

Mathematics III
Problem Set 2: Differential Equations
Suggested Solutions

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Deadline is *Mon 27 November at 17:00*. Submission via email: jose.elias.gallegos@iies.su.se or in class. By that same time, I will upload solutions to my webpage, www.joseeliasgallegos.com

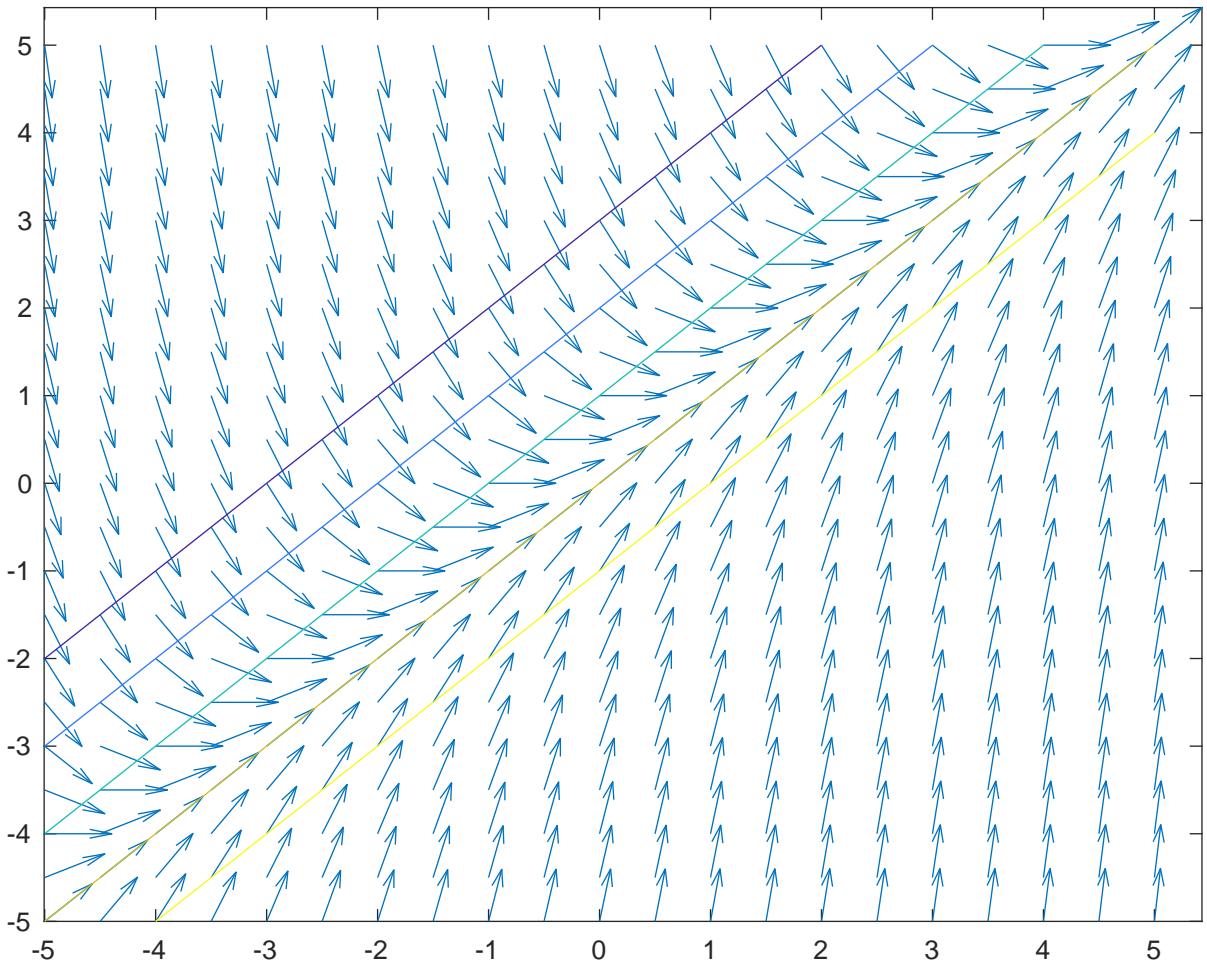
Exercise 1: Direction Fields

(a) Draw the direction field (using isoclines or otherwise) and the curve of $\dot{y} = 1 + t - y$.

