Exercises

Exercise 9.1 At each date $t \geq 1$, an economy consists of overlapping generations of a constant number N of two-period-lived agents. Young agents born in t have preferences over consumption streams of a single good that are ordered by $u(c_t^t) + u(c_{t+1}^t)$, where $u(c) = c^{1-\gamma}/(1-\gamma)$, and where c_t^i is the consumption of an agent born at i in time t. It is understood that $\gamma > 0$, and that when $\gamma = 1$, $u(c) = \ln c$. Each young agent born at $t \geq 1$ has identical preferences and endowment pattern (w_1, w_2), where w_1 is the endowment when young and w_2 is the endowment when old. Assume $0 < w_2 < w_1$. In addition, there are some initial old agents at time 1 who are endowed with w_2 of the time 1

[11] Abel, Mankiw, Summers, and Zeckhauser (1989) propose an empirical test of whether there is capital overaccumulation in the U.S. economy, and conclude that there is not.

consumption good, and who order consumption streams by c_1^0. The initial old (i.e., the old at $t = 1$) are also endowed with M units of unbacked fiat currency. The stock of currency is constant over time.

a. Find the saving function of a young agent.

b. Define an equilibrium with valued fiat currency.

c. Define a stationary equilibrium with valued fiat currency.

d. Compute a stationary equilibrium with valued fiat currency.

e. Describe how many equilibria with valued fiat currency there are. (You are not being asked to compute them.)

f. Compute the limiting value as $t \to +\infty$ of the rate of return on currency in each of the nonstationary equilibria with valued fiat currency. Justify your calculations.

Exercise 9.2 Consider an economy with overlapping generations of a constant population of an even number N of two-period-lived agents. New young agents are born at each date $t \geq 1$. Half of the young agents are endowed with w_1 when young and 0 when old. The other half are endowed with 0 when young and w_2 when old. Assume $0 < w_2 < w_1$. Preferences of all young agents are as in problem 1, with $\gamma = 1$. Half of the N initial old are endowed with w_2 units of the consumption good and half are endowed with nothing. Each old person orders consumption streams by c_1^0. Each old person at $t = 1$ is endowed with M units of unbacked fiat currency. No other generation is endowed with fiat currency. The stock of fiat currency is fixed over time.

a. Find the saving function of each of the two types of young person for $t \geq 1$.

b. Define an equilibrium without valued fiat currency. Compute all such equilibria.

c. Define an equilibrium with valued fiat currency.

d. Compute all the (nonstochastic) equilibria with valued fiat currency.

e. Argue that there is a unique stationary equilibrium with valued fiat currency.

f. How are the various equilibria with valued fiat currency ranked by the Pareto criterion?

Exercise 9.3 Take the economy of exercise *9.1*, but make one change. Endow the initial old with a tree that yields a constant dividend of $d > 0$ units of the consumption good for each $t \geq 1$.

a. Compute all the equilibria with valued fiat currency.

b. Compute all the equilibria without valued fiat currency.

c. If you want, you can answer both parts of this question in the context of the following particular numerical example: $w_1 = 10, w_2 = 5, d = .000001$.

1 a) The problem for an agent of generation t is

$$\max \quad \frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \frac{(c_{t+1}^t)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t^t + s_t^t \leq w_1$$

$$c_{t+1}^t \leq w_2 + s_t^t R_t$$

Forming the lagrangian

$$\mathcal{L} = \frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \frac{(c_{t+1}^t)^{1-\gamma}}{1-\gamma} + \lambda_t (w_1 - c_t^t - s_t^t) + \lambda_{t+1} (w_2 + s_t^t R_t - c_{t+1}^t)$$

Taking the FOC's

$$\left.\begin{aligned} c_t^t &: (c_t^t)^{-\gamma} - \lambda_t = 0 \\ c_{t+1}^t &: (c_{t+1}^t)^{-\gamma} - \lambda_{t+1} = 0 \\ s_t^t &: -\lambda_t + \lambda_{t+1} R_t = 0 \end{aligned}\right\} \quad c_{t+1}^t = R_t^{\frac{1}{\gamma}} c_t^t \qquad (1)$$

Equalizing both budget constraints,

$$s_t^t = s_t^t \Longrightarrow w_1 - c_t^t = \frac{c_{t+1}^t - w_2}{R_t} \qquad (2)$$

