Microeconomics II

Lecture 2: Backward induction and subgame perfection

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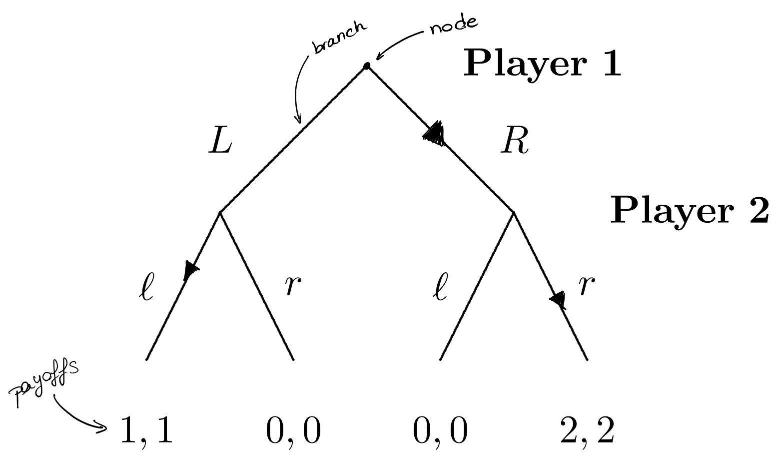
Games in extensive form

So far, we have only considered games where players make their decisions simultaneously and independently of each other. More generally, however, games may have a time order of decisions.

Example (coordination game):



player 1 has one information set

player 2 has two information sets

An extensive form (or *game tree*) consists of

- *nodes*, which represent previous *histories* of the game and points at which some player makes a decision,
- *branches*, which represent actions by a player or by Nature,
- *information sets*, which contain all the nodes at which a player believes he may be at that point in the game, and finally
- *payoffs* for each player at each terminal node.

In the example, all information sets consist of a single node. This means we assume Player 2 observes Player 1's initial move with certainty.

A *strategy* of an extensive form game is a *complete contingent plan of action*, which for each information set a player could be at specifies what action he takes there.

In the example, Player 2 observes Player 1's action, so a strategy for Player 2 in this game specifies one of Player 2's actions ℓ and r for each of Player 1's actions L and R. Hence Player 2's strategy set in this game is $S_2 = \{\ell\ell, \ell r, r\ell, rr\}$ (if we write what Player 2 does if Player 1 played L in the first position and what Player 2 does if Player 1 played R in the second position).

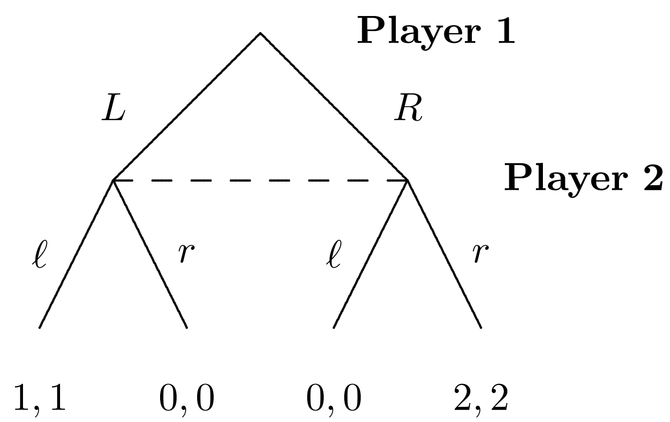
Since Player 1 starts (and therefore cannot make his action contingent on anything), a strategy for Player 1 is simply one of his actions L and R.

A Nash equilibrium is, of course, just as before, a strategy profile such that each player plays a best reply.

We may also write the game in its normal form, i.e., listing all the possible strategy profiles and their corresponding payoffs.

		Player 2			
		$\ell\ell$	ℓr	$r\ell$	rr
Player 1	L	$1, 1$	$1, 1$	$0, 0$	$0, 0$
	R	$0, 0$	$2, 2$	$0, 0$	$2, 2$

Now instead assume that Player 2 *cannot* observe Player 1's initial move, or, equivalently, that they make their moves simultaneously. The extensive form then looks as follows.



Player 2 has only one information set

The dashed line connects the two nodes that form Player 2's single information set. This game has the following normal form.

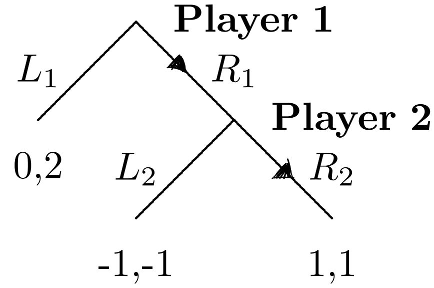
		Player 2	
		ℓ	r
Player 1	L	$1, 1$	$0, 0$
	R	$0, 0$	$2, 2$

P2 cannot have "ℓr" as strategy, that would imply he knows what P1 played!

We can always find all the Nash equilibria of a game by studying its normal form. But, as we shall see, some equilibria of extensive form games may involve behavior that seems implausible in a dynamic perspective, which is not captured by the normal form.

Sequential games and backward induction

Consider the following game tree.



Player 1 first chooses between L_1 and R_1. If Player 1 chooses L_1, that is the end of the game. Otherwise, Player 2, having observed Player 1's move, gets to choose between L_2 and R_2.

Example applications:

- Entry deterrence
- Wage bargaining

The game has the following normal form

		Player 2	
		L_2	R_2
Player 1	L_1	$0, 2$	$0, 2$
	R_1	$-1, -1$	$1, 1$

P1 has one info set → two actions
P2 has <u>also</u> only one info set!

We see that the game has two pure-strategy equilibria, (L_1, L_2) and (R_1, R_2). Player 2 would prefer the former, Player 1 the latter.

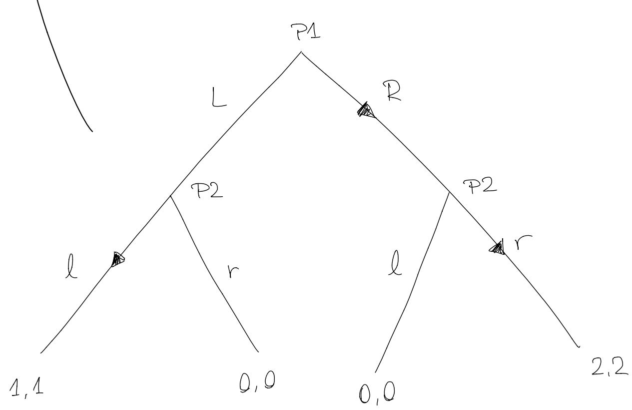
But the normal form does not take into account the sequential structure of the game, i.e., that decisions are not made at the same time. Suppose that a (not very well educated) game theorist has recommended (L_1, L_2) as a solution to the game, but Player 1 gets a strange impulse, deviates, and instead plays R_1. Would Player 2 then actually play L_2? Not if he is rational, since he gets a higher payoff by playing R_2 given that Player 1 has played R_1.

The equilibrium (L_1, L_2) may therefore be said to rely on a *non-credible threat*, which would never actually be carried out. Since Nash equilibria may imply dynamically irrational behavior of this type, we need to supplement the equilibrium concept with some additional, stronger criterion.

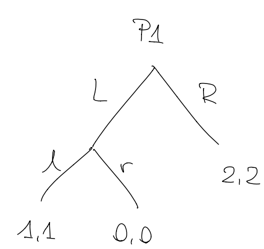
In this case, we were able to solve the game using *backward induction*, i.e., by starting by contemplating the last stage of the game and requiring that all earlier decisions be rational given the later ones. Backward induction is an extension of the notion of iterated elimination of dominated strategies to games with a time dimension. We observe that backward induction also relies on mutual knowledge of rationality. (Up to what level?)