Answer:

(a)

To find the steady-state of equation $\dot{y} = 1 + t - y$, assume it does not vary over time: $\dot{y} = 0$. This implies $y = 1 + t$, which is the red line.

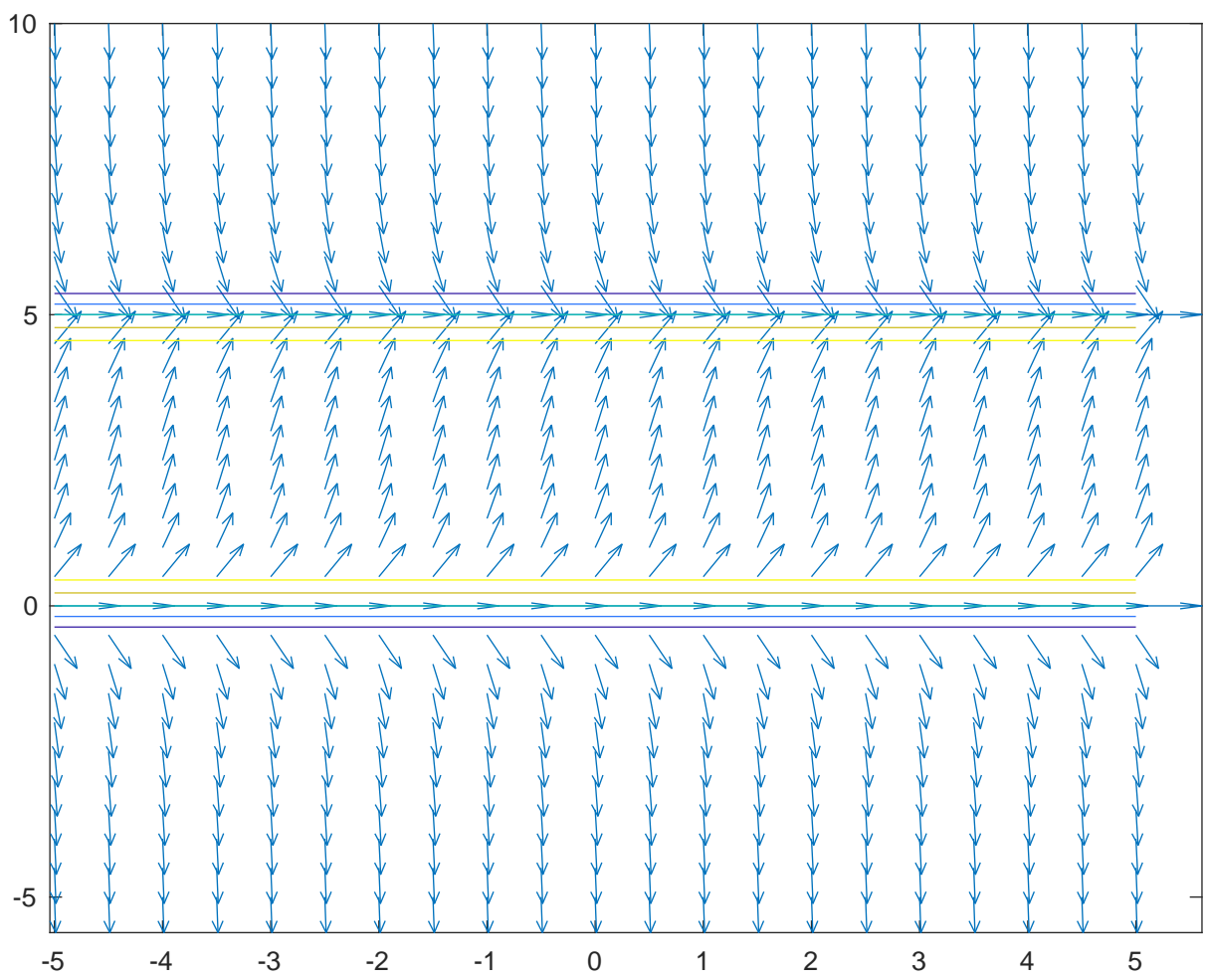


(b) Draw the direction field, phase portrait and stability line of $\dot{y} = y(5 - y)$.

Answer:

(b)

To find the steady-state of equation $\dot{y} = y(5 - y)$, assume it does not vary over time: $\dot{y} = 0$. This implies $y \in \{0, 5\}$, which are the red and orange lines.