Introducing (1) into (2)

$$w_1 - c_t^t = \frac{R_t^{1/\gamma} c_t^t - w_2}{R_t} \Longrightarrow c_t^t = \frac{R_t w_1 + w_2}{R_t\left(1 + R_t^{\frac{1-\gamma}{\gamma}}\right)}$$

and using (1), $c_{t+1}^t = \dfrac{R_t w_1 + w_2}{R_t^{\frac{\gamma-1}{\gamma}}\left(1 + R_t^{\frac{1-\gamma}{\gamma}}\right)}$. Using the budget constraint of the young

$$s_t^t(R_t) = w_1 - \frac{R_t w_1 + w_2}{R_t\left(1 + R_t^{\frac{1-\gamma}{\gamma}}\right)} = \frac{w_1 - R_t^{-\frac{1}{\gamma}} w_2}{1 + R_t^{\frac{\gamma-1}{\gamma}}} \qquad (3)$$

Intuitively, one should expect $s_t^{t\,\prime}(R_t) > 0$ (an increase in the interest rate (reward from saving) increases savings).

$$s_t^{t\,\prime}(R_t) = \frac{R_t^{-\frac{1}{\gamma}} \overbrace{\left(\frac{1-\gamma}{\gamma} + w_2 R_t + w_2 R_t^{1/\gamma}\right)}^{>0 \text{ for sure if } \gamma \in (0,1]}}{\underbrace{\left(1 + R_t^{\frac{\gamma-1}{\gamma}}\right)^2}_{>0 \text{ always}}}$$

hence, our intuition was right!

It only remains to solve the initial old problem.

$$\max \quad \frac{(c_1^0)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_1^0 \leq w_2$$

Hence, $c_1^0 = w_2$ (if we assume $S_0 P_0 = 0$ or no monetary endowment $M = 0$).

b) Def: An equilibrium with valued fiat money M is an allocation $\{c_1^0, \{c_t^t, c_{t+1}^t\}_{\forall t \geq 1}\}$ and a sequence of nominal price $\{P_t > 0\}_{\forall t}$ such that

- Given P_t, household problem solves

$$\max \quad \frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \frac{(c_{t+1}^t)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t^t + \frac{m_t}{P_t} \leq w_1$$

$$c_{t+1}^t \leq w_2 + \frac{m_t}{P_{t+1}} \qquad \forall t \geq 1$$

- Given P_1, initial old solves

$$\max \quad \frac{(c_1^0)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_1^0 \leq w_2 + \frac{m_0}{P_1}$$

- Markets clear

$$c_t^t + c_t^{t-1} = w_1 + w_2$$

$$m_t = M \qquad \forall t$$

One should also impose $P_t < \infty$ so that the problem is well-defined.

c) Def: A stationary equilibrium with valued fiat money is an equilibrium (see (b)) such that

$$c_t^t = c_y$$

$$c_{t+1}^t = c_o$$

c_1^0 might be different

d) From the budget constraint for young

$$c_t^t = w_1 - \frac{m_t}{P_t}$$

Doing the same for generation $t+1$,

$$c_{t+1}^{t+1} = w_1 - \frac{m_{t+1}}{P_{t+1}}$$

Since $c_t^t = c_{t+1}^{t+1} = c_y$ and $m_t = m_{t+1} = M$,

$$\omega_1 - \frac{M}{P_t} = \omega_1 - \frac{M}{P_{t+1}} \Longrightarrow P_t = P_{t+1} = P$$

Hence, since $R_t = \frac{P_t}{P_{t+1}}$, $R_t = R = 1$. Equalizing both budget constraints,

$$m_t = m_t \Longrightarrow c_t^t + \frac{P_{t+1}}{P_t} c_{t+1}^t = \omega_1 + \frac{P_{t+1}}{P_t} \omega_2$$

(comparing it to (2), $R_t = \frac{P_t}{P_{t+1}}$ as we know). Since the problem we solved in (a) is identical with $\frac{m_t}{P_t} = s_t^t$, we take (3) in steady state

$$\frac{M}{P} = \frac{\omega_1 - \omega_2}{2} \Longrightarrow p = \frac{2M}{\omega_1 - \omega_2} \qquad (4)$$

Using both budget constraints, inserting (4),

$$c_y = \omega_1 - \frac{M}{P} = \frac{\omega_1 + \omega_2}{2}$$

$$c_o = \omega_2 + \frac{M}{P} = \frac{\omega_1 + \omega_2}{2}$$

$$c_0^1 = \omega_2 + \frac{M}{P} = \frac{\omega_1 + \omega_2}{2}$$

Notice that $p > 0$ is satisfied from $\omega_2 > \omega_1$ in (4).

e) Rewrite (3) with $R_t = \frac{P_t}{P_{t+1}}$ and $s_t^t = \frac{m_t}{P_t} = \frac{M}{P_t}$

$$\frac{M}{P_t} = \frac{\omega_1 - \left(\frac{P_{t+1}}{P_t}\right)^{\frac{1}{\gamma}} \omega_2}{1 + \left(\frac{P_{t+1}}{P_t}\right)^{\frac{1-\gamma}{\gamma}}} \qquad (5)$$

