begin{array}{ll} < 0 & \mu_i \uparrow \\ > 0 & \mu_i \downarrow \end{array}$$

Recall as in the case of Pareto weights, only ratios of multipliers matter.

Sequential trading

Dynamic Programming

Controls $\quad c$ is consumption today

$\quad \hat{a}(s_{t+1})$ Arrow securities $\forall s_{t+1}$

state $\quad a$ asset at the beginning of the period

$\quad s^t$ (y not necessary!)

$$V_t^i(a, s^t) = \max_{\{c, \{\hat{a}(s_{t+1})\}_{s_{t+1}}\}} \left\{ u_i(c) + \beta \mathbb{E}_t V_{t+1}(\hat{a}(s_{t+1}), s^{t+1}) \right\}$$

$$\text{s.t.} \quad y_t^i(s^t) + a \geq c + \sum_{s_{t+1}} \hat{a}(s_{t+1}) \tilde{Q}_{t+1}(s_{t+1} | s^t)$$

$$- \hat{a}(s_{t+1}) \leq A_{t+1}^i(s^{t+1}) \quad \forall s_{t+1}$$

Solution at equilibrium price

$$c = \tilde{c}_t^i(s^t)$$

$$\hat{a}(s_{t+1}) = \tilde{a}_{t+1}^i(s_{t+1}, s^t)$$

Adopt Markov specialization

state $\quad a_t^i$

$\quad s_t$

seek policy function

$$c_t^i = h^i(a_t^i, s_t)$$

$$a_{t+1}^i(s_{t+1}) = g^i(a_t^i, s_t; s_{t+1})$$

$$V^i(a, s) = \max_{\{c, \{\hat{a}(s')\}_{s'}\}} \left\{ u_i(c) + \beta \sum_{s'} \pi(s'|s) V(\hat{a}(s'), s') \right\}$$

$$\text{s.t.} \quad y^i(s) + a \geq c + \sum_{s'} \hat{a}(s') \tilde{Q}(s'|s)$$

$$- \hat{a}(s') \leq \bar{A}^i(s') \quad \forall s'$$

FOC

$$c: \quad u_i'(c) - \mu^i = 0$$

$$\hat{a}(s'): \quad \beta \pi(s'|s) V_1^i(\hat{a}(s'), s') - \mu^i Q(s'|s) = 0$$

$$c_t: \quad u_i'(c_t) - \mu_t^i = 0$$

$$\hat{a}_{t+1}(s_{t+1}): \quad \beta \pi(s_{t+1}|s_t) \underbrace{V_1^i(\hat{a}_{t+1}(s_{t+1}), s_{t+1})}_{= u_i'\left[c_{t+1}^i(s_{t+1})\right]} - \mu_t^i Q(s_{t+1}|s_t) = 0$$

Solve for Q

$$Q(s_{t+1}|s_t) = \frac{\beta \pi(s_{t+1}|s_t) \, u_i'\left[c_{t+1}(s_{t+1})\right]}{u_i'(c_t)}$$

$$Q(s'|s) = \frac{\beta \pi(s'|s) \, u_i'\left(h^i(g^i(a,s;s'),s')\right)}{u_i'(h^i(a,s))}$$

Def: A recursive competitive equilibrium is an initial distribution of wealth, a pricing function $Q(s'|s)$, set of value functions $\{v^i(a,s)\}_{\forall i}$ and policy function $\{h^i(a,s), g^i(a,s;s')\}_{\forall i}$ such that

(a) For all i, given a_0^i and the pricing function, the policy functions solve the agent's problem

(b) For all realizations of $\{s^t\}$, the consumption and asset portfolio implied by the policy functions satisfy

$$\sum_i c_t^i = \sum_i y_t^i$$

$$\sum_i \hat{a}_{t+1}^i(s') = 0$$

Pricing redundant assets

I. include the asset in agent's budget constraint, take FOC's, impose market clearing and compute equilibrium price.

II. no-arbitrage argument

A. Multiply $(2)_t$ by $Q(s_{t+1}|s_t)$ and sum over s_{t+1}

B. Substitute into $(1)_t$

Risk-free bond (one-period)

quantity L_t^i purchased by agent i (each bond pays one good next period)

price of a bond $\frac{1}{R(s^t)}$, i.e. $R(s^t)$ is the risk-free interest rate

W_t^i financial wealth of agent i at the beginning of period t excluding not endowment