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1 % [t y]=meshgrid(-5:0.5:5,-5:0.5:5);
2 % dy=1+t-y;
3 % dt=ones(size(dy));
4 % figure(1)
5 % quiver(t, y, dt, dy);
6 % hold on
7 % y = @(t) 1+t;
8 % fplot(y,[-5,4])
9 % hold off
10 % print(1, 'ex1a', '-dpdf')
11
12 int1 = [-5:0.5:5];
13 int2 = [-5:0.5:5];
14 [t,y]=meshgrid(int1,int2);
15 f=1+t-y;
16 dy=f;
17 dt=ones(size(dy));
18 L=sqrt(1+dy.^2);
19 figure(3)
20 quiver(t,y, dt./L, dy./L);
21 hold on
22 contour(t,y,1+t-y, [-2 -1, 0, 1 2]);
23 hold off
24 print(3, 'ex1a', '-dpdf')
25
26 % [t y]=meshgrid(-5:0.5:5,-5:0.5:10);
27 % dy=y.*(5-y);
28 % dt=ones(size(dy));
29 % figure(2)
30 % quiver(t, y, dt, dy);
31 % hold on
32 % y1 = @(t) 5;
33 % y2 = @(t) 0;
34 % fplot(y1,[-5,5])
35 % hold on
36 % fplot(y2,[-5,5])
37 % hold off
38 % print(2, 'ex1b', '-dpdf')
39

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40 int1 = [-5:0.5:5];
41 int2 = [-5:0.5:10];
42 [t,y]=meshgrid(int1,int2);
43 f=y.*(5-y);
44 dy=f;
45 dt=ones(size(dy));
46 L=sqrt(1+dy.^2);
47 figure(4)
48 quiver(t,y,dt./L,dy./L);
49 hold on
50 contour(t,y,y.*(5-y),[-2 -1, 0, 1 2]);
51 hold off
52 print(4,'ex1b','-dpdf')

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Exercise 2: Difference Equations

A loan of amount $\$L$ is taken out on January 1 of year 0. Instalment payments for the principal and interest are paid annually, commencing on January 1 of year 1. Let the interest rate be $r < 2$, so that the interest amounts to rL for the first payment.

(a) The contract states that the principal share of the repayment will be half the size of the interest share. Show that the debt after January 1 of year n is $(1 - \frac{r}{2})^n L$.

Answer:

(a) Denote $L_n \equiv$ remaining debt on Jan 1 in year n . Then, $L = L_0$.

The payment on the principal in year n is $L_{n-1} - L_n$.

The interest payment in year n is rL_{n-1} .

Hence, we have

$$L_{n-1} - L_n = \frac{1}{2}rL_{n-1} \implies L_n = \left(1 - \frac{r}{2}\right)L_{n-1} = \left(1 - \frac{r}{2}\right)^2 L_{n-2} = \dots = \left(1 - \frac{r}{2}\right)^n L_0 = \left(1 - \frac{r}{2}\right)^n L$$

(b) Find r when it is known that exactly half the original loan is paid after 10 years.

Answer:

(b) In this case, $n = 10$ and $L_{10} = \frac{1}{2}L$,

$$\begin{aligned}\frac{1}{2}L &= \left(1 - \frac{r}{2}\right)^{10} L \\ \frac{1}{2} &= \left(1 - \frac{r}{2}\right)^{10} \\ \left(\frac{1}{2}\right)^{\frac{1}{10}} &= 1 - \frac{r}{2} \\ 1 - \left(\frac{1}{2}\right)^{\frac{1}{10}} &= \frac{r}{2} \\ r &= 2 \left[2 - \left(\frac{1}{2}\right)^{\frac{1}{10}}\right]\end{aligned}$$

(c) What will the remaining payments be each year if the contract is not changed?

Answer:

(c) Total payment in year n will be

$$TP_n = L_{n-1} - L_n + rL_{n-1}$$

Since $L_n = \left(1 - \frac{r}{2}\right) L_{n-1}$,

$$\begin{aligned}TP_n &= L_{n-1} - \left(1 - \frac{r}{2}\right) L_{n-1} + rL_{n-1} \\ &= \frac{3}{2}rL_{n-1}\end{aligned}$$

and since $L_{n-1} = \left(1 - \frac{r}{2}\right)^{n-1} L$,

$$TP_n = \frac{3}{2}r \left(1 - \frac{r}{2}\right)^{n-1} L$$

As $n \rightarrow \infty$, $L_{n-1} \rightarrow 0$ since $r < 2$.