Since $P_t \geq 0 \ \forall t$, we know that LHS ≥ 0. From RHS, we know that denominator $1 + \left(\frac{P_{t+1}}{P_t}\right)^{\frac{1-\gamma}{\gamma}} \geq 0$. Hence, it must be that

$$\omega_1 - \left(\frac{P_{t+1}}{P_t}\right)^{\frac{1}{\gamma}} \omega_2 \geq 0 \Longrightarrow R_t = \frac{P_t}{P_{t+1}} \geq \left(\frac{\omega_2}{\omega_1}\right)^{\gamma}$$

hence, $R_t = \left(\frac{\omega_2}{\omega_1}\right)^{\gamma}$ is the lower bound for savings return. If this holds,

$$s_t^t(R_t) = \frac{M}{P_t}\left(\frac{P_t}{P_{t+1}}\right) = \frac{\omega_1 - \left[\left(\frac{\omega_1}{\omega_2}\right)^{\frac{1}{\gamma}}\right]^{\gamma} \omega_2}{1 + \left[\left(\frac{\omega_1}{\omega_2}\right)^{\frac{1}{\gamma}}\right]^{\frac{1-\gamma}{\gamma}}} = 0$$

In such R_t we would be in autarky! (no trade) and $c_y = \omega_1$, $c_o = \omega_2$

Let us now find the upper bound. We know that there do not exist equilibria with $R_t > 1$ (explosive), so lets prove that

$$R_t \in \left[\left(\frac{\omega_2}{\omega_1}\right)^{\gamma}, 1\right]$$

We already showed that $s'(R) > 0$ (that is, savings will increase as

R_t increases from its lower bound $\underline{R}_t$). Notice that we can write savings as

$$s_t = \frac{M}{P_t} = \frac{M}{P_{t-1}}\frac{P_{t-1}}{P_t} = s_{t-1}R_{t-1}$$

Hence, if $R_t > 1$, $s_t > s_{t-1}$ $\forall t$. Savings would be increasing over time. Actually, they would grow unboundedly

$$s_t \xrightarrow[t\to\infty]{} \infty$$

This cannot be the case since, from young's budget constraint,

$$c_t^t \leq \omega_1 - s_t \implies s_t \leq \omega_1 - c_t^t$$

The lower bound for c_t^t is 0 $(c_t^t \geq 0)$. Hence, the upper bound of s_t is $\omega_1 < \infty$ $(s_t \leq \omega_1)$. In this economy, you cannot borrow against future endowments!

Let us now check $R_t < 1$. Notice that $s_t = s_{t-1}R_{t-1}$ would imply $s_t < s_{t-1}$. Hence, savings would decrease over time,

$$s_t \xrightarrow[t\to\infty]{} 0$$

the economy would monotonically converge to autarky. Hence, the set of equilibria are those where

$$R_t \in \left[\left(\frac{\omega_2}{\omega_1}\right)^\gamma, 1\right]$$

such that $R_t = 1$ is the steady-state equilibrium, which additionaly requires that $\omega_1 > \omega_2$.

f) If there is money (trade), it must be that the economy is not in autarky

$$R_t = \frac{P_t}{P_{t+1}} > \left(\frac{\omega_2}{\omega_1}\right)^\gamma$$

and to be sustained as an equilibrium it must be that $R_t \leq 1$. We showed above that this case implies that $s_t = \frac{M}{P_t} \xrightarrow[t\to\infty]{} 0$. Thus, keeping money stock M constant, it implies that $P_t \xrightarrow[t\to\infty]{} \infty$. We know that $s_t = 0$ implies

$$s_t = \frac{M}{P_t} = 0 \implies R_t = \frac{P_t}{P_{t+1}} = \left(\frac{\omega_2}{\omega_1}\right)^\gamma$$

hence, as $s_t = \frac{M}{P_t} \xrightarrow[t\to\infty]{} 0$, $\frac{P_t}{P_{t+1}} \xrightarrow[t\to\infty]{} \left(\frac{\omega_2}{\omega_1}\right)^\gamma$.

