Mathematics III
Problem Set 3: Dynamic Optimization I
Suggested Solutions

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Deadline is *Mon 11 December at 14:00.* Submission via email: jose.elias.gallegos@iies.su.se. By that same time, I will upload solutions to my webpage, www.joseeliasgallegos.com. I suggest you to have a look at them before coming to class.

Exercise 1: Bang-Bang Control

Consider the following dynamic optimization problem,

$$\max_{u \in [-1,1]} \int_0^1 (2x - x^2) dt$$
$$s.t. \quad \dot{x} = u$$
$$x(0) = 0$$
$$x(1) = 0$$

(a) Write down the conditions of the maximum principle. Are these conditions sufficient?
Answer:

Pontryagin's Maximum Principle provides necessary conditions for the optimality of an admissible pair to the standard problem.

Theorem A. *Suppose* $(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t))$ *solves the standard problem. Then, there exists a continuous and piece-wise differentiable function* $\lambda(t)$ *and a number* $\lambda_0 \in \{0, 1\}$ *such that for all* $t \in [0, T]$, $(\lambda_0, \lambda(t)) \neq (0, \mathbf{0})$, *and*

- $\boldsymbol{u}^*(t)$ *maximizes* $\mathcal{H}$ $\forall \boldsymbol{u} \in \mathcal{U}$,

$$\mathcal{H}(t, \boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \lambda(t)) \geq \mathcal{H}(t, \boldsymbol{x}^*(t), \boldsymbol{u}(t), \lambda(t)) \qquad \forall \boldsymbol{u} \in \mathcal{U}$$

- *Whenever* $\boldsymbol{u}^*(t)$ *is continuous,*

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}(t)}$$

- *Corresponding to the terminal condition, a transversality condition holds:*
  - $\lambda(T) = \mathbf{0}$ *($x(T)$ free)*
  - $\lambda(T) \geq \mathbf{0}$ *($\boldsymbol{x}(T) \geq \boldsymbol{x}_1$)*
    $\lambda(T)[\boldsymbol{x}(T) - \boldsymbol{x}_1] = \mathbf{0}$
  - $\boldsymbol{x}(T) = \boldsymbol{x}_1$

(b) Show that $\lambda(t)$ is decreasing
Answer:

Form the Hamiltonian

$$\mathcal{H} = 2x(t) - x(t)^2 + \lambda(t)u(t)$$

hence,

$$u^*(t) = \begin{cases} 1 & \text{if } \lambda(t) > 0 \\ -1 & \text{if } \lambda(t) < 0 \end{cases}$$

We know from the maximum principle that $\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}(t)}$. Hence,

$$\dot{\lambda}(t) = -[2 - 2x(t)] = 2[x(t) - 1]$$

Notice that $\dot{x}(t) = u^*(t)$. Hence, $\dot{x}(t) \leq 1$,

$$\frac{dx}{dt} \leq 1 \implies dx \leq dt \implies \int dx \leq \int dt \implies x(t) + C \leq t + D \implies x(t) \leq t + E$$

We need to pin down E. Notice that the above expression can hold with equality, and that $x(0) = 0$. Hence, $0 = 0 + E \implies E = 0$. As a result, $x(t) \leq t$. Therefore, $x(t) \leq 1 \; \forall t \in [0, 1)$, and $x(T) = x(1) = 0$ by the text.

[It is also easily seen if you take $dt = t - 0 = t$ and $dx = x(t) - x(0) = x(t)$]

Finally,

$$\dot{\lambda}(t) = 2[x(t) - 1] < 0 \quad \forall t \in [0, 1]$$

So we have proved that $\lambda(t)$ is strictly decreasing in $[0, 1]$!