Exercise 3: Linear First-Order Difference Equations

Consider the first-order difference equation

$$x_{t+1} = a_t x_t + b_t, \quad t = 0, 1, 2, \dots \quad (1)$$

where $(a_t)_t$ and $(b_t)_t$ are sequences of real numbers. Show (using a proof by induction) that the general solution is given by

$$x_t = \left(\prod_{s=0}^{t-1} a_s\right) x_0 + \sum_{k=0}^{t-1} b_k \left(\prod_{s=k+1}^{t-1} a_s\right), \quad t = 0, 1, 2, \dots$$

Recall: We define $\prod_{s=k}^l a_s = a_k a_{k+1} \cdots a_l$ for $l = k, k+1, k+2, \dots$, and $\prod_{s=k}^l a_s = 1$ for $l < k$.

Answer:

- $t = 0$:

$$x_0 = \underbrace{\left(\prod_{s=0}^{t-1} a_s \right)}_1 x_0 + \underbrace{\sum_{k=0}^{t-1} b_k \left(\prod_{s=k+1}^{t-1} a_s \right)}_0 \stackrel{!}{=} x_0$$

- $t \rightarrow t+1$: Suppose we have

$$x_j = \left(\prod_{s=0}^{j-1} a_s \right) x_0 + \sum_{k=0}^{j-1} b_k \left(\prod_{s=k+1}^{j-1} a_s \right) \quad (2)$$

and it holds. Plugging (2) into (1) for $j = t$,

$$\begin{aligned} x_{t+1} &= a_t \left[\left(\prod_{s=0}^{t-1} a_s \right) x_0 + \sum_{k=0}^{t-1} b_k \left(\prod_{s=k+1}^{t-1} a_s \right) \right] + b_t \\ &= \left(\prod_{s=0}^t a_s \right) x_0 + \underbrace{\sum_{k=0}^{t-1} b_k \left(\prod_{s=k+1}^t a_s \right)}_{b_0 a_1 a_2 \cdots a_t + b_1 a_2 a_3 \cdots a_t + \dots + b_{t-1} a_t + b_t} \\ &= \left(\prod_{s=0}^t a_s \right) x_0 + \sum_{k=0}^t b_k \left(\prod_{s=k+1}^t a_s \right) \end{aligned}$$

Exercise 4: Second-Order Linear Difference Equations

(a) Determine the general solution of the second-order equation

$$x_{t+2} + 4x_t = 0$$

Is the equation globally asymptotically stable?

Answer:

(a) The characteristic equation is

$$m^2 + 0m + 4 = 0 \implies m_{1,2} = \pm \frac{i\sqrt{-16}}{2}$$

with $r = \sqrt{b} = \sqrt{4} = 2$, $\cos(\theta) = \frac{-a}{2\sqrt{b}} = 0 \implies \theta = \frac{\pi}{2}$. The general solution is

$$x_t = r^t [A \cos(\theta t) + B \sin(\theta t)] = 2^t \left[A \cos\left(\frac{t\pi}{2}\right) + B \sin\left(\frac{t\pi}{2}\right) \right]$$

Finally, x_t is not Globally Asymptotically Stable since $r = 2 > 1$.

(b) Find one particular solution of the equation

$$x_{t+2} + 4x_t = 8 \cdot 2^t \cdot \cos\left(\frac{t\pi}{2}\right)$$

and construct the general solution of this equation using your result from part (a).

Answer:

(b) Make an Ansatz: $w_t^* = t8C2^t \cos\left(\frac{t\pi}{2}\right) + t8D2^t \sin\left(\frac{t\pi}{2}\right)$ is a solution, for some coefficients $C, D \in \mathbb{R}$. Hence,