Interest rates are negative $(R_t = 1 + r_t < 1)$ because inflation is high $\left(\frac{P_{t+1}}{P_t} > 1 \Rightarrow P_{t+1} > P_t\right)$. As $P_t \xrightarrow[t\to\infty]{} \infty$, inflation keeps growing until nobody is willing to trade

② a) $\max \quad \ln c_{it}^t + \ln c_{it+1}^t$

$$\text{s.t.} \quad c_{it}^t + s_{it} \leq y_{it}^t$$
$$c_{it+1}^t \leq y_{it+1}^t + R_t s_{it}$$

Forming the lagrangian

$$\mathcal{L} = \ln c_{it}^t + \ln c_{it+1}^t + \lambda_t \left(y_{it}^t - c_{it}^t - s_{it} \right) + \lambda_{t+1} \left(y_{it+1}^t + R_t s_{it} - c_{it+1}^t \right)$$

Taking the FOC's,

$$\left.\begin{aligned} c_{it}^t &: \frac{1}{c_{it}^t} - \lambda_t = 0 \\ c_{it+1}^t &: \frac{1}{c_{it+1}^t} - \lambda_{t+1} = 0 \\ s_{it} &: -\lambda_t + \lambda_{t+1} R_t = 0 \end{aligned}\right\} \quad c_{it+1}^t = R_t c_{it}^t \qquad (6)$$

Combining both budget constraints,

$$s_{it} = s_{it} \Longrightarrow y_{it}^t - c_{it}^t = \frac{c_{it+1}^t - y_{it+1}^t}{R_t} \xrightarrow[(6)]{} y_{it}^t - c_{it}^i = \frac{R_t c_{it}^t - y_{it+1}^t}{R_t} \Longrightarrow$$

$$\Longrightarrow c_{it}^t = \frac{1}{2}\left[y_{it}^t + \frac{y_{it+1}^t}{R_t} \right]$$

and hence $c_{it+1}^t = \frac{1}{2}\left[R_t y_{it}^t + y_{it+1}^t \right]$. Finally, from the young budget constraint,

$$s_{it} = y_{it}^t - c_{it}^t = \frac{1}{2}\left[y_{it}^t - \frac{y_{it+1}^t}{R_t} \right]$$

The problem for the initial old reads,

$$\max \quad c_{i1}^0$$
$$\text{s.t.} \quad c_{i1}^0 \leq y_{i1}^1$$

hence, $c_{i1}^0 = y_{i1}^1$

Assume that $y_{At}^t = \omega_1$, $y_{Bt}^t = 0$, $y_{At+1}^t = 0$ and $y_{Bt+1}^t = \omega_2$. Hence,

$$\begin{aligned} c_{At}^t &= \frac{\omega_1}{2} & c_{Bt}^t &= \frac{\omega_2}{2} \frac{1}{R_t} \\ c_{At+1}^t &= \frac{\omega_1}{2} R_t & c_{Bt+1}^t &= \frac{\omega_2}{2} \\ s_{At} &= \frac{\omega_1}{2} & s_{Bt} &= -\frac{\omega_2}{2} \frac{1}{R_t} \\ c_{A1}^0 &= 0 & c_{B1}^0 &= \omega_2 \end{aligned} \qquad (7)$$

Notice that s_{At} is independent of R_t. On the other hand, $s_B'(R) = \frac{\omega_2}{2R_t^2} > 0$ which is what we were expecting

intuition ?

b) <u>Def</u>: An equilibrium without valued fiat money is an allocation $\{c_{i1}^0, \{c_{it}^t, c_{it+1}^t, s_{it}\}_{\forall t\geq 1, i}\}$ and a price sequence $\{R_t > 0\}_{\forall t}$ such that

- Given R_t, household i problem solves

$$\max \quad \ln c_{it}^t + \ln c_{it+1}^t$$
$$\text{s.t.} \quad c_{it}^t + s_{it} \leq y_{it}^t$$
$$c_{it+1}^t \leq y_{it+1}^t + R_t s_{it}$$

- Initial old solves

$$\max \quad c_{i1}^0$$
$$\text{s.t.} \quad c_{i1}^0 \leq y_{i1}^1$$

- Markets clear

$$\frac{N}{2}\left(c_{At}^t + c_{Bt}^t\right) + \frac{N}{2}\left(c_{At}^{t-1} + c_{Bt}^{t-1}\right) = \frac{N}{2}\omega_1 + \frac{N}{2}\omega_2$$
$$s_{At} + s_{Bt} = 0$$

Notice that I have already solved it in (a). It only remain to show that, through asset market clearing,

$$s_{At} + s_{Bt} = \frac{\omega_1}{2} - \frac{\omega_2}{2}\frac{1}{R_t} = 0 \implies R_t = R = \frac{\omega_2}{\omega_1} < 1 \qquad (\omega_2 < \omega_1)$$

Notice that $R_t = \underline{R}_t$ in the previous exercise. Hence, we are in autarky! There is only <u>intra-generational</u> trade, no <u>inter-generational</u> trade.

c) <u>Def</u>: An equilibrium with valued fiat money M is an allocation $\{c_{i1}^0, \{c_{it}^t, c_{it+1}^t\}_{\forall t\geq 1, i}\}$ and a sequence of nominal prices $\{P_t > 0\}_{\forall t}$ such that

need to add R_t? don't think so!

