Mathematics III
Problem Set 4: Dynamic Optimization II
Suggested Solutions

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December 17, 2017

Deadline is *Tue 19 December at 9:00.* Submission via email: jose.elias.gallegos@iies.su.se. By that same time, I will upload solutions to my webpage, joseeliasgallegos.com. I suggest you to have a look at them before coming to class.

Exercise 1: Value Function Iterations (30 points)

Consider the dynamic optimization problem,

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \quad s.t. \quad k_{t+1} + c_t = Ak_t^{\alpha},\ c_t > 0,$$

where $A > 0$, α, $\beta \in (0, 1)$ and $k_0 > 0$ is given.
(a) Let V_n be the sequence of value function iterations obtained by starting from $V_0 = 0$. Show that these functions all have the format:

$$V_n(k) = X_n + Y_n \log k \quad \text{for some constants } X_n,\ Y_n \in \mathbb{R}.$$

Answer:

(a) We can write

$$\max_{k_{t+1}} \sum_{t=0}^{\infty} \log(Ak_t^{\alpha} - k_{t+1}) \tag{1}$$

guessing $V_n(k) = X_n + Y_n \log k$, and knowing $V_0(k) = 0$, by induction:

1. Check for $n = 0$

$$V_0 = X_0 + Y_0 \log k \overset{!}{=} 0 \implies X_0 = 0, Y_0 = 0$$

*Solutions adapted from previous work by Sebastian Koehne, Alexandre N. Kohlhas and Has van Vlokhoven

2. Assume $V_n(k) = X_n + Y_n \log k$. We must show that we have

$$V_{n+1} = \max_{k'} \left\{ \log(Ak^\alpha - k') + \beta V_n(k') \right\} \overset{!}{=} X_{n+1} + Y_{n+1} \log k$$

Plugging in the guess

$$V_{n+1} = \max_{k'} \left\{ \log(Ak^\alpha - k') + \beta(X_n + Y_n \log k') \right\}$$

Solve the max: taking the FOC wrt k',

$$-\frac{1}{Ak^\alpha - k'} + \beta \frac{Y_n}{k'} = 0 \implies k' = \frac{\beta Y_n A k^\alpha}{1 + \beta Y_n}$$

Hence, substituting

$$\begin{aligned} V_{n+1} &= \log\left(Ak^\alpha - \frac{\beta Y_n A k^\alpha}{1+\beta Y_n}\right) + \beta\left(X_n + Y_n \log \frac{\beta Y_n A k^\alpha}{1+\beta Y_n}\right) \\ &= \alpha(1+\beta Y_n)\log k + \log\left(\frac{A}{1+\beta Y_n}\right) + \beta X_n + \beta Y_n \log\left(\frac{A\beta Y_n}{1+\beta Y_n}\right) \overset{!}{=} X_{n+1} + Y_{n+1}\log k \end{aligned}$$

Hence,

$$\begin{aligned} X_{n+1} =& \log\left(\frac{A}{1+\beta Y_n}\right) + \beta X_n + \beta Y_n \log\left(\frac{A\beta Y_n}{1+\beta Y_n}\right) \\ Y_{n+1} =& \alpha(1+\beta Y_n) \end{aligned}$$

(b) Determine the $\lim_{n\to\infty} Y_n$ and $\lim_{n\to\infty} X_n$.
Use these results to find the policy functions for k' and c.

Answer:

(b) Since $Y_{n+1} = \alpha(1+\beta Y_n)$, we know from difference equations that

$$Y_n = (\alpha\beta)^n \left(Y_0 - \frac{\alpha}{1-\alpha\beta}\right) + \frac{\alpha}{1-\alpha\beta}$$

for $\alpha, \beta < 1$. Hence,

$$\lim_{n\to\infty} Y_n = \frac{\alpha}{1-\alpha\beta}$$

Now, turning to X_n,

$$\begin{aligned} X_{n+1} &= \log\left(\frac{A}{1+\beta Y_n}\right) + \beta X_n + \beta Y_n \log\left(\frac{A\beta Y_n}{1+\beta Y_n}\right) \implies \\ \implies X_{n+1} - \beta X_n &= \log\left(\frac{A}{1+\beta Y_n}\right) + \beta Y_n \log\left(\frac{A\beta Y_n}{1+\beta Y_n}\right) \end{aligned}$$

Taking limits,

$$\lim_{n\to\infty}\{X_{n+1}-\beta X_n\} \simeq (1-\beta)\lim_{n\to\infty} X_n \implies$$
$$\implies \lim_{n\to\infty} X_n = \frac{1}{1-\beta}\left\{\lim_{n\to\infty}\left[\log\left(\frac{A}{1+\beta Y_n}\right)+\beta Y_n \log\left(\frac{A\beta Y_n}{1+\beta Y_n}\right)\right]\right\} =$$
$$= \frac{1}{1-\beta}\left\{\log[A(1-\alpha\beta)] + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta A)\right]$$