We can now return to our first extensive form game and conclude that there is something fishy about the equilibrium that induces the payoff pair $(1, 1)$.

All backward induction solutions are Nash equilibria, but the converse is not true. Backward induction solutions are special cases of the more powerful concept of *subgame perfection*, which we shall deal with later on.



$\Rightarrow$



		P2			
		ll	lr	rl	rr
P1	L	1,1	1,1	0,0	0,0
	R	0,0	2,2	0,0	2,2

	l	r
L	1,1	0,0
R	2,2	2,2

(L, l) [or (L, ll) in previous case] is not sustained in this subgame. Hence, $(L, ll) \notin$ SPE

Stackelberg oligopoly (1934)

Consider the same model as in our earliest Cournot example, but assume one firm makes its quantity decision before the other one does.

Time order:

1. Firm 1 chooses q_1.
2. Firm 2 observes q_1 and chooses q_2.
3. Firm i gets payoff ← even firm 2, who has more info, does worse than firm 1

$$\pi_i(q_i, q_j) = q_i(p(q_1, q_2) - c) = q_i(a - q_1 - q_2 - c).$$

We start at the end with Firm 2's decision, given q_1. Firm 2's optimal choice depends on q_1 in the same manner as in the Cournot model, i.e., its best reply given any q_1 it observes from Firm 1 is given by

$$q_2^\star(q_1) = \frac{a - q_1 - c}{2}.$$

Now Firm 1 has to take into account that Firm 2 will adapt optimally later. That is, Firm 1 wishes to maximize

$$\pi_1(q_1, q_2^\star(q_1)) = q_1(a - q_1 - q_2^\star(q_1) - c) = q_1\frac{a - q_1 - c}{2}.$$

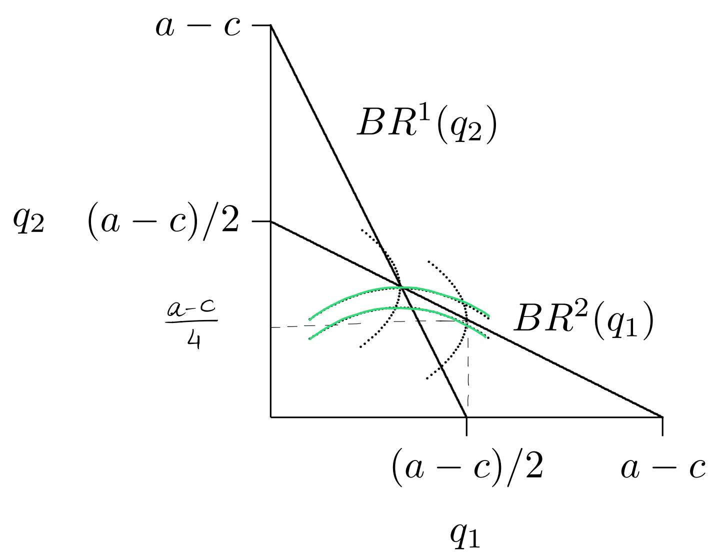
We differentiate with respect to q_1, set the derivative equal to zero, and solve for Firm 1's optimal output

$$\boxed{q_1^\star = \frac{a - c}{2}}.$$

It follows that

$$\boxed{q_2^\star(q_1^\star) = \frac{a-c}{4}.}$$

We may draw the firms' iso-profit curves in the Cournot diagram.



We now see that the solution of the Stackelberg model involves letting Firm 1 pick its favorite point on Firm 2's best-reply curve. We note that Firm 1's profit is higher and Firm 2's lower than under Cournot competition with simultaneous decisions. Firm 1 benefits from going first—it has a *first-mover advantage*. Equivalently, Firm 1 gains from being able to *commit* itself, i.e., from, in contrast with Firm 2, not being able to react to its opponent's decision.

This is also an example of how *better information may hurt a player* in a game, something that cannot happen in a single-person decision problem.

Bargaining with alternating offers

Two players take turns suggesting how to split \$ 1. Both have the discount factor $\delta \in (0, 1)$, i.e., \$ x in the next period is only worth \$ δx now.

Time order:

Period 1.

a. Player 1 proposes s_1 for himself, $1 - s_1$ for Player 2.

b. Player 2 accepts or rejects the proposal. If he rejects, the game continues to period 2.

Period 2.

a. Player 2 proposes s_2 for Player 1, $1 - s_2$ for himself.

b. Player 1 accepts or rejects the proposal. If he rejects, the game continues to period 3.

Period 3. Player 1 gets s, Player 2 gets $1 - s$, where s is exogenously given.

We begin backward inducting by assuming play has reached period 2.

Period 2. Player 1 can get s in period 3 if he rejects Player 2's proposal. Hence he only accepts if $s_2 \geq \delta s$. Therefore the most Player 2 can get in period 2 is $1 - \delta s$. Otherwise he gets $1 - s$ in period 3, which is worth $\delta(1 - s)$ now. Since we have $\delta(1-s) < 1-\delta s$, he must propose $s_2^\star = \delta s$.

Period 1. Player 1 knows that Player 2 is guaranteed $1 - s_2^\star$ in period 2, which is worth $\delta(1 - s_2^\star)$ now. It follows that $s_1^\star = 1 - \delta(1 - s_2^\star) = 1 - \delta(1 - \delta s)$.

$\boxed{t=3}$ P1: s

P2: $1-s$

$\boxed{t=2}$ P1: $s_2 \geq \delta s \Rightarrow s_2^* = \delta s$

important

P2: $1 - s_2^* \Rightarrow 1 - \delta s > \delta(1-s)$

$\boxed{t=1}$ P2: $1 - s_1 \geq (1-\delta s)\delta \Rightarrow 1 - s_1 = \delta(1 - \delta s)$

P1: $s_1 = 1 - \delta(1 - \delta s) > \delta s$

$\boxed{t=5}$ P1: s

P2: $1-s$

$\boxed{t=4}$ P1: $s_4 \geq \delta s \Rightarrow s_4^* = \delta s$

P2: $1-s_4^* \Rightarrow 1-\delta s > \delta(1-s)$ important

$\boxed{t=3}$ P2: $1-s_3 \geq (1-\delta s)\delta \Rightarrow 1-s_3 = \delta(1-\delta s)$

P1: $s_3 = 1-\delta(1-\delta s) > \delta s$

$\boxed{t=2}$ P1: $s_2 \geq \delta[1-\delta(1-\delta s)] \Rightarrow s_2^* = \delta[1-\delta(1-\delta s)]$

P2: $1-s_2^* = 1-\delta[1-\delta(1-\delta s)]$

$\boxed{t=1}$ P2: $1-s_1 = \delta\{1-\delta[1-\delta(1-\delta s)]\}$

P1: $s_1 = 1-\delta\{\underbrace{1-\delta\underbrace{[1-\delta(1-\delta s)]}_{1-\delta+\delta^2 s}}_{1-\delta+\delta^2-\delta^3 s}\}$

$\underbrace{\qquad}_{1-\delta+\delta^2-\delta^3+\delta^4 s}$

Rubinstein (1982) discusses the case of an infinite time horizon.* Here is a (possibly somewhat suspect) way of handling this.