(c) Suppose that $\lambda(t)$ has a unique crossing with the zero-line at $\hat{t} = 0.5$. Use this to derive $u^*(t), \lambda(t)$ and $x^*(t)$.
Answer:

Thanks to the hint $\lambda(\hat{t}) = 0$ and $\lambda(t)$ decreasing, we know that $\lambda(t) > 0$ in $[0, \hat{t}]$ and $\lambda(t) < 0$ in $[\hat{t}, T] = [\hat{t}, 1]$. Lets do the two cases

- $[0, \hat{t}]$: in this case $u^*(t) = \dot{x}^*(t) = 1$. This implies that $x(t) = t + C$. We need to pin down C: since $x(0) = 0 \implies 0 = 0 + C \implies C = 0$, and therefore $x^*(t) = t$. Finally

$$\begin{aligned} \dot{\lambda}(t) &= 2[x(t) - 1] = 2(t - 1) \\ d\lambda &= 2(t-1)dt \\ \int d\lambda &= 2\int (t-1)dt \\ \lambda(t) &= t(t-2) + D \end{aligned}$$

  We need to pin down D. We know that $\lambda(\hat{t}) = \lambda(0.5) = 0 \implies 0 = 0.5(0.5 - 2) + D \implies D = \frac{3}{4}$. Hence, $\lambda(t) = t(t-2) + \frac{3}{4}$.

- $[\hat{t}, T] = [\hat{t}, 1]$: in this case $u^*(t) = \dot{x}^*(t) = -1$. This implies that $x(t) = -t + C$. We need to pin down C: since $x(T) = x(1) = 0 \implies 0 = -1 + C \implies C = 1$, and therefore $x^*(t) = 1 - t$. Finally

$$\begin{aligned} \dot{\lambda}(t) &= 2[x(t) - 1] = -2t \\ d\lambda &= -2tdt \\ \int d\lambda &= -2\int tdt \\ \lambda(t) &= -t^2 + D \end{aligned}$$

  We need to pin down D. We know that $\lambda(\hat{t}) = \lambda(0.5) = 0 \implies 0 = -(0.5)^2 + D \implies D = \frac{1}{4}$. Hence, $\lambda(t) = -t^2 + \frac{1}{4}$.

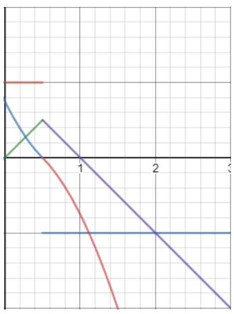

Figure 1: u^*: strong red and strong blue, x^*: green and purple, λ^*: light blue and light red

Exercise 2: A Distance Problem

Suppose $x_0 < x_1 < x_0 + T$. Solve the problem

$$\max_{u\in[0,1]} \int_0^T x \ dt$$
$$s.t. \quad \dot{x} = u$$
$$x(0) = x_0$$
$$x(T) = x_1$$

What classes of problems does this problem attempt to solve? Discuss.

Answer:

A cake-eating problem. Let us now solve it. Form the Hamiltonian

$$\mathcal{H} = x(t) + \lambda(t)u(t)$$

hence,

$$u^*(t) = \begin{cases} 1 & \text{if } \lambda(t) > 0 \\ 0 & \text{if } \lambda(t) < 0 \end{cases}$$

By the maximum principle we know that $\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}(t)}$. Hence,

$$\dot{\lambda}(t) = -1 \implies d\lambda = -dt \implies \lambda(t) = -t + C$$

We know that $\lambda(t)$ is decreasing, and we know $u(t)$ in terms of $\lambda(t)$. Let us now guess how does $\lambda(t)$ behave

- Suppose $\lambda(t) > 0 \ \forall t$. This implies $u^*(t) = 1 = \dot{x}(t)$, which in turn gives $x(t) = t + D$. We need to pin down D,

$$x(0) = x_0 = D \implies x(t) = t + x_0$$
$$x(T) = x_1 \implies D = x_1 - T \implies x(t) = t + x_1 - T$$

  which is not consistent since $x_0 \neq x_1 - T$.

- Suppose $\lambda(t) < 0 \ \forall t$. This implies $u^*(t) = 0 = \dot{x}(t)$, which in turn gives $x(t) = D$. We need to pin down D,

$$x(0) = x_0 = D \implies x(t) = x_0$$
$$x(T) = x_1 = D \implies x(t) = x_1$$

which is not consistent since $x_0 \neq x_1$.