$$\begin{aligned} w_{t+2}^* + 4w_t^* &= (t+2)8C2^{(t+2)} \underbrace{\cos\left(\frac{(t+2)\pi}{2}\right)}_{\cos\left(\frac{t\pi}{2}+\pi\right)} + (t+2)8D2^{(t+2)} \underbrace{\sin\left(\frac{(t+2)\pi}{2}\right)}_{\sin\left(\frac{t\pi}{2}+\pi\right)} + \\ &\quad + 4t8C2^t \cos\left(\frac{t\pi}{2}\right) + 4t8D2^t \sin\left(\frac{t\pi}{2}\right) \\ &= (t+2)8C2^t 2^2 \left[\cos\left(\frac{t\pi}{2}\right) \underbrace{\cos(\pi)}_{-1} - \sin\left(\frac{t\pi}{2}\right) \underbrace{\sin(\pi)}_0 \right] + \\ &\quad + (t+2)8D2^t 2^2 \left[\sin\left(\frac{t\pi}{2}\right) \underbrace{\cos(\pi)}_{-1} + \cos\left(\frac{t\pi}{2}\right) \underbrace{\sin(\pi)}_0 \right] + \\ &\quad + 4t8C2^t \cos\left(\frac{t\pi}{2}\right) + 4t8D2^t \sin\left(\frac{t\pi}{2}\right) \\ &= 8 \cdot 2^t \cos\left(\frac{t\pi}{2}\right) (-8C) + 8 \cdot 2^t \sin\left(\frac{t\pi}{2}\right) (-8D) \\ &\stackrel{!}{=} 8 \cdot 2^t \cdot \cos\left(\frac{t\pi}{2}\right) \end{aligned}$$

w_t^* guess solves the equation when $-8C = 1$ and $-8D = 0$, hence $C = -\frac{1}{8}$ and $D = 0$. Therefore,

$$w_t^* = -t2^t \cos\left(\frac{t\pi}{2}\right)$$

and the particular solution is

$$x_t = 2^t \left[A \cos\left(\frac{t\pi}{2}\right) + B \sin\left(\frac{t\pi}{2}\right) \right] - t2^t \cos\left(\frac{t\pi}{2}\right)$$

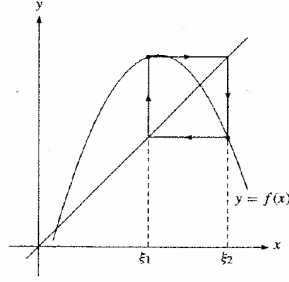
Exercise 5: Cycle of Period 2

The function f in the figure is given by $f(x) = -x^2 + 4x - \frac{4}{5}$.

Find the values of the cycle points ξ_1 and ξ_2 , and use

$$|f'(\xi_1)f'(\xi_2)| < 1 \iff 4 < (b-1)^2 - 4ac < 6$$

to determine whether the cycle is stable. It is clear from the figure that the difference equation



$x_{t+1} = f(x_t)$ has two equilibrium states. Find these equilibria, show that they are both unstable, and *verify* that if $f : I \rightarrow I$ is continuous and the difference equation $x_{t+1} = f(x_t)$ admits a cycle ξ_1, ξ_2 of period 2, it also has at least one equilibrium solution between ξ_1 and ξ_2 . (*Hint: Consider the function $f(x) - x$ over the interval with endpoints ξ_1 and ξ_2 .*)

Answer:

Let me first find the two equilibrium states. There is a stable point if $x = f(x)$,

$$x = -x^2 + 4x - \frac{4}{5} \implies -x^2 + (4 - 1)x - \frac{4}{5} = 0$$

(in our case, $a = -1$, $b = 4$ and $c = -\frac{4}{5}$). Hence,

$$x_{1,2} = \frac{1 - b \pm \sqrt{(b - 1)^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{\frac{29}{5}}}{-2}$$

it follows that x_1 is always unstable, and x_2 is also unstable if

$$(b - 1)^2 - 4ac = \frac{29}{5} > 4$$

According to FMEA, the cycle points are the roots of the equation $g(x) = a^2x^2 + a(b + 1)x + ac + b + 1 = 0$,

$$\xi_{1,2} = \frac{-(b + 1) \pm \sqrt{(b - 1)^2 - 4ac - 4}}{2a}$$

Hence, $\xi_{1,2} = \frac{-5 \pm \sqrt{\frac{41}{5}}}{-2}$.

To show that there is a cycle of period 2,

$$|f'(\xi_1)f'(\xi_2)| = |4ac - (b - 1)^2 + 5| = \left| -\frac{4}{5} \right| < 1$$

Hence, this function is presents a stable cycle of period 2.

Finally, I will verify that there is at least one equilibrium solution between ξ_1 and ξ_2 ,

$$x_1 = 0.2958\dots$$

$$x_2 = 2.7042\dots$$

$$\xi_1 = 1.0682\dots$$

$$\xi_2 = 3.9318\dots$$

Notice that $x_2 \in [\xi_1, \xi_2]$.