Suppose we start by adding 2 more periods of alternating bidding along the same lines. Note that in period 3 a game then begins that is identical to the 2-period game, and therefore has the same solution. In the game starting in period 3 Player 1 is therefore guaranteed $1 - \delta(1 - \delta s)$. Following similar logic as before, the solution of the four-period problem is then

$$s_1^\star = 1 - \delta(1 - \delta(1 - \delta(1 - \delta s))).$$

If we continue adding 2 periods at a time, Player 1's equilibrium payoff approaches

$V = 1-\delta+\delta^2-\delta^3+\dots$

$V = 1-\delta+\delta^2 V \Rightarrow V = \frac{1-\delta}{1-\delta^2} = \frac{1-\delta}{(1+\delta)(1-\delta)} = \frac{1}{1+\delta}$

$$1 - \delta + \delta^2 - \delta^3 \ldots = \sum_{i=0}^{\infty} (-\delta)^i = \frac{1}{1+\delta},$$

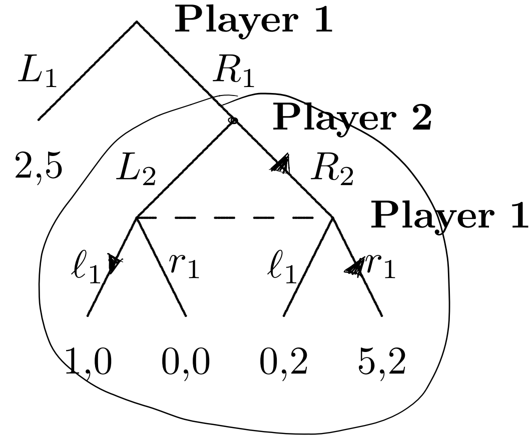
since we know that $\sum_{i=0}^{\infty} x^i = 1/(1-x)$ for $0 \leq x < 1$ (a fact we shall shortly use again).

* Ståhl (1972) pioneered the study of the finite time horizon problem.

Subgame perfection

So far, we have studied extensive form games with *perfect information* in the sense that each player can observe all actions taken previously in the game. Unless this is the case, we cannot use backward induction.

Example:



The dashed line signifies that Player 1 does not know which of the connected nodes he is at. This could be, e.g., because both players move simultaneously after Player 1 has played R_1. We say that both of the connected nodes are in Player 1's *information set* if he gets to act again.

That is, a player's information set at some point in a game is the set of nodes at which he could be at that point. If the player knows for certain that he is at a particular node, then the information set consists of that node only.

Our example game has the following normal form.

		Player 2	
		L_2	R_2
Player 1	$L_1\ell_1$	$\cancel{2, 5}$	$2, 5$
	L_1r_1	$\cancel{2, 5}$	$2, 5$
	$R_1\ell_1$	$1, 0$	$0, 2$
	R_1r_1	$0, 0$	$5, 2$

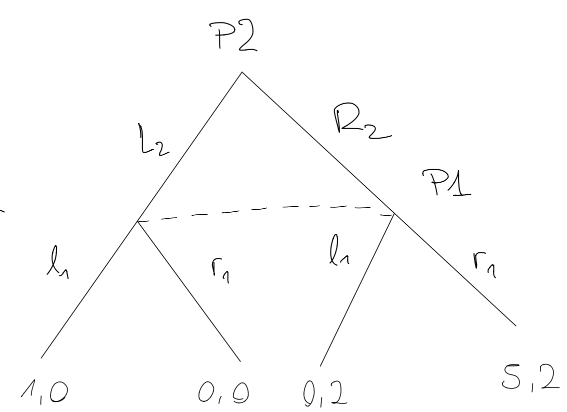
We also see that we cannot use backward induction in the extensive form, since Player 1, if it became his turn again, would like to play ℓ_1 at the left node of his information set and r_1 at the right node. There is no one action which is best in both cases.

But note that the part of the game that starts with Player 2 acting may be considered a game in its own right, with the following normal form.

		Player 2	
		L_2	R_2
Player 1	ℓ_1	$\underline{1}, 0$	$0, \underline{2}$
	r_1	$0, 0$	$\underline{5}, \underline{2}$

normal form of subgame

This *subgame* has the unique pure Nash equilibrium (r_1, R_2), with payoffs $(5, 2)$. Hence Player 1 should play R_1 and get 5, rather than play L_1 and get 2.



Selten (1965) introduced the notion of a *subgame perfect* equilibrium, a generalization of the idea of backward induction to games with imperfect information. Subgame perfect equilibria are Nash equilibria that are not based on non-credible threats.

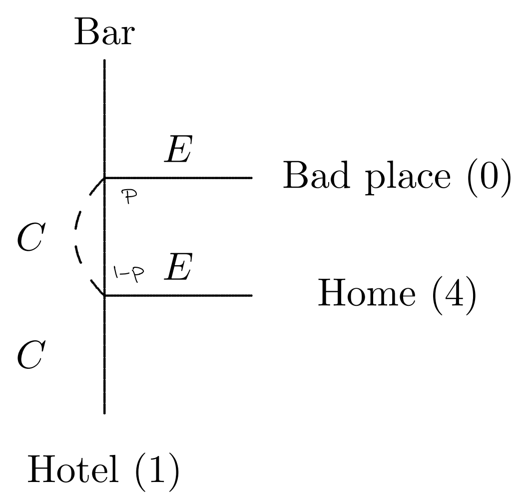
We now also note that two different types of randomization are possible in extensive form games. A *mixed strategy* is, as before, a probability distribution over pure strategies of the normal form. But since a player now may have more than one information set, we can also allow a player to assign local, independent probabilities over the actions available in a particular information set. A set of such probability distributions, one for each information set, is called (for obscure historical reasons) a *behavior strategy*. Since these two notions are equivalent in games with *perfect recall*, the only type we shall study, in the following we shall consider behavior strategies. This allows us to define subgame perfection also with randomization.

Mixed strategies = Behavioural strategies (in games of perfect recall)

The Absent-Minded Driver

Here is a single-person decision problem with imperfect information (due to Piccione and Rubinstein 1997).

A man is sitting in a bar late at night. He knows that once he is on the road and comes to an exit, he will be unable to recall whether he has passed an exit already.



What is an optimal pure strategy for this problem? Since the driver should realize he will never be able to get home, he should decide to (C)ontinue at an exit.

But then when he actually gets to an exit, it may be either one with equal probability. Then it is now optimal to (E)xit. Hence optimal plans may become inconsistent over time, without any new information, change of preferences, or the like.

There is also an optimal nontrivial behavior strategy, which you will be asked to find. Note that in standard games against Nature, randomizing can never be optimal.

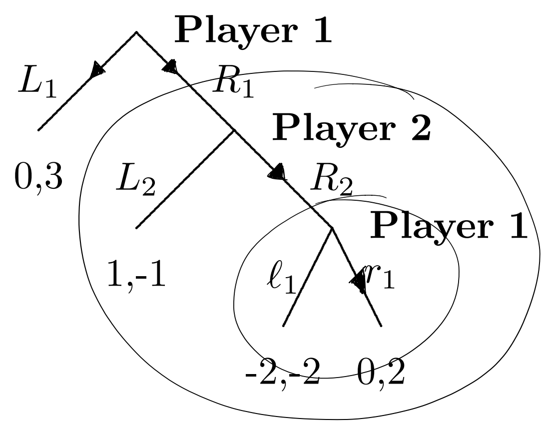
Definition **(Subgame).** *A subgame consists of a node that is the only one in its information set and is not a terminal node, and all descendant nodes of this node, and contains all nodes of the information sets that are part of the subgame.*

It follows that the entire game itself is always a subgame.

