CH1 Difference Equations

$x_{t+1} = ax_t + b_t \iff x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_{k-1}$

$x_{t+1} = ax_t + b \iff \begin{cases} x_t = a^t\left(x_0 - \frac{b}{1-a}\right) + \underbrace{\frac{b}{1-a}}_{SS} & \text{if } a \neq 1 \\ x_t = x_0 + tb & \text{if } a = 1 \end{cases}$

GAS if $|a|<1$

$x_{t+1} = a_t x_t + b_t \iff x_t = \prod_{s=0}^{t-1} a_s x_0 + \sum_{k=0}^{t-1}\left(\prod_{s=k+1}^{t-1} a_s\right) b_k$

Prop 2: Let x^* be an eq. state of $x_{t+1} = f(x_t)$. Suppose f is C^1 around x^*. (a) If $|f'(x^*)|<1$ x^* is LAS. (b) If $|f'(x^*)|>1$, x^* unstable. Proof: Let $\{x_t\}$ be a sol. Then $|x_{t+1} - x^*| = |f(x_t) - f(x^*)| = |f'(c)(x_t - x^*)|$. (a) Since $|f'(x^*)|<1$, $\exists \varepsilon > 0$, $k \in (0,1)$ with $|f'(x)| \leq k$ $\forall x \in (x^*-\varepsilon, x^*+\varepsilon)$. Suppose
mvth $x_t \in (x^*-\varepsilon, x^*+\varepsilon) \Rightarrow |x_{t+1} - x^*| \leq k|x_t - x^*| \leq \dots \leq k^{t+1}|x_0 - x^*| \Rightarrow \lim_{t\to\infty} x_t = x^*$ if $x_0 \in (x^*-\varepsilon, x^*+\varepsilon)$.
(b) Since $|f'(x^*)|>1$, $\exists \varepsilon>0$, $k>1$ with $|f'(x)|>k$ $\forall x \in (x^*-\varepsilon, x^*+\varepsilon)$. Let $\delta > 0$ and pick any $x_0 \in (x^*-\varepsilon, x^*+\varepsilon)$. Suppose that $x_t \in (x^*-\varepsilon, x^*+\varepsilon)$. Then, $|x_{t+1} - x^*| \geq k|x_t - x^*| \geq \dots \geq k^{t+1}|x_0 - x^*|$
$\Rightarrow |x_t - x^*| \geq \varepsilon$ ↯. Remark: $|f'(\cdot)| < 1 \Rightarrow x^*$ GAS

Prop 3: Suppose (ξ_1, ξ_2) is cycle for $x_{t+1} = f(x_t)$. Suppose f is C^1 around ξ_1 and ξ_2. (a) If $|f'(\xi_1)f'(\xi_2)| < 1$, cycle is LAS. (b) If $|f'(\xi_1)f'(\xi_2)| > 1$, cycle is unstable. Proof: (ξ_1, ξ_2) is LAS (unstable) iff (ξ_1, ξ_2) are LAS (unstable) eq. points of $x_{t+1} = \underbrace{f(f(x_t))}_{g}$. By Prop 2, (ξ_1, ξ_2) are LAS (unstable) if $|g'(\xi_1)g'(\xi_2)| < 1$ ($|g'(\xi_1)g'(\xi_2)| > 1$) where
$g'(\xi_1) = f'(\underbrace{f(\xi_1)}_{\xi_2}) f'(\xi_1) = f'(\xi_2)f'(\xi_1)$

Induction: $x_0 \overset{!}{=} x_0$; x_τ for some $\tau = 0,1,\dots,t$. Show it is true for $\tau = t+1$: $x_{t+1} = [\dots]$ ← insert x_t

CH2: Second-order

$x_{t+2} + a_t x_{t+1} + b_t x_t = c_t$ (1)
$x_{t+2} + a_t x_{t+1} + b_t x_t \overset{!}{=} 0$ (2)

① Gen. sol. of (2) is $x_t = Au_t + Bv_t$ with $\nexists(\alpha_1, \alpha_2) \neq (0,0)$ s.t. $\alpha_1 u_t + \alpha_2 v_t = 0$ and $\nexists \alpha$ s.t. $u_t = \alpha v_t$.

② Gen. Sol. of (1) is $x_t = Au_t + Bv_t + u_t^*$

CH3: Constant coeff.