- Given P_t, household problem solves

$$\max \quad \ln c_{it}^t + \ln c_{it+1}^t$$
$$\text{s.t.} \quad c_{it}^t + \frac{m_{it}}{P_t} \leq y_{it}^t$$
$$c_{it+1}^t \leq y_{it+1}^t + \frac{m_{it}}{P_{t+1}}$$

- Given P_1, initial old solves

$$\max \quad c_{i1}^0$$
$$\text{s.t.} \quad c_{i1}^0 \leq y_{i1}^1 + \frac{m_0}{P_1}$$

differs a lot from book and previous years solutions. I claim they are solving an economy with valued money AND BONDS!

- Markets clear

$$\frac{N}{2}\left(c^t_{At} + c^t_{Bt}\right) + \frac{N}{2}\left(c^{t-1}_{At} + c^{t-1}_{Bt}\right) = \frac{N}{2}\omega_1 + \frac{N}{2}\omega_2 \quad \forall t$$

$$\frac{N}{2} m_{At} + \frac{N}{2} m_{Bt} = NM$$

One should also impose $P_t < \infty$ so that the problem is well-defined.

d) Instead of solving the exercise again, mapping the solution in (7) of an economy without money to an economy with, and setting $s_{it} = \frac{m_{it}}{P_t}$ and $R_t = \frac{P_t}{P_{t+1}}$,

$$c^t_{At} = \frac{\omega_1}{2} \qquad c^t_{Bt} = \frac{\omega_2}{2}\frac{P_{t+1}}{P_t}$$

$$c^t_{At+1} = \frac{\omega_1}{2}\frac{P_t}{P_{t+1}} \qquad c^t_{Bt+1} = \frac{\omega_2}{2} \qquad (8)$$

$$\frac{m_A}{P_t} = \frac{\omega_1}{2} \qquad \frac{m_{Bt}}{P_t} = -\frac{\omega_2}{2}\frac{P_{t+1}}{P_t}$$

$$c^0_{1A} = \frac{M}{P_1} \qquad c^0_{B1} = \omega_2 + \frac{M}{P_1}$$

It only remains to find the price sequence. Using the money market clearing,

$$m_{At} + m_{Bt} = 2M \implies \frac{\omega_1}{2}P_t - \frac{\omega_2}{2}P_{t+1} = 2M \implies P_{t+1} = \frac{\omega_1}{\omega_2}P_t - \frac{4M}{\omega_2}$$

Let us solve this in two ways:

1) Difference equations: $x_{t+1} = a x_t + b \implies x_t = a^t\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$

$$P_t = \left(\frac{\omega_1}{\omega_2}\right)^t\left(P_0 + \frac{4M}{\omega_2 - \omega_1}\right) - \frac{4M}{\omega_2 - \omega_1} \qquad (9)$$

2) Lag operator: $L x_t = x_{t-1}$, $L^{-1}x_t = x_{t+1}$

$$2M = \frac{\omega_1}{2}P_t - \frac{\omega_2}{2}P_{t+1} \implies 4M = \omega_1 P_t - \omega_2 P_{t+1} = \left[\omega_1 - \omega_2 L^{-1}\right]P_t \implies$$

$$\implies \frac{4M}{\omega_1} = \left[1 - \frac{\omega_2}{\omega_1}L^{-1}\right]P_t$$

$$\implies P_t = \frac{4M}{\omega_1\left[1 - \frac{\omega_2}{\omega_1}L^{-1}\right]} = \frac{4M}{\omega_1\left[1 - \frac{\omega_2}{\omega_1}\right]} + K\left(\frac{\omega_1}{\omega_2}\right)^t$$

$$\implies P_t = \frac{4M}{\omega_1 - \omega_2} + K\left(\frac{\omega_1}{\omega_2}\right)^t \qquad (*)$$

check: $\left[1 - \frac{\omega_2}{\omega_1}L^{-1}\right]K\left(\frac{\omega_1}{\omega_2}\right)^t = K\left(\frac{\omega_1}{\omega_2}\right)^t - \frac{\omega_2}{\omega_1}K\left(\frac{\omega_1}{\omega_2}\right)^{t+1} = 0$

In order to restrict the equilibria to those with money, we exclude the case with $K = \infty$ (autarky, with $P_t \xrightarrow[t\to\infty]{} \infty$) where money has no value

e) Notice that $\left(\frac{w_1}{w_2}\right)^t \xrightarrow[t\to\infty]{} \infty$ since $w_1 > w_2$. Let's study the two possible cases