Definition **(Subgame perfect equilibrium).** *A Nash equilibrium is subgame perfect if it prescribes a Nash equilibrium for every subgame.*

Since every backward induction solution is a subgame perfect equilibrium, we shall use the latter, more general term in the following.

Example:



The game has the following normal form.

		Player 2	
		L_2	R_2
Player 1	$L_1\ell_1$	$0,3$	$0,3$
	L_1r_1	$0,3$	$0,3$
	$R_1\ell_1$	$1,-1$	$-2,-2$
	R_1r_1	$1,-1$	$0,2$

SPE (explained later)

There are four equilibria in pure strategies, $(L_1\ell_1, R_2)$, (L_1r_1, R_2), $(R_1\ell_1, L_2)$, and (R_1r_1, R_2).

There are three subgames: The game itself, the subgame starting at Player 2's node, and the subgame starting at Player 1's last node.

In the last subgame, Player 1 has to play r_1.

The subgame starting at Player 2's node has the following normal form.

		Player 2	
		L_2	R_2
Player 1	ℓ_1	$1, -1$	$-2, -2$
	r_1	$1, -1$	$0, 2$

This subgame has two equilibria in pure strategies, (ℓ_1, L_2) and (r_1, R_2). Since subgame perfection implies that Player 1 must not play ℓ_1, only (r_1, R_2) is subgame perfect in the subgame starting at Player 2's node. It follows that only $(L_1 r_1, R_2)$ and $(R_1 r_1, R_2)$ are subgame perfect equilibria of the game as a whole.

Bertrand competition with entry costs

We study a model with two firms and two time periods. In the first period, the firms simultaneously and independently decide whether they each want to make an investment $0 < k < ((a-c)/2)^2$, which is necessary in order to produce in the market. In the second period, a firm that has decided to enter sets its price and meets demand given by $q(p) = a - p$. If both firms have entered, everything is like in the Bertrand example with homogenous goods and common constant average cost c. A firm that has not entered gets zero profit.

We now look for a subgame perfect equilibrium of this game. We start by considering the four possible period 2 subgames that start when both firms have entered, Firm 1 has entered but not Firm 2, Firm 2 has entered but not Firm 1, and neither has entered, respectively.

$\boxed{t=1}$

$k \in \left(0, \frac{(a-c)^2}{4}\right)$

$\boxed{t=2}$

sets p

$q(p) = a - p$

Case 1: Both firms have entered. In the unique equilibrium of this subgame, we know from before that both firms must set price equal to marginal cost c. $p^* = c$
$\pi = (p-c)q = (p-c)(a-p) = 0$

Case 2: Firm i has invested, but Firm $j \neq i$ has not invested. Firm i then has the profit function

$$\pi_i(p_i) = (p_i - c)(a - p_i),$$

which is maximized when

$$\frac{\partial \pi_i}{\partial p_i} = a - 2p_i + c = 0, \quad \Rightarrow$$

$p_i = \frac{a+c}{2}$
$\pi_i = (p-c)(a-p) = \left(\frac{a+c}{2} - c\right)\left(a - \frac{a+c}{2}\right) =$
$= \left(\frac{a-c}{2}\right)\left(\frac{a-c}{2}\right) = \frac{(a-c)^2}{4}$

i.e., when Firm i sets its monopoly price $(a + c)/2$ and hence gets profit $((a - c)/2)^2$. (Note that the fixed cost k is "sunk" at this stage.)

Case 3: Neither of the firms has invested. Since they are not operating in the market, neither one gets any profit.

Viewed from the perspective of the start of the game, *given rational behavior later* (in the sense that equilibria are played), the game can now be reduced to the following normal form.

		r	$1-r$
		Firm 2	
		Invest	Don't invest
Firm 1	Invest	$-k, -k$	$\left(\frac{a-c}{2}\right)^2 - k, 0$
	Don't invest	$0, \left(\frac{a-c}{2}\right)^2 - k$	$0, 0$

Hence there is only one type of subgame perfect equilibrium in pure strategies in this game, namely that one firm stays out and the other one enters and sets the monopoly price. Even for k arbitrarily small, the basic Bertrand result has disappeared completely. Instead of price equal to marginal cost, we get monopoly.

Since the game is symmetric, having two asymmetric solutions may be considered unsatisfying, since there is no method for determining which of the firms should enter. However, there is also a symmetric equilibrium in behavior strategies.

Let r be a firm's probability of investing in period 1. In a subgame perfect equilibrium in behavior strategies, we must have that

$$r(-k) + (1-r)\left(\left(\frac{a-c}{2}\right)^2 - k\right) = r \cdot 0 + (1-r) \cdot 0,$$

i.e., that

$$r = \frac{((a-c)/2)^2 - k}{((a-c)/2)^2},$$

and of course that the firms play the equilibria in period 2.

Does this game have any other Nash equilibria? Yes. For example, the subgame perfection criterion has rejected equilibria of the following type: Firm 1 invests, and sets the monopoly price if Firm 2 does not invest, but if Firm 2 invests, Firm 1 sets its price lower than c. Firm 2 does not invest. This is a best reply for Firm 2, since $-k$ is the best payoff it can get if it invests. Firm 1's strategy is a best reply since Firm 2 does not invest, which gives Firm 1 the monopoly profit minus the fixed cost. But this equilibrium is based on the non-credible threat on the part of Firm 1 to set price less than average cost, something a rational Firm 1 would never actually do if Firm 2 entered after all.

F1 invests

- F2 doesn't invest : $P_1 = \frac{a+c}{2}$
- F2 invest : $P_1 < c$

F2 doesn't invest ($\pi = 0$, k cost)

A finitely repeated game

The following game is played twice without discounting, i.e., payoffs for the whole game are simply the sum of payoffs from each period. The players can observe what happened in the first period before the second period starts. Is it possible to get the payoff profile $(4, 4)$ in the first period in a subgame perfect equilibrium in pure strategies?

		Player 2		
		L	C	R
Player 1	T	$3, 1$	$0, 0$	$5, 0$
	M	$2, 1$	$1, 2$	$3, 1$
	B	$1, 2$	$0, 1$	$4, 4$

Yes, that is indeed possible. Note that the complicating factor is that B is not a best reply for Player 1 against R in the one-period game. Player 2 would not mind (B, R). Hence the equilibrium must be such that Player 1 is given an incentive to play B in the first period, e.g., by a threat of punishment in period 2.

We know that subgame perfection requires that a one-period equilibrium is played in period 2. Hence (T, L) or (M, C) has to be played in period 2. Player 1 would prefer (T, L). Could one, perhaps, let him have (T, L) if he behaves nicely in the first period and (M, C) otherwise?

Consider the following strategy pair:

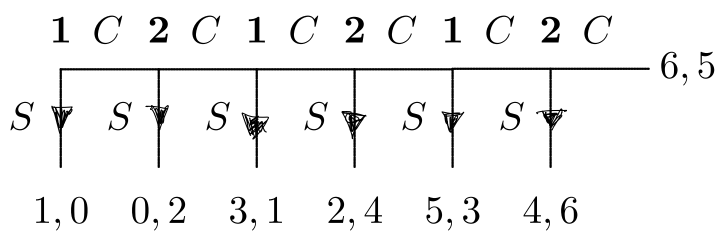
Player 2: Play R in period 1. If (B, R) was played in period 1, play L in period 2. If something other than (B, R) was played in period 1, play C in period 2.

Player 1: Play B in period 1. If (B, R) was played in period 1, play T in period 2. If something other than (B, R) was played in period 1, play M in period 2.