(c) Show that the policy function for c satisfies an extended Euler equation.
Is the solution unique?
Answer:

(c) The value function is

$$V(k) = \max_{k'}\left\{\log(Ak^\alpha - k') + \beta V(k')\right\}$$

Taking the FOC,

$$\frac{1}{Ak^\alpha - k'} = \beta \underbrace{\frac{\partial V(k')}{\partial k'}}_{?}$$

But what is "?"? First, get $\frac{\partial V(k)}{\partial k}$. Define optimal savings $k'_* = g(k)$. Then, we can write

$$V(k) = \log[Ak^\alpha - g(k)] + \beta V[g(k)]$$

Taking the FOC wrt k,

$$\frac{\partial V(k)}{\partial k} = \frac{A\alpha k^{\alpha-1} - g'(k)}{Ak^\alpha - g(k)} + \beta V'[g(k)]g'(k) = \frac{A\alpha k^{\alpha-1}}{Ak^\alpha - g(k)} + g'(k)\underbrace{\left[\beta V'[g(k)] - \frac{1}{Ak^\alpha - g(k)}\right]}_{0}$$

Where the bracket part is 0 (it is the FOC!) Hence,

$$\frac{\partial V(k)}{\partial k} = \frac{A\alpha k^{\alpha-1}}{Ak^\alpha - g(k)}$$

Moving it one period forward,

$$\frac{\partial V(k')}{\partial k'} = \frac{A\alpha (k')^{\alpha-1}}{A(k')^\alpha - g(k')}$$

Hence, we can finally come back to the Euler equation and plug it in,

$$\frac{1}{Ak^\alpha - k'} = \beta\frac{A\alpha [g(k)]^{\alpha-1}}{A[g(k)]^\alpha - g[g(k)]}$$

We know that the policy function for k' satisfies

$$k' = \frac{\beta Y_n A k^\alpha}{1 + \beta Y_n} \underset{n\to\infty}{\rightarrow} \alpha\beta A k^\alpha$$

and c is,

$$c = Ak^\alpha - k' \underset{n\to\infty}{\rightarrow} (1 - \alpha\beta) A k^\alpha$$

and plug it in the Euler equation, we see that it is satisfied.

Notice that the solution is not unique: the Euler equation is a second-order difference equation, we need to initial values but we only have one (k_0). This is why we need the transversality condition!

Exercise 2: Guess and Verify (20 points)

Consider a problem with a binary shock $\theta \in \{\theta_1, \theta_2\} > 0$. The transition from the current shock i to next period's shock j happens with probability π_{ij}. The Bellman equation for the problem is:

$$V(k, i) = \max_{c,k'} \left\{ \log c + \beta \sum_j \pi_{ij} V(k', j) \right\} \quad s.t. \quad c + k' = \theta_i k^\alpha.$$

Find the value function using the guess-and-verify method with the guess:

$$V(k, i) = \gamma_{1,i} + \gamma_2 \log k + \gamma_3 \log \theta_i.$$

Note: *1)* The solution for $\gamma_{1,i}$ will depend on the transition probabilities π_{ij} and cannot be solved in closed form. *2)* You don't need to verify that the solution to the Bellman equation is unique.
Answer:

Plug the guess into the Bellman equation and take the FOC wrt k'. The FOC is

$$\frac{1}{\theta_i k^\alpha - k'} = \beta\gamma_2 \frac{1}{k'} \implies k'_* = \frac{\theta_i \beta \gamma_2}{1 + \beta\gamma_2} k^\alpha$$

Plug k'_* into the Bellman equation,

$$
\begin{aligned}
V(k,i) &= \log\left(\frac{\theta_i}{1+\beta\gamma_2}k^\alpha\right) + \beta\sum_j \pi_{ij}\left[\gamma_{1,j} + \gamma_2\log\left(\frac{\theta_i\beta\gamma_2}{1+\beta\gamma_2}k^\alpha\right) + \gamma_3\theta_j\right] = \\
&= (1+\beta\gamma_2)\alpha\log k + (1+\beta\gamma_2)\log\theta_i - \log(1+\beta\gamma_2) + \beta\sum_j \pi_{ij}\left[\gamma_{1,j} + \gamma_2\log\left(\frac{\beta\gamma_2}{1+\beta\gamma_2}\right) + \gamma_3\theta_j\right] \\
&= (1+\beta\gamma_2)\alpha\log k + (1+\beta\gamma_2)\log\theta_i - \log(1+\beta\gamma_2) + \beta\gamma_2\log(\beta\gamma_2) - \beta\gamma_2\log(1+\beta\gamma_2) + \\
&\qquad + \beta\sum_j \pi_{ij}\gamma_{1,j} + \beta\gamma_3\sum_j \pi_{ij}\theta_j \\
&= (1+\beta\gamma_2)\alpha\log k + (1+\beta\gamma_2)\log\theta_i - (1+\beta\gamma_2)\log(1+\beta\gamma_2) + \beta\gamma_2\log(\beta\gamma_2) + \beta\sum_j \pi_{ij}\gamma_{1,j} + \\
&\qquad + \beta\gamma_3\sum_j \pi_{ij}\theta_j
\end{aligned}
$$