Budget constraint

$$c_t^i + L_t^i \frac{1}{R(s^t)} + \sum_{s_{t+1}} Q(s_{t+1}|s_t)\, a_{t+1}^i(s_{t+1}) \leq y_t^i + W_t^i \qquad (1)$$

Next period wealth

$$\sum_{s_{t+1}} Q(s_{t+1}|s_t)\, W_{t+1}^i(s_{t+1}) = \sum_{s_{t+1}} \left[L_t^i + a_{t+1}^i(s_{t+1})\right] Q(s_{t+1}|s_t) \qquad (2) \quad (2')$$

From (2')

$$\sum_{s_{t+1}} Q(s_{t+1}|s_t)\, a_{t+1}^i(s_{t+1}) = \sum_{s_{t+1}} Q(s_{t+1}|s_t)\, W_{t+1}^i(s_{t+1}) - \sum_{s_{t+1}} Q(s_{t+1}|s_t)\, L_t^i$$

Substituting into (1)

$$c_t^i + L_t^i \frac{1}{R(s_t)} + \sum_{s_{t+1}} Q(s_{t+1}|s_t)\, W_{t+1}^i(s_{t+1}) - \sum_{s_{t+1}} Q(s_{t+1}|s_t)\, L_t^i \leq y_t^i + W_t^i$$

$$\Rightarrow \underbrace{\left\{\frac{1}{R(s_t)} - \sum_{s_{t+1}} Q(s_{t+1}|s_t)\right\}}_{=0} L_t^i + c_t^i + \sum_{s_{t+1}} Q(s_{t+1}|s_t)\, W_{t+1}^i(s_{t+1}) \leq y_t^i + W_t^i$$

if >0, go short on L_t^i!

No arbitrage: $\frac{1}{R(s_t)} = \sum_{s_{t+1}} Q(s_{t+1}|s_t)$

Approach I

$$v^i(W_t^i, s_t) = \max_{\{c_t^i, \{a_{t+1}^i(s_{t+1})\}_{s_{t+1}}, L_t^i\}} \left\{ u(c_t^i) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\, v^i\left(W_{t+1}^i(s_{t+1}), s_{t+1}\right)\right\}$$

FOC

$$L_t^i: \quad u'(c_t^i)\frac{1}{R(s_t)} = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\, v_1^i\left(W_{t+1}^i(s_{t+1}), s_{t+1}\right)$$

$$a_{t+1}^i(s_{t+1}): \quad u'(c_t^i)\, Q(s_{t+1}|s_t) = \beta\pi(s_{t+1}|s_t)\, v_1^i\left(W_{t+1}^i(s_{t+1}), s_{t+1}\right)$$

By Benveniste - Schenkman formula

$$\frac{1}{R(s_t)} = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \frac{u'[c_{t+1}^{i*}(s_{t+1})]}{u'(c_t^{i*})}$$

$$Q(s_{t+1}|s_t) = \beta \pi(s_{t+1}|s_t) \frac{u'[c_{t+1}^{i*}(s_{t+1})]}{u'(c_t^{i*})}$$

Evaluate at competitive equilibrium allocation $*$

$$\Rightarrow \frac{1}{R(s_t)} = \sum_{s_{t+1}} Q(s_{t+1}|s_t)$$

Complete markets with heterogeneous beliefs

Recall Debreu's general preferences

$$u^i\left(\{c_t^i(s^t)\}_{\forall t, s^t} \right)$$

Our specialization

$$U^i(\cdot) = \sum_{t=0}^{\infty} \sum_{s_t} \pi_t^i(s^t)\, u_i[c_t^i(s^t)]$$

Recall social planner solution

$$\frac{\pi_t^i(s^t)}{\pi_t^1(s^t)} \cdot \frac{u_i'[c_t^i(s^t)]}{u_1'[c_t^1(s^t)]} = \frac{\lambda_1}{\lambda_i}$$