Suppose Player 2 plays the proposed strategy, but Player 1 deviates optimally. The best thing he can do in period 1 is then to play T and get the payoff 5. Then Player 2 will play C in period 2, so Player 1 has to play his best reply M, and in total gets 5+1=6. If instead he plays the proposed strategy, he gets 4+3=7. So the proposed strategy is indeed a best reply for Player 1. It is easy to see than the proposed strategy for Player 2 is also a best reply. Hence we have found an equilibrium. It is subgame perfect, since it always prescribes play of a one-period equilibrium in period 2.

The Centipede (Rosenthal 1982)

Although backward induction is an *a priori* appealing principle of rationality, it can lead to conclusions that are intuitively unsatisfying. Here is a popular example.



Players 1 and 2 participate in a mysterious process where the potential gain for both keeps increasing if no player stops. The players take turns choosing between stopping (S) or continuing (C). Each player would prefer stopping when it is his turn over his opponent stopping the next time.

The unique backward induction solution of this game is for Player 1 to stop immediately, even though both would have been better off if the game had continued. This seems particularly unreasonable if the game is even bigger.

The backward induction solution is based on the idea that a player always expects his opponent to stop the next time it is his turn—even if he has continued many times before. This reveals a paradox in the application of the notion of common knowledge of rationality. The argument depends on, among other things, the idea that Player 2 would act rational at his last decision node. But this node is never reached unless Player 2 has acted irrationally earlier!

This paradox is still a controversial issue at the frontiers of game theory research. Feel free to contribute.

Ultimatum bargaining

Actual human beings often act in ways that appear to contradict game-theoretical rationality, e.g., as it is embodied in the notion of backward induction.

A game that really brings this out is the *ultimatum bargaining* game of Werner Güth. This 2-player game concerns the division of a sum of money $M > 2$. Player 1 proposes a split that gives him $M_1 \leq M - 1$ cents and Player 2 the rest. Player 2 then either accepts or rejects the proposal. In case Player 2 rejects, nobody gets anything.

Any value of M_1 could result from some Nash equilibrium of this game. For instance, consider only strategies on the part of Player 2 of the form

- accept if $M - M_1 \geq M - M_1^\star$,
- reject otherwise,

for some $M_1^\star$.

Given that Player 2 plays such a strategy, it is a best reply for Player 1 to suggest $M_1 = M_1^\star$. Given that Player 1 suggests $M_1 = M_1^\star$, Player 2's strategy is a best reply (since it is *always* best for Player 2 to accept).

But the unique backward induction solution is the one where $M_1 = M - 1$ and Player 2 accepts, since a rational Player 2 must accept all proposals.

In experiments carried out by Güth and others, where anonymous subjects play ultimatum games, Player 2 typically rejects some proposals that give him more than one cent (or the lowest specified payoff). Many explanations have been offered for this type of behavior. Güth thinks this shows that people are simply not rational in the game-theoretical sense, but are instead guided by social norms concerning what is a "fair" division in situations like this. Others think differently.

Problem. *Find the optimal behavior strategy for the absent-minded driver problem.*

Problem. *Firm 1 and Firm 2 produce imperfect substitutes. Demand is given by*

$$q_1 = 1 - 2p_1 + p_2$$

and

$$q_2 = 1 - 2p_2 + p_1,$$

where q_i and p_i are Firm i's quantity and price, respectively. Each firm has a large unsold stock of its good, and hence no production cost. Now consider a game with two stages. In stage one, the firms simultaneously and independently decide whether to be price-setters or quantity-setters. In stage two, they play the market game, controlling the decision variables they selected in the first stage. Find a subgame perfect equilibrium outcome of this game.

Problem. *Consider the following game.*

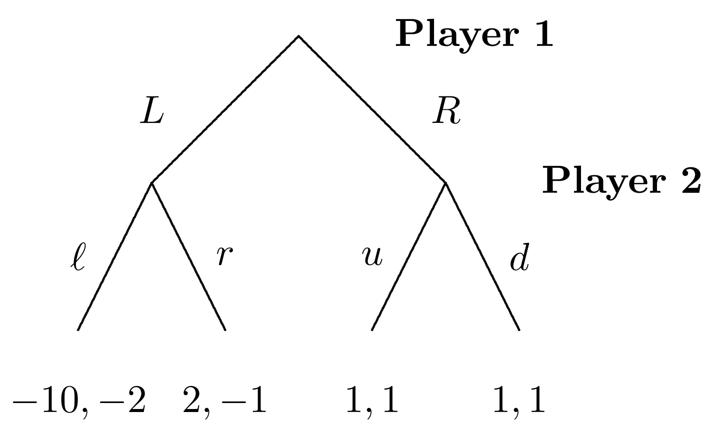
		Player 2	
		S_2	C_2
Player 1	S_1	5, 2	3, 1
	C_1	6, 3	4, 4

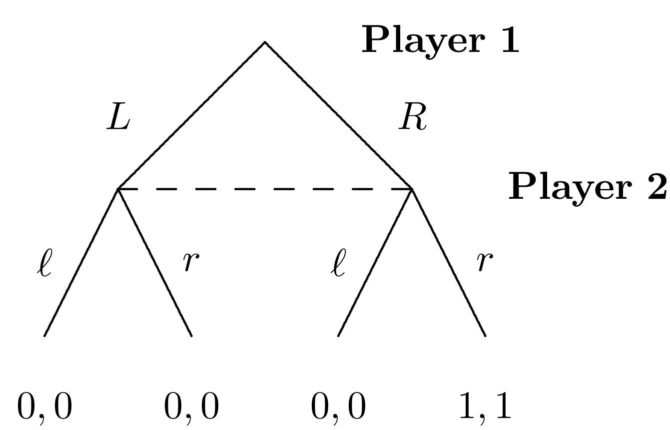
a) Suppose the players make their choices simultaneously. Find the unique equilibrium.

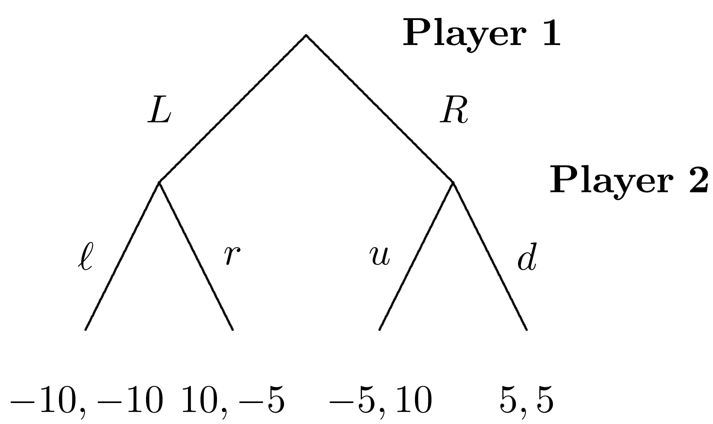
b) Suppose Player 1 makes his decision before Player 2 makes his, and Player 2 observes Player 1's decision. Find the normal form and the extensive form of this game. Find the backward induction solution.

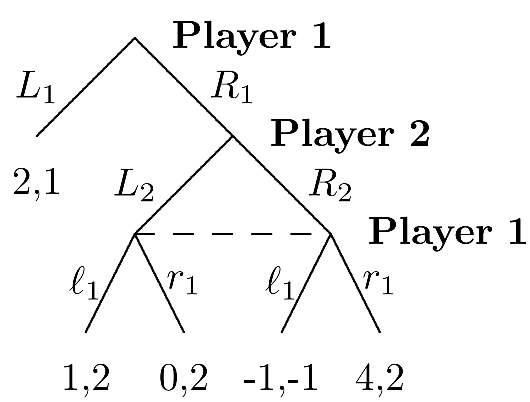
c) Suppose Player 1 makes his decision first, but Player 2 does not directly observe Player 1's decision. Instead, Player 2 gets a signal ϕ, which with some very small probability ε gives the wrong indication about Player 1's choice. That is, we have $P(\phi = S_1|S_1) = 1-\varepsilon$ and $P(\phi = C_1|C_1) = 1-\varepsilon$. Find the normal form of this game. Find the unique equilibrium. (Note that we cannot apply backward induction in this case.)