- Hence, it must be that $\lambda(t)$ is sometimes positive and sometimes negative. Since we know that it is decreasing, it must be positive in the beginning and negative later: $\lambda(t) > 0$ in $[0, \hat{t}]$ and $\lambda(t) < 0$ in $[\hat{t}, T]$

  – $[0, \hat{t}]$: since $\lambda(t) > 0$, $u^*(t) = 1 = \dot{x}(t)$, which in turn gives $x(t) = t + D$. We need to pin down D,

  $$x(0) = x_0 = D \implies x^*(t) = t + x_0$$

  and we need to find $\lambda(t)$. We need to pin down C. We know that $\lambda(\hat{t}) = 0$. Hence, $0 = -\hat{t} + C \implies \hat{t} = C$, so that $\lambda^*(t) = -t + \hat{t}$.

  – $[\hat{t}, T]$: since $\lambda(t) < 0$, $u^*(t) = 0 = \dot{x}(t)$, which in turn gives $x(t) = D$. We need to pin down D,

  $$x(T) = x_1 = D \implies x^*(t) = x_1$$

  and we need to find $\lambda(t)$. How can we pin down $\hat{t}$ (that is, the critical point)? Well, it seems reasonable that the state $x^*(t)$ is the same in both cases at $x^*(\hat{t})$ (if not, there would be a discontinuity in the state function...that cannot happen!) Hence,

  $$\underbrace{x^*(\hat{t})}_{t\in[0,\hat{t}]} = \underbrace{x^*(\hat{t})}_{t\in[\hat{t},T]} \implies \hat{t} + x_0 = x_1 \implies \hat{t} = x_1 - x_0$$

  Therefore, $\lambda^*(t) = -t + x_1 - x_0$.

Exercise 3: Growth in Consumption

Suppose a person seeks to maximize,

$$\max_{x(t)} \int_0^\infty u(c)e^{-\delta t}dt \quad s.t. \quad \dot{k} = b(1-x)k, \quad x \in [0,1]$$

where $c = f(xk)$, $f(0) = 0$, $f'(\cdot) > 0$ and $f''(\cdot) < 0$.

(a) Derive the sufficient first-order conditions associated with the maximization problem.

Answer:

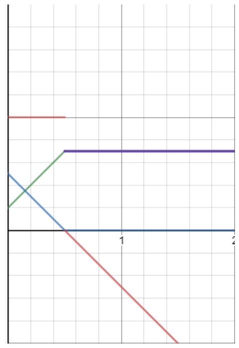

Figure 2: u^*: strong red and strong blue, x^*: green and purple, λ^*: light blue and light red (using $x_0 = 0.2$ and $x_1 = 0.7$).

(a) Forming the Hamiltonian,

$$\mathcal{H} = u\{f[x(t)k(t)]\}e^{-\delta t} + \lambda(t)\{b[1 - x(t)]k(t)\}$$

Taking the FOC's,

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial x(t)} &\stackrel{!}{=} 0 \implies \lambda(t) = \frac{u'\{f[x(t)k(t)]\}f_x[x(t)k(t)]k(t)e^{-\delta t}}{b} \\
\frac{\partial \mathcal{H}}{\partial k(t)} &\stackrel{!}{=} -\dot{\lambda}(t) \implies u'\{f[x(t)k(t)]\}f_k[x(t)k(t)]x(t)e^{-\delta t} + \lambda(t)b[1 - x(t)] = -\dot{\lambda}(t) \\
\lim_{t\to\infty} \lambda(t)k(t) &= 0
\end{aligned}$$

(b) Assume that $f(xk) = A(xk)^\alpha$ where $\alpha \in (0, 1)$ and that $u(c) = \log c$. Show that on the optimal path the growth of c is given by $\alpha(b - \delta)$. When does c grow forever?