- $K > 0$: in this case $P_0 > \frac{4M}{w_1 - w_2}$, and we obtain a continuum of non-stationary equilibria with fiat money, such that $P_t > P_{t-1}$ and $P_t \xrightarrow[t\to\infty]{} \infty$
- $K = 0$: in this case $P_0 = \frac{4M}{w_1 - w_2}$ we obtain a unique stationary equilibrium with (from (9))

$$P_t = P = \frac{4M}{w_1 - w_2} \qquad (10)$$

(which also implies $R_t = \frac{P_t}{P_{t+1}} = \frac{P}{P} = 1 = R$). The equilibrium is characterized by (using (8) and (10)),

$$c^t_{At} = c_{Ay} = \frac{w_1}{2} \qquad c^t_{Bt} = c_{By} = \frac{w_2}{2}$$

$$c^t_{At+1} = c_{Ao} = \frac{w_1}{2} \qquad c^t_{Bt+1} = c_{Bo} = \frac{w_2}{2}$$

$$\frac{m_A}{P_t} = \frac{m_A}{P} = \frac{w_1}{2} \qquad \frac{m_{Bt}}{P_t} = \frac{m_B}{P} - \frac{w_2}{2}$$

differ from previous solution

$$c^0_{1A} = \frac{m_A}{P} = \frac{w_1}{2} \qquad c^0_{B1} = w_2 + \frac{m_B}{P} = \frac{w_2}{2}$$

$$m_A + m_B = 2M \Rightarrow \frac{M}{P} = \frac{w_1 - w_2}{4}$$

In the text it is written that all are endowed with M. Then $m_A \overset{!}{=} m_B \Rightarrow$

$$\Rightarrow \frac{w_1}{2}P = -\frac{w_2}{2}P \Rightarrow w_1 = -w_2$$

this would require either $w_1 < 0$ or $w_2 < 0$ $\rightarrow$ eq not possible!

f) Along equilibrium path, the utility of an agent of type $i = A$ is

$$U_A = \log\left(\frac{w_1}{2}\right) + \log\left(\frac{w_1}{2}R_t\right) \qquad \forall t \geq 1$$

where $U'_A(R) > 0$. For type $i = B$,

$$U_B = \log\left(\frac{w_2}{2}\frac{1}{R_t}\right) + \log\left(\frac{w_2}{2}\right) \qquad \forall t \geq 1$$

where instead $U'_B(R) < 0$. To find K in $(\star)$, set for example $t = 1$,

$$P_1 = \frac{4M}{w_1 - w_2} + K\frac{w_1}{w_2} \Rightarrow K = \frac{w_2}{w_1}\left(P_1 - \frac{4M}{w_1 - w_2}\right)$$

hence one could rewrite $(\star)$ as

$$P_t = \frac{4M}{w_1 - w_2} + \left(P_1 - \frac{4M}{w_1 - w_2}\right)\left(\frac{w_1}{w_2}\right)^{t-1} \qquad (11)$$

Rewriting (11) as

$$R_t = \frac{P_t}{P_{t+1}} = \frac{\frac{4M}{w_1 - w_2} + \left(P_1 - \frac{4M}{w_1 - w_2}\right)\left(\frac{w_1}{w_2}\right)^{t-1}}{\frac{4M}{w_1 - w_2} + \left(P_1 - \frac{4M}{w_1 - w_2}\right)\left(\frac{w_1}{w_2}\right)^{t}}$$

shows that $R'(P_1) < 0$. Hence, $\tilde{P}_1 > P_1 \Rightarrow \tilde{R}_t < R_t$. Along equilibrium path, utility for initial old is

$$u_A^0 = \frac{m_A}{P_1} \quad , \quad u_B^0 = \frac{m_B}{P_1} + \omega_2$$

Let us focus on the four sets of individuals:

- $i = A$, excluding initial old: an increase (decrease) in P_1 will decrease (increase) R_t, and they would be worse (better)-off
- $i = B$, excluding initial old: an increase (decrease) in P_1 will decrease (increase) R_t, and they would be better (worse)-off
- Initial old: by increasing (decreasing) P_1 both would be worse (better)-off

3 a) Even though it is not required, let me define the equilibrium

Def: An equilibrium with valued fiat money M is an allocation $\{c_1^0, \{c_t^t, c_{t+1}^t\}_{\forall t \geq 1}\}$, demand for tree shares $\{\alpha_t \in (0,1)\}_{\forall t}$ a sequence of nominal prices $\{P_t > 0\}_{\forall t}$ and a sequence of prices for the tree $\{q_t > 0\}_{\forall t}$ such that