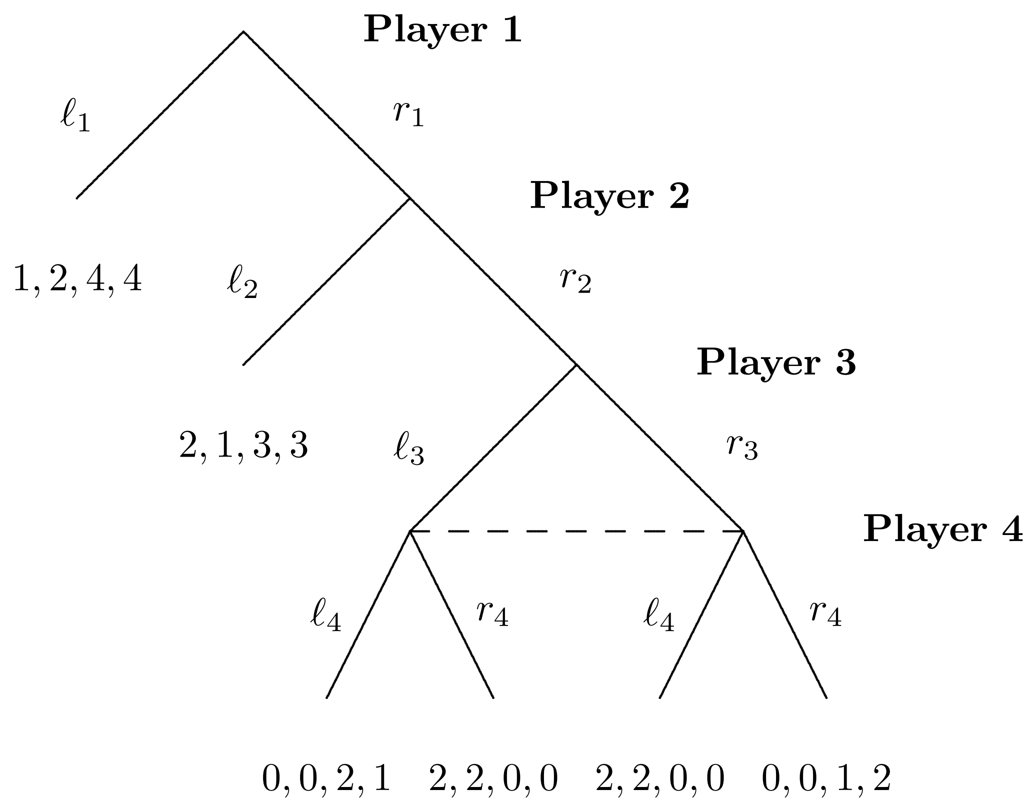
Problem. *Find subgame perfect equilibria in the following five games.*



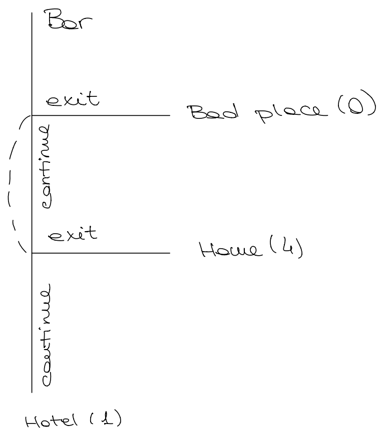



Player 1
L
R
Player 2
ℓ
r
u
d
$-10, -10$
$10, -5$
$-5, 10$
$5, 5$


Player 1
L_1
R_1
Player 2
2,1
L_2
R_2
Player 1
ℓ_1
r_1
ℓ_1
r_1
1,2
0,2
-1,-1
4,2


Player 1
ℓ_1
r_1
Player 2
1, 2, 4, 4
ℓ_2
r_2
Player 3
2, 1, 3, 3
ℓ_3
r_3
Player 4
ℓ_4
r_4
ℓ_4
r_4
0, 0, 2, 1
2, 2, 0, 0
2, 2, 0, 0
0, 0, 1, 2

PROBLEM 1



Consider that player P exits with prob p and continues with prob 1-p. His expected utility is

$$EU = \underbrace{(1-p)0}_{\text{exit}} + \underbrace{p[(1-p)4}_{\text{continue - exit}} + \underbrace{p1]}_{\text{continue-continue}} =$$

$$= 4p - 4p^2 + p^2 = 4p - 3p^2$$

max utility : $\frac{dEU}{dp} = 4 - 6p = 0 \Rightarrow \boxed{p^* = \frac{2}{3}}$

The optimal behaviour strategy is to assign prob p^* to continue.

PROBLEM 2

$$q_1 = 1 - 2p_1 + p_2 \qquad (1)$$

$$q_2 = 1 - 2p_2 + p_1 \qquad (2)$$

$$c = 0 \Rightarrow \pi_i = p_i q_i$$

Since the decision is made simultaneously, the game has no more subgames than the game itself. There are three possible scenarios

- Price - setters (symmetric)
  maximize $\pi_i = p_i q_i = p_i(1 - 2p_i + p_j)$:

$$\frac{d\pi_i}{dp_i} = 1 - 2p_i + p_j - 2p_i = 0 \underset{\text{sym}}{\Rightarrow} 1 - 3p = 0 \Rightarrow \boxed{p^* = \frac{1}{3}}$$

$$q_i = q_j = 1 - 2p_i + p_j = 1 - p \Rightarrow \boxed{q^* = \frac{2}{3}}$$

Profits are $\pi = p \cdot q = \frac{1}{3} \cdot \frac{2}{3} \Rightarrow \boxed{\pi^* = 2/9} \simeq 0.2\hat{2}$

- Quantity - setters (symmetric)
  rewrite the demand function on price function

$$q_i = 1 - 2p_i + p_j \Rightarrow p_i = \frac{1 + p_j - q_i}{2} \qquad \Rightarrow p_i = 1 - \frac{q_j}{3} - \frac{2q_i}{3} \qquad (3)$$

$$p_j = \frac{1 + p_i - q_j}{2}$$

maximize $\pi_i = p_i q_i = \left(1 - \frac{q_j}{3} - \frac{2q_i}{3}\right) q_i$:

$$\frac{d\pi_i}{dq_i} = 1 - \frac{q_j}{3} - \frac{2q_i}{3} - \frac{2}{3}q_i = 0 \underset{\text{sym}}{\Rightarrow} 1 - \frac{5}{3}q = 0 \Rightarrow \boxed{q^* = \frac{3}{5}}$$

$$p_i = p_j = 1 - q_j/3 - 2/3\, q_i = 1 - q \Rightarrow \boxed{p^* = \frac{2}{5}}$$

Profits are $\pi = p \cdot q = \frac{3}{5} \cdot \frac{2}{5} \Rightarrow \boxed{\pi^* = \frac{6}{25}} \approx 0.24$