$x_{t+2} + ax_{t+1} + bx_t = 0$ $(m^2 + am + b = 0)$ (3)

Gen. sol. of (3) is (1) $x_t = A m_1^t + B m_2^t$, $m_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$ if $\frac{a^2}{4} - b > 0$. (2) $x_t = (A + Bt)m_1^t$,
$m = -\frac{a}{2}$ if $\frac{a^2}{4} - b = 0$. (3) $x_t = r^t[A\cos(\theta t) + B\sin(\theta t)]$, $r = \sqrt{b}$, $\cos(\theta) = -\frac{a}{2\sqrt{b}}$ if $\frac{a^2}{4} - b < 0$

(4) Solve $x_{t+2} + ax_{t+1} + bx_t = c_t$? Solve (3) → MUC ($c_t = t^2 + 3$). Guess $u_t^* = Ct^2 + Dt + E$ sol.
for $C,D,E \in \mathbb{R}$: $u_{t+2}^* + au_{t+1}^* + bu_t^* = C(t+2)^2 + D(t+2) + E + a[C(t+1)^2 + D(t+1) + E] + (b\ Ct^2 + Dt + E) \overset{!}{=} t^2 + 3$
→ $2Ct^2 + (2D - 6C)t + (2E - C - 3D) \overset{!}{=} t^2 + 3 \Rightarrow u_t^* = \frac{1}{2}t^2 + \frac{3}{2}t + 4$ part. sol. → gen sol: $x_t = Am_1^t + Bm_2^t + u_t^*$

(4) GAS ↔ roots of $m^2 + am + b$ have mod < 1 ⟷ $|a| < 1 + b$, $b < 1$ — $\mathrm{mod}(z) = \sqrt{x^2 + y^2}$ if $z = x + iy$; $\mathrm{mod}(x) = \sqrt{x^2} = |x|$

SS: $x_t = x^*$ solves $x_{t+2} + ax_{t+1} + bx_t = c$ (← not t!) ⟷ $x^* = \frac{c}{1 + a + b}$ ⇒ gen. sol: $x_t = Au_t + Bu_t + x^*$
x^* GAS ⟷ $\lim_{t\to\infty} x_t = x^*$ ⟷ $\lim_{t\to\infty} Au_t + Bu_t = 0$ ⟷ $x_{t+2} + ax_{t+1} + bx_t = c$ GAS

CH4: Systems

$x(t) = \underbrace{\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{B}$

eigenvalue: $|A - I\lambda| \overset{!}{=} 0 \Rightarrow \lambda_{1,2} \in \{-3, 2\}$

eigenvectors: $\lambda_1 = 2$: $(A - \lambda_1 I)v = 0 \iff \begin{matrix} -3v_1 + 3v_2 = 0 \\ 2v_1 - 2v_2 = 0 \end{matrix} \iff v_1 = v_2 \iff v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3$: $(A - \lambda_2 I)v = 0 \iff \begin{matrix} 2v_1 + 3v_2 = 0 \\ 2v_1 + 3v_2 = 0 \end{matrix} \iff v_1 = -\frac{3}{2}v_2 \iff v = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$P = (v) = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}$; $P^{-1} = \begin{pmatrix} -2 & -3 \\ -1 & 1 \end{pmatrix} \frac{1}{|P|} = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & -0.2 \end{pmatrix}$; Homog eq: $x(t+1) = Ax(t) \Rightarrow x(t) = A^t x(0) = P\begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} P^{-1}\begin{pmatrix} x(0) \\ x(0) \end{pmatrix}$
Non-H. eq: $x(t+1) = Ax(t) + B \Rightarrow x(t) = \underbrace{A^t x(0)}_{\text{sol. H}} + \sum_{k=1}^{t} A^{t-k} B$ → find one part. sol.
→ SS → $x^* = Ax^* + B$ → $x^* = (I - A)^{-1}B = \begin{pmatrix} 2 & -3 \\ -2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ → $x(t) = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2^t & 0 \\ 0 & (-3)^t \end{pmatrix}\begin{pmatrix} 0.4 & 0.6 \\ 0.2 & -0.2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2^t(0.4c_1 + 0.6c_2) + (-3)^t 0.6(c_1 - c_2) \\ 2^t(0.4c_1 + 0.6c_2) + (-3^t)0.4(-c_1 + c_2) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$x(t+1) = Ax(t) + B(t)$ (5) ⇒ gen. sol → $x(t) = A^t x(0) + \sum_{k=1}^{t} A^{t-k} B(k-1)$ → GAS if $\lim_{t\to\infty} A^t x(0) = 0$ / all eigenvalues of A have mod < 1 ($|\lambda_i| < 1$)

CH5 Differential eqs.