- Given P_t and q_t, household problem solves

$$\max \quad \frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \frac{(c_{t+1}^t)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_t^t + \frac{m_t}{P_t} + \alpha_t q_t \leq \omega_1 \qquad \forall t \geq 1$$

$$c_{t+1}^t \leq \omega_2 + \frac{m_t}{P_{t+1}} + \alpha_t (q_{t+1} + d)$$

- Given P_1, initial old solves

$$\max \quad \frac{(c_1^0)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_1^0 \leq \omega_2 + \frac{m_0}{P_1} + q_1 + d$$

- Markets clear

$$c_t^t + c_t^{t-1} = \omega_1 + \omega_2 + d$$

$$m_t = M \qquad \forall t$$

$$\alpha_t = 1$$

One should also impose $P_t < \infty$ and $q_t < \infty$ so that the problem is well-defined.

Taking the Lagrangian,

$$\mathcal{L} = \frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \frac{(c_{t+1}^t)^{1-\gamma}}{1-\gamma} + \lambda_t\left(\omega_1 - c_t^t - \frac{m_t}{P_t} - \alpha_t q_t\right) + \lambda_{t+1}\left[\omega_2 + \frac{m_t}{P_{t+1}} + \alpha_t(q_{t+1}+d) - c_{t+1}^t\right]$$

Taking the FOC's,

$$c_t^t : (c_t^t)^{-\gamma} - \lambda_t = 0 \quad (12)$$

$$c_{t+1}^t : (c_{t+1}^t)^{-\gamma} - \lambda_{t+1} = 0 \quad (13)$$

$$m_t : -\frac{\lambda_t}{P_t} + \frac{\lambda_{t+1}}{P_{t+1}} = 0 \quad (14)$$

$$\alpha_t : -\lambda_t q_t + \lambda_{t+1}(q_{t+1}+d) = 0 \quad (15)$$

combining (12)-(14),

$$c_t^t = \left(\frac{P_{t+1}}{P_t}\right)^{1/\gamma} c_{t+1}^t \quad (16)$$

combining (12)-(13) and (15)

$$c_t^t = \left(\frac{q_t}{q_{t+1}+d}\right)^{1/\gamma} c_{t+1}^t \quad (17)$$

combining (14)-(15),

$$R_t \equiv \frac{P_t}{P_{t+1}} = \frac{q_{t+1}+d}{q_t} \quad (18)$$

which implies that both assets (money and tree) must yield the same return. Let us now find $q_t(R_t)$. Rewritting (18),

$$q_t = \frac{q_{t+1}+d}{R_t} = \underbrace{\frac{\frac{q_{t+2}+d}{R_{t+1}}+d}{R_t}}_{\frac{d}{R_t}+\frac{d+q_{t+2}}{R_tR_{t+1}}} = \dots = \sum_{k=0}^{\infty}\prod_{s=0}^{k}\frac{d}{R_{t+s}} \quad (19)$$

where we have assumed $q_t < \infty$. Notice that such an imposition implies that $R_t > 1 \ \forall t$ (otherwise $q_t \xrightarrow[t\to\infty]{} \infty$ since it would be an unbounded sum).

Separate now $\frac{m_t}{P_t} + \alpha_t q_t$ in both budget constraints

$$c_t^t + \frac{m_t}{P_t} + \alpha_t q_t = \omega_1 \implies \frac{m_t}{P_t} + \alpha_t q_t = \omega_1 - c_t^t$$

$$c_{t+1}^t = \omega_2 + \frac{m_t}{P_t}\underbrace{\frac{P_t}{P_{t+1}}}_{R_t} + \underbrace{\frac{(q_{t+1}+d)}{q_t}}_{R_t}\alpha_t q_t \implies \frac{m_t}{P_t} + \alpha_t q_t = \frac{c_{t+1}^t - \omega_2}{R_t}$$

equalizing both budget constraints

$$\frac{m_t}{P_t} + \alpha_t q_t = \frac{m_t}{P_t} + \alpha_t q_t \implies \omega_1 - c_t^t = \frac{c_{t+1}^t - \omega_2}{R_t}$$