Inserting our guess on the LHS,

$$
\gamma_{1,i} + \gamma_2\log k + \gamma_3\log\theta_i \stackrel{!}{=} (1+\beta\gamma_2)\alpha\log k + (1+\beta\gamma_2)\log\theta_i - (1+\beta\gamma_2)\log(1+\beta\gamma_2) + \beta\gamma_2\log(\beta\gamma_2) + \\
+ \beta\sum_j \pi_{ij}\gamma_{1,j} + \beta\gamma_3\sum_j \pi_{ij}\theta_j
$$

Hence,

$$
\begin{aligned}
\gamma_2 &= (1+\beta\gamma_2)\alpha \implies \gamma_2 = \frac{\alpha}{1-\alpha\beta} \\
\gamma_3 &= 1+\beta\gamma_2 = \frac{1}{1-\alpha\beta} \\
\gamma_{1,i} &= -(1+\beta\gamma_2)\log(1+\beta\gamma_2) + \beta\gamma_2\log(\beta\gamma_2) + \beta\sum_j \pi_{ij}\gamma_{1,j} + \beta\gamma_3\sum_j \pi_{ij}\theta_j \\
&= \frac{1}{1-\alpha\beta}\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta) - \frac{\alpha\beta}{1-\alpha\beta}\log(1-\alpha\beta) + \beta\sum_j \pi_{ij}\gamma_{1,j} + \frac{\beta}{1-\alpha\beta}\sum_j \pi_{ij}\theta_j \\
&= \log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta) + \beta\sum_j \pi_{ij}\gamma_{1,j} + \frac{\beta}{1-\alpha\beta}\sum_j \pi_{ij}\theta_j
\end{aligned}
$$

The last condition needs to hold for all $i = 1, ..., m$, which generates a system of m equations with m unknowns $\gamma_{1,1}, ..., \gamma_{1,m}$. We cannot simplify this further without knowledge of transition probabilities π_{ij}.

Exercise 3: Contraction Mapping (20 points)

Show that the Bellman equation for *McCall's Job Search Model* covered in lectures defines a contraction using *Blackwell's Sufficient Conditions.* What are the conclusions from proving this?

(Sketch) Answer: Consider the operator T defined on our Bellman Equation from class:

$$(TV)(w) = \max_{accept,\ reject} \left\{ \frac{\bar{w}}{1-\beta},\ b + \beta \int_0^B V(w')dF(w') \right\}. \tag{2}$$

We want to show that this operator satisfies Blackwell's Sufficient Conditions:

- *$T : \mathcal{B}(X) \to \mathcal{B}(X)$: Suppose $V(w)$ is bounded for all $w \in [0, B]$. Then, since $\beta \in (0, 1)$, $\bar{w} \in [0, B]$, b is finite and so too is the integral of a bounded function over a closed set, TV is also bounded. Thus, $T : \mathcal{B}(X) \to \mathcal{B}(X)$.*

- *Monotonicity: Suppose $H(w) \geq V(w)$ for all $w \in [0, B]$. Then,*

$$(TH)(w) = \max_{accept,\ reject} \left\{ \frac{\bar{w}}{1-\beta},\ b + \beta \int_0^B H(w')dF(w') \right\}. \tag{3}$$

  Now notice: (a) that the left-hand side of the max-operator in (3) is equal to the left-hand side in (2); (b) that the right-hand side of the max-operator in (3) by contrast is always larger than or equal to the right-hand side in (2). The latter follows from the fact that integral operator itself preserves monotonicity . Combined (a) and (b) show that $TH(w) \geq TV(w)$. This shows that the operator satisfies the monotonicity condition.