Separable: $\dot{x} = -2tx^2$
1) $\frac{dx}{dt} = -2tx^2$
2) $\frac{dx}{x^2} = -2t\,dt$
3) $\int \frac{dx}{x^2} = -2\int t\,dt \rightarrow x = \frac{1}{t^2 - C}$; $x = 0$

Linear eq (cnt): $\dot{x} + ax = b$
1) $\dot{x}e^{at} + axe^{at} = be^{at}$
2) $\frac{d\,xe^{at}}{dt} = be^{at}$
3) $\int d\,xe^{at} = b\int e^{at}\,dt \rightarrow xe^{at} = \frac{b}{a}e^{at} + C \rightarrow x = \frac{b}{a} + Ce^{-at}$

Substitution: $t\dot{x} = x - e^{at+b}x^2$
1) transformation $x = tz$, $\dot{x} = z + t\dot{z}$
2) $t(z + t\dot{z}) = tz - e^{at+b}(tz)^2 \rightarrow \frac{\dot{z}}{z^2} = -e^{at-b}$
3) as usual and de-transform

Uniqueness: consider the initial value problem $\dot{x} = F(t,x)$, $x(t_0) = x_0$ and suppose F and $F'(x)$ are continuous around (t_0, x_0). Then there exists one local sol., with $x(t_0) = x_0$

Second-order: $\ddot{x} + a(t)\dot{x} + b(t)x = f(t)$. Assoc. H: $\ddot{x} + a\dot{x} + bx = 0$ (← cnt). Gen. sol: $x = Ae^{r_1 t} + Be^{r_2 t}$,
$r_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$; $x = (A + Bt)e^{rt}$, $r = -\frac{a}{2}$; $x = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)]$, $\alpha = -\frac{a}{2}$, $\beta = \sqrt{b - \frac{a^2}{4}}$

Series: $\sum_{k=0}^{t-1} k = \frac{t(t-1)}{2}$; $\sum_{k=1}^{t} a^{t-k} = \frac{1 - a^t}{1 - a}$; $\sum_{k=1}^{t} k(k-1) = \frac{(t-1)t(t+1)}{3}$; $\prod_{s=0}^{t-1}(s+1) = t!$; $\prod_{s=k+1}^{t-1}(s+1) = \frac{t!}{(k+1)!}$;
$a^{\sum_{k=0}^{t-1}} = a^t - 1$ ← check HW1,2b

Guess:
$7t^2 + 2t - 6 \rightarrow g: Ct^2 + Dt + E$.
$t2^t \rightarrow g: Ct2^t + D2^t$.
$8 2^t\cos\left(\frac{t\pi}{2}\right) \rightarrow g: Ct2^t\cos\left(\frac{t\pi}{2}\right) + Dt2^t\sin\left(\frac{t\pi}{2}\right)$

Trigonometrics:
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

sin/cos	0/1	$\frac{1}{2}/\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}, \frac{1}{2}$	1/0	0/-1
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π

CH6: Sequential Methods

Finite Horizon: $\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$ (← sum of concave functions) s.t. $c_t + k_{t+1} \leq f(k_t)$, $c_t \geq 0$, $k_{t+1} \geq 0$, k_0 given (← convex constraints)

Assume $\lim_{c\to 0} u'(c) \rightarrow \infty$

$\mathcal{L} = \sum_{t=0}^{T} \beta^t [u(c_t) - \lambda_t(c_t + k_{t+1} - f(k_t)) + \mu_t k_{t+1}]$, $c_t > 0$

$\frac{\partial\mathcal{L}}{\partial c_t} = 0 \rightarrow u'(c_t) - \lambda_t = 0$; $\frac{\partial\mathcal{L}}{\partial k_{t+1}} = 0 \rightarrow \beta^t(\mu_t - \lambda_t) + \lambda_{t+1}\beta^{t+1}f'(k_{t+1}) = 0$; $\frac{\partial\mathcal{L}}{k_{T+1}} = 0 \rightarrow \mu_T - \lambda_T = 0$

slackness conditions: $\mu_t k_{t+1} = 0$, $\lambda_t, \mu_t, k_{t+1} \geq 0$
$u'(c) > 0 \rightarrow \lambda_t > 0 \rightarrow \lambda_T > 0 \rightarrow \mu_T > 0 \rightarrow k_{T+1} = 0$

Infinite Horizon
Thm 1: consider dynamic prob $\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1})$ (7), $k_{t+1} \geq 0$. If $\{k_{t+1}^*\}_{t=0}^{\infty}$ & $\{\mu_t^*\}_{t=0}^{\infty}$ satisfy: (1) $k_{t+1}^* \geq 0$; (2) EE: $F_2(k_t^*, k_{t+1}^*) + \beta F_1(k_t^*, k_{t+1}^*) + \mu_t^* = 0$, (3) $\mu_t^* \geq 0$, $\mu_t^* k_{t+1}^* = 0$,
(4) transversality condition: $\lim_{t\to\infty} \beta^t F_1(k_t^*, k_{t+1}^*)k_t^* = 0$; and $F(k_t, k_{t+1})$ is concave in (k_t, k_{t+1}) and increasing in its first argument, then $\{k_{t+1}^*\}_{t=0}^{\infty}$ solves (7).