If $\pi_t^i(s^t) > \pi_t^1(s^t) \Rightarrow$ social planner gives relatively more consumption to individual i

Milton Friedman (among others) conjectured "natural selection" in market dynamics: those who behave irrationally will be driven out by rational agents (including those who have more accurate beliefs)

Consider the case of <u>iid uncertainty</u>

Let $\pi(s) > 0$ be the true probability for event $s \in S$ to be realized

$$\sum_{s \in S} \pi(s) = 1$$

Let $\pi^i(s)$ be the belief of agent i, where $\pi^i(s) > 0 \ \forall s$ and

$$\sum_{s \in S} \pi^i(s) = 1 \qquad \text{for } i = 1, \dots, I$$

Let agent 1 have the strict lowest "relative entropy"

$$\text{ent}(\pi, \pi^1) < \text{ent}(\pi, \pi^i) \qquad \forall i \in I$$

Relative entropy of the actual distribution π relative to agent i's distribution π^i

$$ent(\pi, \pi^i) = \sum_{s \in S} \pi(s) \log\left[\frac{\pi(s)}{\pi^i(s)}\right]$$

$$-ent(\pi, \pi^i) = \sum_{s \in S} \pi(s) \log\left[\frac{\pi^i(s)}{\pi(s)}\right] \leq \log\left[\sum_{s \in S} \pi(s) \frac{\pi^i(s)}{\pi(s)}\right] = \log\Big[\underbrace{\sum_{s \in S} \pi^i(s)}_{1}\Big] = 1$$

By Jensen's inequality
(log is strictly concave function)

hence, $ent(\pi, \pi^i) \geq 0$!

$$\frac{u_i'[c_t^i(s^t)]}{u_1'[c_t^1(s^t)]} = \frac{\pi_t^1(s^t)}{\pi_t^i(s^t)} \frac{\lambda_1}{\lambda_i} = \frac{\lambda_1}{\lambda_i} \frac{\prod_{\tau=0}^{t} \pi^1(s_\tau)}{\prod_{\tau=0}^{t} \pi^i(s_\tau)} = \frac{\lambda_1}{\lambda_i} \frac{\prod_{\tau=0}^{t} \frac{\pi(s_\tau)}{\pi^i(s_\tau)}}{\prod_{\tau=0}^{t} \frac{\pi(s_\tau)}{\pi^1(s_\tau)}} =$$

$$\Longrightarrow \frac{1}{t} \log\left\{\frac{u_i'[c_t^i(s^t)]}{u_1'[c_t^1(s^t)]}\right\} = \frac{1}{t} \log\left\{\frac{\lambda_1}{\lambda_i}\right\} + \frac{1}{t} \sum_{\tau=0}^{t} \left[\log\left[\frac{\pi(s_\tau)}{\pi^i(s_\tau)}\right] - \log\left[\frac{\pi(s_\tau)}{\pi^1(s_\tau)}\right]\right]$$

$$\frac{1}{t} \sum_{\tau=0}^{t} \log\left[\frac{\pi(s_\tau)}{\pi^i(s_\tau)}\right] \xrightarrow[t \to \infty]{a.s.} \sum_{s \in S} \pi(s) \log\left[\frac{\pi(s)}{\pi^i(s)}\right] \equiv ent(\pi, \pi^i)$$

$$\frac{1}{t} \log\left\{\frac{u_i'[c_t^i(s^t)]}{u_1'[c_t^1(s^t)]}\right\} \xrightarrow[t \to \infty]{} ent(\pi, \pi^i) - ent(\pi, \pi^i) > 0$$

$$\Longrightarrow \log\left[\frac{u_i'[c_t^i(s^t)]}{u_1'[c_t^1(s^t)]}\right] \xrightarrow[a.s]{} \infty$$

Does $u_i'[c_t^i(s^t)] \longrightarrow \infty$ finite resources
$u_1'[c_t^1(s^t)] \longrightarrow 0$

Bloom & Easley, Econometrica 2006

Beker & Chattopadhyay, JET 2010