- *Discounting: Consider,*

$$\begin{aligned} [T(V+a)](w) &= \max_{accept,\ reject} \left\{ \frac{\bar{w}}{1-\beta},\ b + \beta \int_0^B \left[V(w') + a\right] dF(w') \right\}. \\ &= \max_{accept,\ reject} \left\{ \frac{\bar{w}}{1-\beta},\ b + \beta \int_0^B V(w')dF(w') + \beta a \right\} \\ &\leq \max_{accept,\ reject} \left\{ \frac{\bar{w}}{1-\beta},\ b + \beta \int_0^B V(w')dF(w') \right\} + \beta a = TV(w) + \beta a. \end{aligned}$$

  This shows that the operator also satisfies the discounting property.

Thus, the operator T satisfies all of Blackwell's Sufficient Conditions for a Contraction Mapping. There therefore exists a unique value function V that solves our Bellman Equation, and for any V_0 in $\mathcal{B}(X)$ we converge towards said fixed point (i.e. $T^n V_0 \to V$).

Exercise 4: Savings Under Uncertainty (30 points)

Suppose a person seeks to maximize,

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\alpha_0 c_t - \frac{\alpha_1}{2} c_t^2 \right) \quad s.t. \quad a_{t+1} = (1+r)(a_t + y_t - c_t), \tag{4}$$

where $y_{t+1} = y + \epsilon_{t+1}$, $\epsilon_{t+1} \sim i.i.d.F\left(0,\ \sigma_\epsilon^2\right)$ and $\alpha_0\,, \alpha_1 > 0$.

(a) Formulate this problem as a dynamic programming problem.

(b) Show that there exits a value function $V(k)$ and that $V(k)$ is continuous and strictly concave.

(c) Assuming that $V(k)$ is differentiable, characterize the extended Euler equation that determines the optimal path of c and a.

(d) Is this Euler equation enough to determine the path of c and a? If not, what other condition do we need to impose? Write down this condition and explain it intuitively.

(Sketch) Answer:

(a) Let's first find the state variables to our problem. State variables describes the location of our system at time t and change over time. In this case $\{a_t, y_t\}$*. The Bellman Equation for our problem can therefore be written as,*

$$\begin{aligned} V(a, y) &= \max_{c, a'} \left\{ \alpha_0 c - \frac{\alpha_1}{2} c^2 + \beta \mathbb{E}\left[V(a', y')\right] \right\} && (5) \\ a' &= (1+r)(a+y-c) && (6) \\ y' &= y + \epsilon', \quad \epsilon' \sim i.i.d.F\left(0, \sigma_\epsilon^2\right). \end{aligned}$$

(b) Consider the per-period utility function $u(c) = \alpha_0 c - \frac{\alpha_1}{2} c^2$*. This function is only monotonically increasing (and positive) for* $c \in \left[0, \frac{\alpha_0}{\alpha_1}\right]$*. We will assume that this condition always holds below. There are several equivalent answers to this question: First, one can show using the method of undetermined coefficients that V is a quadratic function in a and y that is strictly concave. Second, one can argue that our problem satisfies Assumptions A.1 − A.5 in the lecture notes.*[1] *Specifically,*

- *The infinite sum in (4) always exits and so to does a choice of* a'.
- *The choice set and equation-of-motion correspondence can be made compact under suitable assumptions (see footnote 1). The equation for* a' *is continuous.*
- *The utility function is strictly increasing in a once one substitutes (6) into (5). The equation-of-motion for* a' *is also monotone.*
- *The utility function is strictly concave and differentiable.*

(c) Substitute the equation for a' *into (5). The (sufficient) first order condition to our problem then becomes,*

$$\alpha_0 - \alpha_1 c = \beta(1+r)\mathbb{E}\left[V_{a'}\left(a', y'\right)\right].$$

The Envelope Theorem however tell us that,

$$V_a(a, y) = u'(c) \rightarrow V_{a'}\left(a', y'\right) = u'(c').$$

Thus,

$$\alpha_0 - \alpha_1 c = \beta(1+r)\mathbb{E}\left[\alpha_0 - \alpha_1 c'\right],$$

[1] *Here, with additional points for any discussion of how to make the equation-of-motion correspondence for* a' *compact (see also Problem 2 from our last lecture and the assumption that* $c \in \left[0, \frac{\alpha_0}{\alpha_1}\right]$*).*

which provides us with our extended Euler equation.

(d) No, we also need the transversality condition (see Theorem 16 in the Lecture Notes and p. 627 in Acemoglu (2008) for the stochastic case):

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[(\alpha_0 - \alpha_1 c_t)\, a_t\right] = 0. \tag{7}$$

(Notice the presence of the expectation operator in (7) since income, and hence assets and consumption, are now stochastic.) The interpretation of the transversality condition is as follows: the expected present discounted utility of assets "left-over" needs to converge to zero. Otherwise, we could have used those assets to increase expected utility in finite time.