CH7: Optimal control Theory

Finite Horizon: a system at time t is described by a state vector $x(t)$ (cont and piece-wise diff vector function). Initial state $x(0) = x_0 \in \mathbb{R}^n$. Terminal condition (either): (i) $x(T)$ free; (ii) $x(T) \geq x_1$, x_1 given; (iii) $x(T) = x_1$, x_1 given (8). At time t control vector $u(t) \in U$ is chosen, $\dot{x}(t) = g(t, x(t), u(t))$ (9) is the eq. of motion.
(piecewise cont. vector func. — compact, convex & time-invariant)
$(x(t), u(t))$ admissible if satisfy (8)–(9) and $x(0)$. $J = \int_0^T f(t, x(t), u(t))\,dt \equiv$ objective function (← cont. & diff. func.)
Hamiltonian: $\mathcal{H}(t, x(t), u(t), \lambda(t)) = f(t, x(t), u(t)) + \lambda(t)\,g(t, x(t), u(t))$ (← co-state vector)

(*) Pontryagin's Max. Principle (necessary condition): If $(x^*(t), u^*(t))$ solve the standard problem, then $\exists$ a cont. and piece-wise diff function $\lambda(t)$ s.t. $\forall t \in [0,T]$
(1) $\mathcal{H}(t, x^*(t), u^*(t), \lambda(t)) \geq \mathcal{H}(t, x^*(t), u(t), \lambda(t))$ $\forall u \in U$ (10); (2) If $u^*(t)$ is diff., $\dot{\lambda} = -\frac{\partial\mathcal{H}}{\partial x}$ (11); (3) A transversality condition holds: (i) $\lambda(T) = 0$ (x_T free); (ii) $\lambda(T) \geq 0$, $\lambda(T)(x(T) - x_1)$ $(x(T) \geq x_1)$; (iii) $x_T = x_1$ (12)

If U convex & $\mathcal{H}$ strictly concave in u → $u^*(t)$ is cont. $\forall t$.

Consider acquiring more x at t, and holding it for Δt before disposing. The change in value is $\underbrace{\frac{\partial\mathcal{H}}{\partial x}\Delta t}_{\text{direct increase}} - \underbrace{\lambda(t)}_{\text{cost of buying}} + \underbrace{\lambda(t + \Delta t)}_{\text{benefit of selling}} \overset{!}{=} 0$ (← at optimum) $\longrightarrow \frac{\partial\mathcal{H}}{\partial x} = -\frac{\lambda(t + \Delta t) - \lambda(t)}{\Delta t} \xrightarrow[\Delta t \to 0]{} -\dot{\lambda}$

Mangasarian (sufficient cond): If $(x^*(t), u^*(t), \lambda(t))$ satisfies (10)–(12) and $\mathcal{H}$ is concave in (x,u) $\forall t \in [0,T]$, then $(x^*(t), u^*(t), \lambda(t))$ is optimal

Infinite Horizon: the standard problem is $\max_{u(t) \in U} \int_0^\infty f(t, x(t), u(t))\,dt$, $\dot{x}(t) = g(t, x(t), u(t))$, $x(t_0) = x_0$, $\lim_{t\to\infty} x(t) = x_1$ (16) (the integral converges whenever $(x(t), u(t))$ satisfy $\dot{x}(t)$.

Max. Principle: If $(x^*(t), u^*(t))$ satisfy $\dot{x}(t)$, x_0 and trans. cond in (16) and is optimal, then it must satisfy all the conditions in (*), except the trans. cond.

Thm 5 (Suff. cond.): If $(x^*(t), u^*(t), \lambda(t))$ satisfies the necessary conditions, $\mathcal{H}$ is concave in (x,u) $\forall t \in [0,T]$ and $\lim_{t\to\infty} \lambda(t)(x(t) - x^*(t)) \geq 0$ $\forall$ admissible $x(t)$, then $(x^*(t), u^*(t), \lambda(t))$ is optimal

CH8: out of place (end)

Stochastic: $z_t \sim$ Markov chain: $\mathbb{P}[z_t = z_i | z_0, \dots, z_{t-1}] = \mathbb{P}[z_t | z_{t-1}]$
BE: the value function in the infinite-horizon case satisfies
$V(x,z) = \max_{x' \in \Gamma(x,z)} \{F(x, x', z) + \beta\mathbb{E}[V(x', z') | z]\}$ $\forall x \in X$ and $\forall z \in Z$

$\mathcal{H} = e^{-\delta t}\frac{c(t)^{1-\sigma}-1}{1-\sigma} + \lambda(t)(ra-c)$

$\frac{\partial \mathcal{H}}{\partial c(t)} = 0 \Rightarrow e^{-\delta t}c(t)^{-\sigma} - \lambda(t) = 0 \Rightarrow e^{-\delta t}c(t)^{-\sigma} = \lambda(t)$ (1) $\quad \frac{\partial \mathcal{H}}{\partial a(t)} = -\dot\lambda \Rightarrow \lambda(t)r + \dot\lambda = 0 \Rightarrow -\dot\lambda = \lambda(t)r$ (2)