- Price / Quantity – setters

Assume firm i is price-setter and firm j is quantity-setter.
Firm i maximises profits $\pi_i = p_i q_i = p_i(1 - 2p_i + p_j)$

$$\frac{\partial \pi_i}{\partial p_i} = 1 - 2p_i + p_j - 2p_i = 0 \Rightarrow \boxed{p_i = \frac{1 + p_j}{4}} \quad (4)$$

Firm j maximises profits $\pi_j = p_j q_j = (1 - q_i/3 - 2q_j/3)\, q_j$

$$\frac{\partial \pi_j}{\partial q_j} = 1 - \frac{q_i}{3} - \frac{2q_j}{3} - \frac{2q_j}{3} = 0 \Rightarrow \boxed{q_j = \frac{3 - q_i}{4}} \quad (5)$$

Using (1)-(3) into (4)-(5)

$$(5) \Rightarrow 4(1 - 2p_j + p_i) = 3 - (1 - 2p_i + p_j) \Rightarrow$$
$$\Rightarrow 4 - 8p_j + 4p_i = 3 - 1 + 2p_i - p_j \Rightarrow$$
$$\Rightarrow 2 - 7p_j + 2p_i = 0 \Rightarrow$$
$$\Rightarrow 2 + 2p_i = 7p_j \Rightarrow$$
$$\overset{(4)}{\Rightarrow} 2 + 2\left(\frac{1+p_j}{4}\right) = 7p_j \Rightarrow$$
$$\Rightarrow \boxed{p_j^* = \frac{5}{13}}$$

$$(4)\quad p_i = \frac{1 + p_j}{4} = \frac{1 + 5/13}{4} \Rightarrow \boxed{p_i^* = \frac{9}{26}}$$

$$(1)\quad q_i = 1 - 2p_i + p_j = 1 - 2\,\frac{9}{26} + \frac{5}{13} \Rightarrow \boxed{q_i^* = \frac{18}{26}}$$

$$(2)\quad q_j = 1 - 2p_j + p_i = 1 - 2\,\frac{5}{13} + \frac{9}{26} \Rightarrow \boxed{q_j^* = \frac{15}{26}}$$

Profits are $\pi_i = p_i q_i = \frac{9}{26} \cdot \frac{18}{26} \Rightarrow \boxed{\pi_i^* = \frac{81}{338}} \approx 0.23964497\ldots$

$\pi_j = p_j q_j = \frac{5}{13} \cdot \frac{15}{26} \Rightarrow \boxed{\pi_j^* = \frac{75}{338}} \approx 0.22189349\ldots$

We can finally write this game in its normal form

1 \ 2	price	quantity
price	2/9 , 2/9	81/338 , 75/338
quantity	75/338 , 81/338	6/25 , 6/25

Unique SPE $= s^* = (s_1^*, s_2^*)$, s.t. $s_i^* = \left(\text{quantity}, \begin{cases} q_i = 3/5 & \text{for quantity, quantity} \\ p_i = 5/13 & \text{for price, quantity} \\ q_i = 9/13 & \text{for quantity, price} \\ p_i = 1/3 & \text{for price, price} \end{cases}\right)$

PROBLEM 3

a)

		P2	
		p S_2	$1-p$ C_2
P1	r S_1	5, 2	3, 1
	$1-r$ C_1	6, 3	4, 4

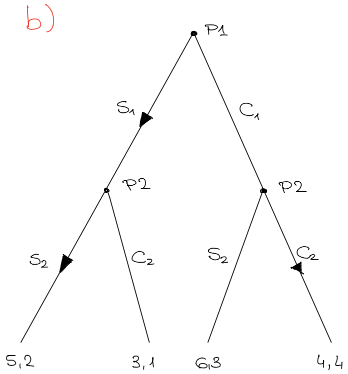
NE : • pure: $\langle C_1, C_2 \rangle$

• mixed : $\emptyset$ (empty set)

$S_1 \equiv C_1 \Rightarrow 5p + 3(1-p) = 6p + 4(1-p) \Rightarrow$ no solution (notice that C_1 dominates S_1!)

$S_2 \equiv C_2 \Rightarrow 2r + 3(1-r) = r + 4(1-r) \Rightarrow \boxed{r = 1/2}$

b)



		S_2S_2	S_2C_2	C_2S_2	C_2C_2
P1	S_1	5,2	5,2	3,1	3,1
	C_1	6,3	4,4	6,3	4,4

NE : $\langle S_1, S_2C_2 \rangle$, $\langle C_1, C_2C_2 \rangle$

SPE : $\langle S_1, S_2C_2 \rangle$

c) Consider the previous SPE. P2 played S_2 if P1 played S_1, and C_2 if C_1. Now we include a signal: P2 plays S_2 if $\phi = S_1$ and C_2 if $\phi = C_1$. The payoffs are

(P1 plays S_1)

P2 : $2(1-\varepsilon) + 1\varepsilon = 2 - \varepsilon$

P1 : $5(1-\varepsilon) + 3\varepsilon = 5 - 2\varepsilon$

We now proceed similarly with the rest of the possible strategy combinations

$\langle S_1, S_2S_2 \rangle$ ← nothing changes: P2 plays S_2 no matter what signal!

P2 : $2(1-\varepsilon) + 2\varepsilon = 2$

P1 : $5(1-\varepsilon) + 5\varepsilon = 5$

$\langle S_1, C_2S_2 \rangle$

P2 : $1(1-\varepsilon) + 2\varepsilon = 1+\varepsilon$

P1 : $3(1-\varepsilon) + 5\varepsilon = 3+2\varepsilon$

$\langle C_1, S_2C_2 \rangle$

P2 : $3\varepsilon + 4(1-\varepsilon) = 4-\varepsilon$

P1 : $6\varepsilon + 4(1-\varepsilon) = 4+2\varepsilon$

$\langle C_1, C_2S_2 \rangle$

P2 : $3(1-\varepsilon) + 4\varepsilon = 3+\varepsilon$

P1 : $6(1-\varepsilon) + 4\varepsilon = 6-2\varepsilon$

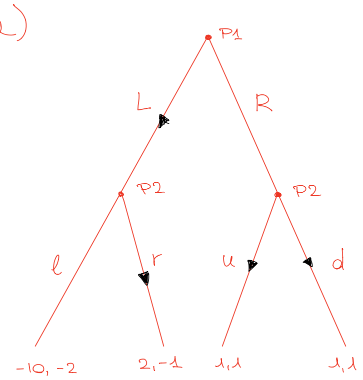
Hence, the game is now

		S_2S_2	S_2C_2	C_2S_2	C_2C_2
P1	S_1	5,2	$5-2\varepsilon$, $2-\varepsilon$	$3+2\varepsilon$, $1+\varepsilon$	3,1
	C_1	6,3	$4+2\varepsilon$, $4-\varepsilon$	$6-2\varepsilon$, $3+\varepsilon$	4,4

← the previous SPE is no longer NE! Only $\langle C_1, C_2C_2 \rangle$ is maintained

PROBLEM 4

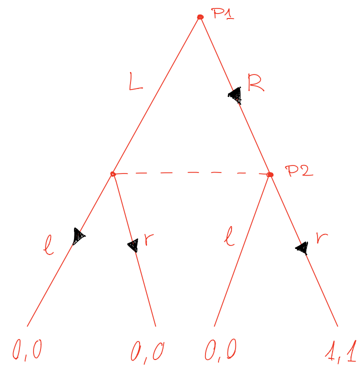
a)



		P2			
		lu	ld	ru	rd
P1	L	-10, -2	-10, -2	2, -1	2, -1
	R	1,1	1,1	1,1	1,1

SPE: $\langle L, rd \rangle$ (choosing l is never optimal in SPE)
$\langle L, ru \rangle$