Answer:

(b) Let me now me sloppy and don't write everything in terms of (t). Expressions will be more transparent. Forming the Hamiltonian,

$$\mathcal{H} = \log[A(xk)^\alpha]e^{-\delta t} + \lambda b(1 - x)k$$

Taking the FOC's,

$$\frac{\partial \mathcal{H}}{\partial x(t)} \stackrel{!}{=} 0 \implies \lambda bk = \frac{\alpha}{x}e^{-\delta t} \tag{1}$$

$$\frac{\partial \mathcal{H}}{\partial k(t)} \stackrel{!}{=} -\dot{\lambda} \implies \dot{\lambda} = -\frac{\alpha}{k}e^{-\delta t} - \lambda b(1 - x) \tag{2}$$

$$\lim_{t\to\infty} \lambda k = 0$$

we want to obtain $c = A(xk)^\alpha$. Hence, lets find xk. Rearraging (1),

$$xk = \frac{\alpha}{\lambda b} e^{-\delta t} \tag{3}$$

we need λ now. Rearranging again (1),

$$\lambda b x = \frac{\alpha}{k} e^{-\delta t} \tag{4}$$

Introducing (4) into (2),

$$\dot{\lambda} = -\lambda b$$

Hence, multiplying both sides by dt and integrating,

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} = -b \implies & \int_0^t \frac{1}{\lambda} \frac{d\lambda}{dt} dt = -\int_0^t b \; dt \implies \\ \implies & [\log \lambda]_0^t = -[bs]_0^t \implies \\ \implies & \log \lambda(t) = \log \lambda(0) - bt \implies \\ \implies & \lambda(t) = \lambda(0) e^{-bt} \end{aligned} \tag{5}$$

where I have written λ in terms of t just to clarify the result of the integral. From now on, I will denote $\lambda(0) = \lambda_0$. Introducing (5) into (3),

$$xk = \frac{\alpha}{\lambda_0 b} e^{(b-\delta)t}$$

Therefore,

$$c(t) = A(xk)^{\alpha} = A \left(\frac{\alpha}{\lambda_0 b} \right)^{\alpha} e^{\alpha(b-\delta)t}$$

Finally, in continuous time we define growth as

$$g_c = \frac{\dot{c}}{c} = \frac{1}{c} \frac{dc}{dt} \underset{elasticities}{\equiv} \frac{d \log c}{dt}$$

Taking therefore the log (it just simplifies calculations, you could also do it without taking the log),

$$\log c = \alpha \log \frac{A\alpha}{\lambda_0 b} + \alpha(b-\delta)t$$

and we can get the growth rate

$$g_c = \frac{d \log c}{dt} = \alpha(b-\delta)$$

hence, $g_c > 0$ if $b > \delta$.

Exercise 4: Autonomous Systems

Consider the optimal control problem

$$\max_{x(t)} \int_0^1 \left(-\frac{1}{2}u^2 - x\right) dt \quad s.t. \quad \dot{x} = 2(1-u), \quad x(0) = 1$$

where $x(1)$ is free.

(a) State the necessary and sufficient conditions for a candidate solution

Answer:

(a) **Necessary**: if there exists a continuous and piecewise differentiable function $\lambda(t)$ s.t. $\forall t \in [0,1]$,

1. $u^*(t)$ maximizes $\mathcal{H}$ $\forall u \in \mathcal{U}$,

$$\mathcal{H}\left[t, x^*(t), u^*(t), \lambda(t)\right] \geq \mathcal{H}\left[t, x(t), u(t), \lambda(t)\right]$$

2. Whenever $u^*(t)$ is differentiable, $\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x}$.
3. One of the transversality conditions hold,
   - $\lambda(1) = 0$ (x_1 free)
   - $\lambda(1) \geq 0$; $\lambda(1)[x(1) - x_1] = 0$ ($x(1) \geq x_1$)
   - $x(1) = x_1$

Sufficient: suppose $[t, x^*(t), u^*(t), \lambda(t)]$ satisfies the above necessary conditions. Moreover, suppose that $\mathcal{H}\left[t, x(t), u(t), \lambda(t)\right]$ is concave in (x, u) $\forall t \in [0,1]$. Then, $[t, x^*(t), u^*(t), \lambda(t)]$ is optimal.