$\lim_{t\to\infty} \lambda(t)a(t) = 0$

Differentiating (1)

$\dot\lambda = -\delta e^{-\delta t}c(t)^{-\sigma} + e^{-\delta t}(-\sigma)c(t)^{-\sigma-1}\dot c \Rightarrow \dot\lambda = -e^{-\delta t}c(t)^{-\sigma}\left(\delta + \sigma\frac{\dot c}{c(t)}\right)$ (3)

(2) $\Rightarrow \lambda(t)r = -\dot\lambda \overset{(3)}{\Rightarrow} \lambda(t)r = \underbrace{e^{-\delta t}c(t)^{-\sigma}}_{\lambda(t)\ \text{by (1)}}\left(\delta + \sigma\frac{\dot c}{c(t)}\right) \Rightarrow r = \delta + \sigma\frac{\dot c}{c(t)} \Rightarrow \frac{\dot c}{c(t)} = \frac{r-\delta}{\sigma}$

Taking $\int_0^t (\cdot)\,ds$

$\int_0^t \frac{1}{c(s)}\frac{dc}{ds}\,ds = \int_0^t \frac{r-\delta}{\sigma}\,ds \Rightarrow \int_0^t \frac{1}{c(s)}\,dc = \int_0^t \frac{r-\delta}{\sigma}\,ds \rightarrow \left[\log c(t)\right]_0^t = \left[\frac{r-\delta}{\sigma}t\right]_0^t$

$\Rightarrow \log c(t) - \log c(0) = \frac{(r-\delta)}{\sigma}t \rightarrow \log c(t) = \log c(0) + \frac{(r-\delta)}{\sigma}t \rightarrow \boxed{c(t) = c(0)\,e^{\frac{(r-\delta)}{\sigma}t}}$

Also, by (2),

$\lambda(t)r = -\dot\lambda \Rightarrow r = -\frac{\dot\lambda}{\lambda(t)} \Rightarrow \int_0^t r\,ds = -\int_0^t \frac{1}{\lambda(t)}\frac{d\lambda}{ds}\,ds \Rightarrow \left[rt\right]_0^t = -\left[\log\lambda(t)\right]_0^t \Rightarrow rt = -\log\lambda(t) + \log\lambda(0)$

$\Rightarrow \log\lambda(t) = \log\lambda(0) - rt \Rightarrow \boxed{\lambda(t) = \lambda(0)\,e^{-rt}}$

CH8: Dynamic Programming

Finite time:

Principle of optimality: the sequence of value functions satisfies

$V_S(x) = \max_{u\in u(s,x)} \left\{ f(s,x,u) + V_{s+1}\left(g(s,x,u)\right)\right\}$, $s = 0,\dots,T-1$; $V_T(x) = \max_{a\in u(T,x)} f(T,x,u)$

(solved in backward induction)

Infinite-Horizon:

Bellman equation: the value function in the infinite-horizon problem satisfies

$V(x) = \max_{x'\in\Gamma(x)} \left\{ F(x,x') + \beta V(x')\right\}$ ← functional equation where the unknown function V appears on both sides (fixed point)

Functional Analysis

Contraction Mapping Thm: If (S,ρ) is a complete metric space and $T: S\to S$ is a cont. mapp. with modulus β then, (1) T has exactly one fixed-point $V\in S$ ($TV = V$); (2) For any $V_0\in S$, $\rho(T^n V_0, V) \le \beta^n \rho(V_0, V)$ $\quad (T^n V_0 \to V)$

Blackwell's Suff. Conds: Let $X\subseteq\mathbb{R}^l$ and $B(X) \equiv$ space bounded functions with sup-norm. Let $T: B(X)\to B(X)$ be an operator satisfying: (1) Monotonicity: If $f,g\in B(X)$ and $f(x)\le g(x)\ \forall x\in X$, then $Tf(x)\le Tg(x)\ \forall x\in X$; (2) $\exists\,\beta\in(0,1)$ s.t. $T(f+a)(x) \le Tf(x) + \beta a$, $\forall f\in B(X)$, $a\ge 0$, $x\in X$; Then T is a cont. mapp. with modulus β. **Proof:** For any $f,g\in B(X)$ we have that $f = g + f - g \le g + \|f-g\|$. Then, by monotonicity $Tf \le T(g + \|f-g\|)$; and by discounting $Tf \le Tg + \beta\|f-g\| \Rightarrow$ Reverse the order of f and g: $Tg \le T(f + \|g-f\|) \le Tf + \beta\|g-f\|$; thus $\|Tf - Tg\| \le \beta\|f-g\| = \sup_{x\in X}|Tf(x) - Tg(x)|$