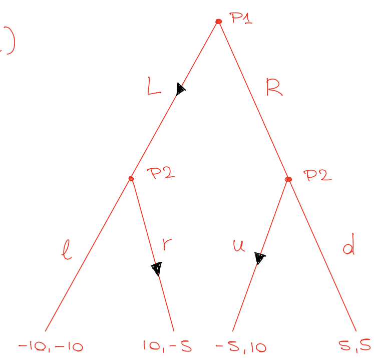
b)



		P2	
		l	r
P1	L	0,0	0,0
	R	0,0	1,1

Since the only subgame is the game itself,
NE = SPE

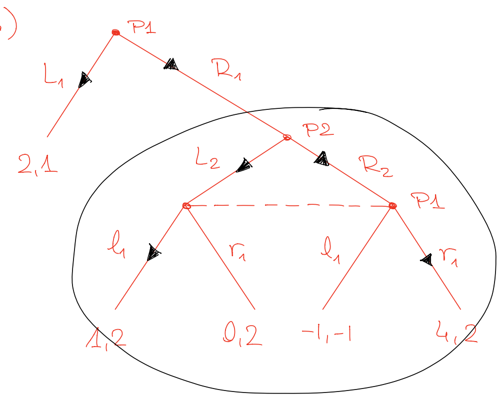
c)



		P2			
		lu	ld	ru	rd
P1	L	-10,-10	-10, -10	10, -5	~~10,-5~~
	R	~~-5,10~~	5,5	-5,10	5,5

SPE: $\langle L, ru \rangle$ (if L is played, r has to be played
if R " " , u " " " ")

d)

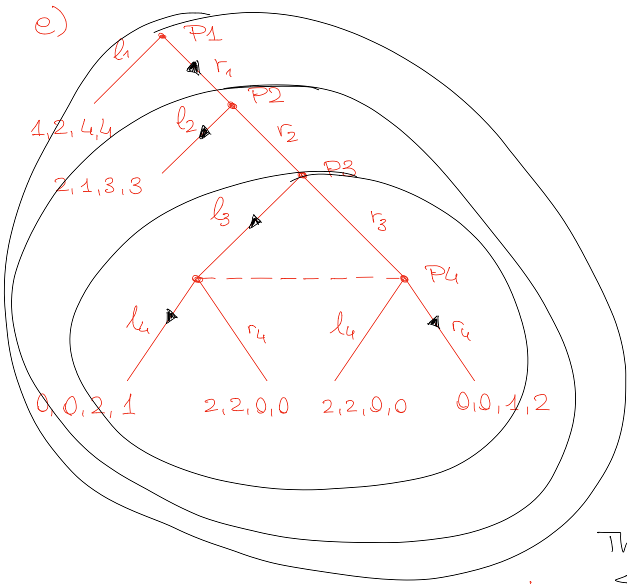


		P2	
		L_2	R_2
P1	$L_1 l_1$	2,1	2,1
	$L_1 r_1$	~~2,1~~	2,1
	$R_1 l_1$	1,2	-1,-1
	$R_1 r_1$	0,2	4,2

		P1	
		l_1	r_1
P2	L_2	1,2	0,2
	R_2	-1,-1	4,2

in the subgame only $\langle L_2, l_1 \rangle$ and $\langle R_2, r_1 \rangle$ are played $\Rightarrow$ play r_1 after L_2 not SPE!

e)



We have to check the 3 different subgames. Let us start with the smallest one

		P4	
		l_4 (p)	r_4 ($1-p$)
P3	l_3 (r)	2,1	0,0
	r_3 ($1-r$)	0,0	1,2

Let us check mixed strategies:

$l_3 \equiv r_3 \Rightarrow 2p + 0(1-p) = 0p + 1(1-p) \Rightarrow p^* = 1/3$

$l_4 \equiv r_4 \Rightarrow 1r + 0(1-r) = 0r + 2(1-r) \Rightarrow r^* = 2/3$

There are 3 NE: $< l_3, l_4 >$, $< r_3, r_4 >$ and

unsure! $< \frac{2}{3} l_3 + \frac{1}{3} r_3 \, , \, \frac{1}{3} l_4 + \frac{2}{3} r_4 >$

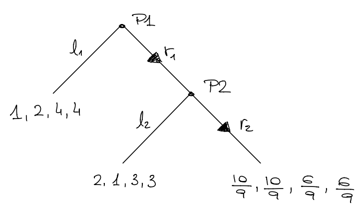
Therefore, only these three NE can potentially be a SPE. Consider first the MNE. If that part of the tree is played, payoffs would be

$$\frac{2}{3}\left(\frac{1}{3}(0,0,2,1) + \frac{2}{3}(2,2,0,0)\right) + \frac{1}{3}\left(\frac{1}{3}(2,2,0,0) + \frac{2}{3}(0,0,1,2)\right) =$$

$$= \frac{2}{3}\left(\left(0,0,\frac{2}{3},\frac{1}{3}\right) + \left(\frac{4}{3},\frac{4}{3},0,0\right)\right) + \frac{1}{3}\left(\left(\frac{2}{3},\frac{2}{3},0,0\right) + \left(0,0,\frac{2}{3},\frac{4}{3}\right)\right) =$$

$$= \frac{2}{3}\left(\frac{4}{3},\frac{4}{3},\frac{2}{3},\frac{1}{3}\right) + \frac{1}{3}\left(\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{4}{3}\right) = \left(\frac{8}{9},\frac{8}{9},\frac{4}{9},\frac{2}{9}\right) + \left(\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{4}{9}\right) = \left(\frac{10}{9},\frac{10}{9},\frac{6}{9},\frac{6}{9}\right)$$

Let us first finish with the MNE. With this randomization, the game becomes



Backward induction:

P2: $r_2 \succ l_2$ since $\frac{10}{9} > 1$

P1: $r_1 \succ l_1$ since $\frac{10}{9} > 1$

Hence, we already have the first SPE: MSPE $= < r_1, r_2, \frac{2}{3} l_3 + \frac{1}{3} r_3 \, , \, \frac{1}{3} l_4 + \frac{2}{3} r_4 >$

Focus now on pure NE. Let us now move to the second subgame

		P4	
		l_4	r_4
P3	l_3	1,3,3	1,3,3
	r_3	1,3,3	1,3,3

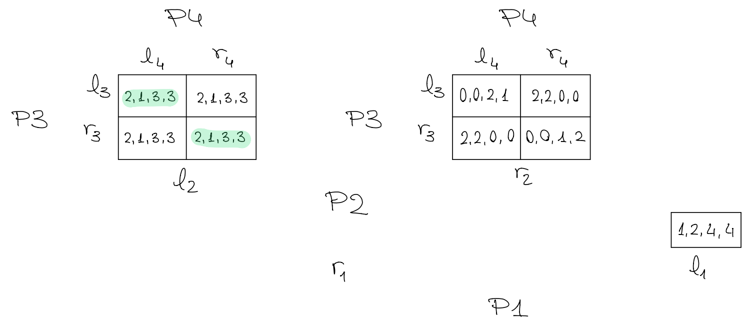
l_2

		P4	
		l_4	r_4
P3	l_3	0,2,1	2,0,0
	r_3	2,0,0	0,1,2

r_2

P2

Two NE: $< l_2, l_3, l_4 >$ and $< l_2, r_3, r_4 >$. Hence, both of the initial PNE survive the SPE criterion so far. Consider now the total game



So the initial PNE are SPE: $\langle r_1, l_2, l_3, l_4 \rangle$ and $\langle r_1, l_2, r_3, r_4 \rangle$