(b) Find $u^*(t)$, $x^*(t)$ and $\lambda^*(t)$ using these conditions.

Answer:

(b) Forming the hamiltonian,

$$\mathcal{H} = -\frac{1}{2}u^2 - x + \lambda 2(1-u)$$

Before starting the analysis, lets check if the hamiltonian is concave,

$$\frac{\partial \mathcal{H}}{\partial u} = -u - \lambda 2$$
$$\frac{\partial^2 \mathcal{H}}{\partial u^2} = -1$$
$$\frac{\partial \mathcal{H}}{\partial x} = -1$$
$$\frac{\partial^2 \mathcal{H}}{\partial x^2} = 0$$
$$\frac{\partial^2 \mathcal{H}}{\partial u \partial x} = \frac{\partial^2 \mathcal{H}}{\partial x \partial u} = 0$$

Hence, it is concave! Again, taking the FOC of the hamiltonian,

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial u} &\overset{!}{=} 0 \implies u = -\lambda 2 \\ \frac{\partial \mathcal{H}}{\partial x} &\overset{!}{=} -\dot{\lambda} \implies \dot{\lambda} = 1 \end{aligned} \tag{6}$$

Hence, $\lambda = t + C$. Since we know that $\lambda(T) = \lambda(1) = 0$,

$$\lambda(1) = 1 + C = 0 \implies C = -1$$

and $\lambda^*(t) = t - 1$. Introducing λ^* into (6) we obtain

$$u^*(t) = -2(t-1)$$

From the law of motion (equivalent to resource constraint),

$$\dot{x} = 2(1-u) = 2[1 + 2(t-1)] = 2t - 1$$

and hence, $x^*(t) = 2t(t-1) + D$. We know that $x(0) = 1$, hence

$$x(0) = 0 + D = 1 \implies D = 1$$

and $x^*(t) = 2t(t-1) + 1$.

(c) Show that the Hamiltonian is constant along the optimal trajectory. Explain why.

Answer:

(c) Plugging in the values we obtain in (b), $\mathcal{H}^* = -1$. Let me explain why,

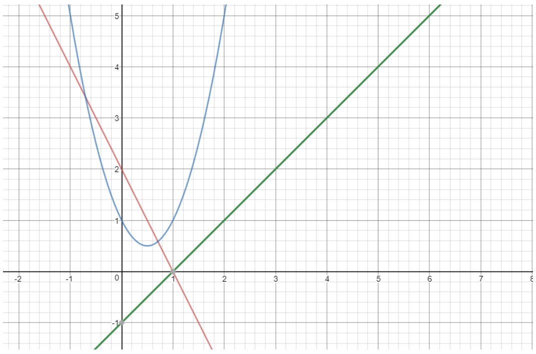

Figure 3: u^*: red, x^*: blue, λ^*: green.

$$\begin{aligned}
\dot{\mathcal{H}} &= \frac{d\mathcal{H}\left[t, x(t), u(t), \lambda(t)\right]}{dt} = \\
&= \frac{d\mathcal{H}}{dt} + \underbrace{\frac{d\mathcal{H}}{du}}_{=0 \text{ if optimum is interior}} \dot{u} + \frac{d\mathcal{H}}{dt}\dot{x} + \frac{d\mathcal{H}}{dt} \underbrace{\dot{\lambda}}_{-\frac{d\mathcal{H}}{dx}} = \\
&= \frac{d\mathcal{H}}{dt} + \frac{d\mathcal{H}}{dt}\big(\dot{x} - \underbrace{\frac{d\mathcal{H}}{d\lambda}}_{\dot{x}}\big) = \\
&= \frac{d\mathcal{H}}{dt} = 0
\end{aligned}$$

Hence, if $\mathcal{H}$ does not depend directly on t, then $\mathcal{H} \equiv cte$ along optimal trayectory.