Introducing either (16) or (17) (thanks to (18) we know they are identical),

$$\omega_1 - c_t^t = \frac{c_t^t R_t^{\frac{1}{\gamma}} - \omega_2}{R_t} \implies c_t^t = \frac{\omega_1 R_t + \omega_2}{R_t\left(1 + R_t^{\frac{1-\gamma}{\gamma}}\right)}$$

and $c_{t+1}^t = \dfrac{\omega_1 R_t + \omega_2}{R_t^{\frac{\gamma-1}{\gamma}}\left(1 + R_t^{\frac{1-\gamma}{\gamma}}\right)}$. It lasts to obtain the savings function

$$s_t \equiv \frac{m_t}{P_t} + \alpha_t q_t = \omega_1 - \frac{\omega_1 R_t + \omega_2}{R_t\left(1 + R_t^{\frac{1-\gamma}{\gamma}}\right)} = \frac{\omega_1 - R_t^{-\frac{1}{\gamma}} \omega_2}{1 + R_t^{\frac{\gamma-1}{\gamma}}} \tag{20}$$

We can now apply the market clearing conditions: since we are dealing with a representative agent model, $m_t = M$ and $\alpha_t = 1$. Hence, (20) reads

$$s_t \equiv \frac{M}{P_t} + q_t = \frac{\omega_1 - R_t^{-\frac{1}{\gamma}} \omega_2}{1 + R_t^{\frac{\gamma-1}{\gamma}}} \equiv \frac{\omega_1 - \left(\frac{P_{t+1}}{P_t}\right)^{\frac{1}{\gamma}} \omega_2}{1 + \left(\frac{P_{t+1}}{P_t}\right)^{\frac{1-\gamma}{\gamma}}} \tag{21}$$

which happens to be a first-order difference equation in P_{t+1}.

The initial old solves

$$\max \quad \frac{(c_1^0)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_1^0 \leq \omega_2 + \frac{m_0}{P_1} + q_1 + d$$

Hence, $c_1^0 = \omega_2 + \dfrac{M}{P_1} + q_1 + d$

Notice that, as we showed in exercise 9.1, $s'(R) > 0$.

From (19) we know that $q_t \longrightarrow \infty$ if $R_t \leq 1$. Hence, that case cannot be an equilibrium.

Notice that we can write s_t as

$$s_t = R_{t-1} s_{t-1} - d \qquad \left[\frac{M}{P_t} + q_t = R_{t-1}\left(\frac{M}{P_{t-1}} + q_{t-1}\right) - d \equiv \frac{M}{\cancel{P_{t-1}}} \frac{\cancel{P_{t-1}}}{P_t} + \frac{q_t + \cancel{d}}{\cancel{q_{t-1}}} \cancel{q_{t-1}} - \cancel{d} \right] \tag{22}$$

Hence, if $R_t > 1$, savings increase over time. As a result, $s_t \xrightarrow[t \to \infty]{} \infty$, which is again not an equilibrium.

So there are no non-stationary equilibria. Let us now check for the stationary state. In this case, (22) looks like

$$s = Rs - d \implies s = \frac{d}{R-1} \tag{23}$$

(19) collapses to

$$q = \sum_{t=1}^{\infty} \left(\frac{1}{R}\right)^t d = \frac{1}{R} \sum_{t=0}^{\infty} \left(\frac{1}{R}\right)^t d = \frac{d}{R-1} \tag{24}$$

Introducing (23)-(24) into (21) in steady-state

$$s = \frac{M}{p} + q \implies \frac{M}{p} = \frac{d}{R-1} - \frac{d}{R-1} = 0$$

which implies that $p \to \infty$, since $M < \infty$. Hence, there is no valued fiat money equilibrium.

b) From (a) we know that there will only be one equilibrium, which happens to be stationary, where $s = q = \frac{d}{R-1}$. Plugging it into (21),

$$s \equiv \frac{d}{R-1} = \frac{w_1 - R^{-\frac{1}{\gamma}} w_2}{1 + R^{\frac{\gamma-1}{\gamma}}} \tag{25}$$

and the consumption allocation

$$c_y = \frac{w_1 R + w_2}{R\left(1 + R^{\frac{1-\gamma}{\gamma}}\right)} \qquad c_o = \frac{w_1 R + w_2}{R^{\frac{\gamma-1}{\gamma}}\left(1 + R^{\frac{1-\gamma}{\gamma}}\right)}$$

c) I understand that $\gamma = 1$. Rewritting (25), we find

$$w_1 R^2 - (2d + w_1 + w_2) R + w_2 = 0 \implies R_{1,2} = \frac{2d + w_1 + w_2 \pm \sqrt{(2d + w_1 + w_2)^2 - 4 w_1 w_2}}{2 w_1} = \begin{cases} R_1 = 1.0000004 \\ R_2 = 0.50 \end{cases}$$

since we require $R > 1$, $R = 1.000000399999840$. Also,

$$c_y = \frac{w_1 R + w_2}{2R} = 7.499990$$

$$c_o = \frac{w_1 R + w_2}{2} = 7.50000199999990$$

$$c_1^0 = d + q = d\left(\frac{R}{R-1}\right) = 2.50000199999990$$