Algorithms

Value Function Iteration: given $V_0\in C(X)$. Use operator $V_1 = TV_0 = \max_{x'\in\Gamma(x)}\left\{F(x,x') + \beta V_0(x)\right\}$ to find V_1. Check if $V_1 \simeq V_0$. If not, repeat until $V_{n+1}\simeq V_n$. The cont. mapp. thm ensures that this procedure converges to unique fixed point $V(x)$

$V(k) = \max_{k'\in[0,Ak^\alpha]}\left\{\log(Ak^\alpha - k') + \beta V(k')\right\}$, start with $V_0(k) = 0$, use $V_1(k)$, obtain $k' = 0$, write $V_2(k)$, find k', write $V_3(k)$, write $V_3(k)$...

Guess and Verify: guess $V(k) = E + F\log k$. Substitute in BE

$V(k) = \max_{k'\in[0,Ak^\alpha]}\left\{\log(Ak^\alpha - k') + \beta E + \beta F\log k'\right\} \Rightarrow k' = \frac{\beta F}{1+\beta F}Ak^\alpha$, reintroduce in BE and obtain E and F. $\rightarrow k' = \alpha\beta Ak^\alpha$

Euler and Transversality

For interior solution $\rightarrow$ EE: $D_{x'}F(x,x'_*) + \beta D_x \overbrace{V(x'_*)}^{?} = 0$ (9) $\overset{\text{envelope theorem}}{\rightarrow} D_x V(x) = D_x F(x,x'_*)$

Using notation $x'_* = x'_*(x)$ to denote optimal policy function, and $D_x V(x'_*) = D_x V(x'_*(x))$

$\rightarrow D_{x'}F(x,x'_*) + \beta D_x F(x'_*, x'_*(x'_*)) = 0$

We also need TC: $\lim_{t\to\infty}\beta^t D_x F(x_*, x'_*)\,x_* = 0$

CH9

Permanent Income Hypothesis: $\max_{\{c_t, a_{t+1}\}} \mathbb{E}_0\sum_{t=0}^{T}\beta^t u(c_t)$ s.t. $c_t + a_{t+1} = (1+r)a_t + z_t$ (markov)

BE: $V(a,z_t) = \max_{a'}\left\{u\left[(1+r)a + z_t - a'\right] + \beta\sum_{j=1}^{n} P_{ij}V(a',z_j)\right\}$

FOC: $\frac{\partial V}{\partial a'} = -u'(c) + \beta E\,\frac{\partial V(a',z_j)}{\partial a'} = 0$ (EE); envelope: $\frac{\partial V(a,z_i)}{\partial a} = (1+r)u'(c)$ $\rightarrow$ theorem $\Rightarrow \frac{\partial V(a',z_j)}{\partial a'} = (1+r)u'(c')$

$\rightarrow$ (EE): $u'(c) = \beta(1+r)\mathbb{E}[u'(c')]$

In ss $\Rightarrow \beta(1+r) = 1 \Rightarrow c = c' = \bar c$ $\Big\}$ $\forall a\in\infty$

$\beta(1+r) > 1 \rightarrow c < c'$

$\beta(1+r) < 1 \rightarrow c > c'$

Asset Pricing: $\max_{\{c_t,a_{t+1}\}} \mathbb{E}_0\sum_{t=0}^{\infty}\left(\frac{1}{1+\delta}\right)^t u(c_t)$ s.t. $a_{t+1} = (a_t - c_t)\left[(1+r)w_t + (1+z_t)(1-w_t)\right]$ (stochastic asset return)

BE: $V(a,z_t) = \max_{c,w}\left\{u(c) + \frac{1}{1+\delta}\mathbb{E}\left[V(a',z')\right]\right\}$ s.t. $a' = (a-c)\left[(1+r)w + (1+z_t)(1-w)\right]$; envelope: $V_a(a',z') = u'(c')$

FOCs: a': $u'(c) = \frac{1}{1+\delta}\mathbb{E}\left[V_a(a',z')\right]\left[(1+r)w + (1+z_t)(1-w)\right]$; w: $\mathbb{E}\left[V_a(a',z')(r-z_t)\right] = 0$

Suppose: $u(c) = \log c$ $\Rightarrow \frac{u'(c)}{u'(c')}(1+z) = \frac{1}{1+\delta}\mathbb{E}\left\{\frac{\alpha}{\alpha'}\left[(1+r)w + (1+z)(1-w)\right]\right\}$ ← c, α are proper RVs

z i.i.d $\Rightarrow \frac{1}{c} = \frac{1}{1+\delta}\frac{\alpha}{\alpha'}\,\mathbb{E}\left[(1+r)w + (1+z)(1-w)\right]$

$V(a) = \gamma + \alpha\log a \qquad \frac{1}{c} = \frac{\alpha}{1+\delta}\frac{1}{a-\bar c} \Rightarrow a = \left(\frac{\alpha}{1+\delta}\right)c_t + \underbrace{\frac{\gamma}{1+\delta}}_{c\ \text{as proper RVs}\ \to\ 0}$

For ss, we have $V(a) = u(c) + \frac{1}{1+\delta}V(a) = \log c + \frac{1}{1+\delta}V(a) \Rightarrow V(a) = \frac{1}{\delta}\log a - \frac{1-\delta}{\delta}\log\left(\frac{\alpha}{1+\delta}\right)$

To define w, thanks to both FOCs: $\mathbb{E}\left\{(r-z_t)\left[(1+r)w + (1+z_t)(1-w)\right]^{-1}\right\} = 0$

Continuous Time: $\max \int_0^\infty e^{-\delta t}u[c(t),k(t)]\,dt$ s.t. $\dot k = G(c,k)$, k_0 given, $c>0$

VF: $V(k_0) = \int_0^\infty e^{-\delta t}u(c^*,k^*)\,dt \overset{\text{principle of optimality}}{=} \max_{c_0}\left\{u(c_0)\,dt + e^{-\delta t}V(k^*(T))\right\}$

Describe: $\max \sum_{t=0}^{\infty} e^{-\delta t}u(c_t,k_t)$ s.t. $k_{t+1} - k_t = hG(c_t,k_t) \rightarrow W(k_t) = \max_{c_t}\left\{u(c_t,k_t)h + e^{-\delta h}V(k_{t+1})\right\}$

Make h change: (1) $-V(k_t)$; (2) $k_{t+1} = k_t + hG$; (3) $e^{-\delta h} = 1 - \delta h + \frac{(\delta h)^2}{2} - \dots$; (4) $\frac{1}{h}$

$\max_c\left\{u(c,k) - (\delta - \frac{(\delta h)^2}{2} + \dots)V(k_t + hG) + \frac{[V(k_t+hG) - V(k_t)]}{hG}\right\} = 0 \longrightarrow \delta V(k) - u(c,k) + V'(k)G(c,k) = 0$ (HJB)

Interior optimum: $u_c(c,k) = -G_c(c,k)V'(k)$

Diff (HJB) wrt k: $u_c(c,k) + V'(k)\left[G_k(c,k) - \delta\right] + V''(k)G(c,k) = 0 \Big\} \Rightarrow \delta = \frac{u_k + \lambda G_k + \dot\lambda}{\lambda}$ (Ramsey rule)

Define $\lambda = V'(k) \rightarrow V''(k) = \frac{\partial V(k)}{\partial k} = \frac{\partial\lambda}{\partial k}$ $\Bigg\}$ $\rightarrow \frac{\partial\lambda}{\partial k}\frac{\partial k}{\partial t} = \dot\lambda$

Example: $\max \int_0^\infty e^{-\delta t}u(c_t)\,dt$ s.t. $\dot k = f(k) - c$ (AFFE: $\delta = f'(k) + \frac{\dot\lambda}{\lambda}$)

HJB: $\max_c\left\{u(c) + V'(k)\left[G(c,k) - \delta k\right]\right\} = 0$; FOC: $u'(c) = -G_c(c,k)V'(k) = V'(k) = \lambda$

Via AFFE: $\dot c = -\frac{u'(c)}{u''(c)}\left[f'(k) - \delta\right]$ (system of non-linear diff eqs + TVC determines the optimal paths for c and k)

$\dot k = f(k) - c$

Assume $u(c) = \frac{c^{1-\frac{1}{\psi}} - 1}{1-\frac{1}{\psi}}$, $f(k) = rk$. NOTE: $c = rk$. Hence, $\dot k = (r-n)k \Rightarrow \frac{\dot k}{k} = (r-n) \overset{\text{use FOC}}{\Rightarrow} \frac{\dot c}{c} = \psi(r-\delta) \Leftrightarrow c = \left[(1)^{\frac{1}{\psi}}\right]k$

since $\frac{\dot c}{c} = \psi(r-\delta)$, $\mathcal{H} = \frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} + \mu\left[(1)^{\frac{1}{\psi}}k\right]$, $\mathcal{H}_c = 0$, $\mathcal{H}_k = -\dot\mu \rightarrow \frac{\dot\mu}{\mu} = -r$

finally $\frac{\dot\lambda}{\lambda} = \frac{\dot\mu}{\mu} \rightarrow \frac{\dot c}{c} = \psi(r-\delta)$

$\max \int_0^T [1-s(t)]\,a\,k(t)\,dt$ s.t. $\dot k(t) = a s(t) k(t)$, $k(0) = k_0$, $k(T) > k_T > k_0$ $\quad s(t)\in[0,1]$, $a>0$, $T > 1/a$, $k_0 e^{aT} > k_T$ $\quad t = \max\{T - \frac{1}{a}, \frac{1}{a}\ln(\frac{k_T}{k_0})\}$

Two cases together → $s^*(t) = \begin{cases} 1 & \text{if } t\in[0,\bar t] \\ 0 & \text{if } t\in[\bar t, T] \end{cases}$

$\mathcal{H} = [1-s(t)]\,a\,k(t) + p(t)\,a\,s(t)\,k(t) = a k(t)[s(t)(p(t)-1)] + a k(t)$

Hence, $s^*(t) = \begin{cases} 1 & \text{if } p(t) > 1 \\ 0 & \text{if } p(t) < 1 \end{cases}$

$\frac{\partial\mathcal{H}}{\partial k(t)} = -\dot p(t) \Rightarrow \dot p(t) = -a + s^*(t)[1-p(t)]\,a$

Hence $\dot p(t) = \begin{cases} -ap(t) & \text{if } p(t) > 1 \\ -a & \text{if } p(t) < 1 \end{cases}$ both cases $\dot p(t) < 0$ $\rightarrow p(t)$ strictly decreasing

Suppose $p(0) \le 1$ ($s^*(t) = 0\ \forall t$) $\rightarrow k^*(t) = k_0 \not> k(T) > k_T > k_0$

Hence $p(0) > 1$ ($s^*(t) = 1$ beginning, $s^*(t) = 0$ later)

Two cases

1) $p(T) = 0$

unique t^* s.t. $p(t^*) = 1$ with $p(t) > 1$ for $t\in[0,t^*]$

and $p(t) < 1$ for $t\in(t^*,T]$. Hence $s^*(t) = 1$ for $t\in[0,t^*]$

and $s^*(t) = 0$ for $t\in(t^*,T]$. Hence, $\dot p(t) = -ap(t)$ for $t\in[0,t^*]$

and $\dot p(t) = -a$ for $t\in(t^*,T]$

$(t^*,T]$ ($p(T)=0$)

$\dot p(t) = -a \rightarrow p(t) = -a(t-T)$

$p(t^*) = 1 \rightarrow -a(t^*-T) = 1 \rightarrow t^* = T - 1/a$

$k = ka \rightarrow k(t) = Ce^{at} \xrightarrow[k=k(0)]{} k(0) = C = k_0 \rightarrow k(t) = k_0 e^{at}$

$[0,t^*]$

$\frac{\dot p(t)}{p(t)} = -ap(t) \rightarrow p(t) = Ce^{-at} \rightarrow p(t) = e^{a(T-t)-1}$

$p(t^*) = 1 \rightarrow p(T - 1/a) = 1 \rightarrow C = \frac{1}{e^{aT+1}}$

$k = 0 \rightarrow k(t) = k_0 e^{at^*} = k_0 e^{aT-1}$

To finish we need to satisfy $k(T) > k_T \rightarrow k_0 e^{at^*} > k_T \rightarrow$

$\rightarrow t^* = T - \frac{1}{a} > \frac{1}{a}\ln\left(\frac{k_T}{k_0}\right)$

2) $p(T) = 0$ ($k(T) = k_T$)

$t\in[0,t^*]$, $s(t) = 1$, $k(t) = k_0 e^{at}$, $p(t) = e^{a(t^*-t)}$

$t\in(t^*,T]$, $s(t) = 0$, $k(t) = k_0 e^{at^*}$, $p(t) = 1 - a(t-t^*)$

$k(T) = k_T \rightarrow t^* = \frac{1}{a}\ln\left(\frac{k_T}{k_0}\right)$

$t^* < T \rightarrow k_0 e^{at^*} > k_T \xrightarrow[p(t)=0]{} T - \frac{1}{a} < \frac{1}{a}\ln\left(\frac{k_T}{k_0}\right) = t^*$

$\max \int_0^T x(t)\,dt$ s.t. $\dot x(t) = X(t) + u(t)$, $x(0) = 0$, $x(t)$ free, $u(t)\in[-1,1]$

$\mathcal{H} = x(t) + p(t)\left[x(t) + u(t)\right] = \underbrace{(1+p(t))}_{>0}x(t) + p(t)u(t)$

$\frac{\partial\mathcal{H}}{\partial x(t)} = -\dot p(t) \rightarrow \dot p(t) = -p(t) - 1 \rightarrow p(t) = Ae^{-t} - 1$ ← $x(t)$ free

$\rightarrow p^*(t) = e^{1-t} - 1 \rightarrow p(t) > 0\ \forall t\in[0,1) \rightarrow u^*(t) = 1$

$\dot x(t) = x(t) + 1 \rightarrow x(t) = Be^t - 1 \rightarrow x(0) = 0 = B - 1 \rightarrow x^*(t) = e^t - 1$