Exercise 8.4 Consider a pure endowment economy with a single representative consumer; $\{c_t, d_t\}_{t=0}^{\infty}$ are the consumption and endowment processes, respectively. Feasible allocations satisfy

$$c_t \leq d_t.$$

The endowment process is described by [20]

$$d_{t+1} = \lambda_{t+1} d_t.$$

The growth rate λ_{t+1} is described by a two-state Markov process with transition probabilities

$$P_{ij} = \text{Prob}(\lambda_{t+1} = \bar{\lambda}_j | \lambda_t = \bar{\lambda}_i).$$

Assume that

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix},$$

and that

$$\bar{\lambda} = \begin{bmatrix} .97 \\ 1.03 \end{bmatrix}.$$

In addition, $\lambda_0 = .97$ and $d_0 = 1$ are both known at date 0. The consumer has preferences over consumption ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

where E_0 is the mathematical expectation operator, conditioned on information known at time 0, $\gamma = 2, \beta = .95$.

Part I

At time 0, after d_0 and λ_0 are known, there are complete markets in date- and history-contingent claims. The market prices are denominated in units of time 0 consumption goods.

[20] See Mehra and Prescott (1985).

a. Define a competitive equilibrium, being careful to specify all the objects composing an equilibrium.

b. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, \ldots, \lambda_5) = (.97, .97, 1.03, .97, 1.03)$. Please give a numerical answer.

c. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, \ldots, \lambda_5) = (1.03, 1.03, 1.03, 1.03, .97)$.

d. Give a formula for the price at time 0 of a claim on the entire endowment sequence.

e. Give a formula for the price at time 0 of a claim on consumption in period 5, contingent on the growth rate λ_5 being .97 (regardless of the intervening growth rates).

Part II

Now assume a different market structure. Assume that at each date $t \geq 0$ there is a complete set of one-period forward Arrow securities.

f. Define a (recursive) competitive equilibrium with Arrow securities, being careful to define all of the objects that compose such an equilibrium.

g. For the representative consumer in this economy, for each state compute the "natural debt limits" that constrain state-contingent borrowing.

h. Compute a competitive equilibrium with Arrow securities. In particular, compute both the pricing kernel and the allocation.

i. An entrepreneur enters this economy and proposes to issue a new security each period, namely, a risk-free two-period bond. Such a bond issued in period t promises to pay one unit of consumption at time $t+1$ for sure. Find the price of this new security in period t, contingent on λ_t.

Exercise 8.5

An economy consists of two consumers, named $i = 1, 2$. The economy exists in discrete time for periods $t \geq 0$. There is one good in the economy, which is not storable and arrives in the form of an endowment stream owned by each consumer. The endowments to consumers $i = 1, 2$ are

$$\begin{aligned} y_t^1 &= s_t \\ y_t^2 &= 1 \end{aligned}$$

where s_t is a random variable governed by a two-state Markov chain with values $s_t = \bar{s}_1 = 0$ or $s_t = \bar{s}_2 = 1$. The Markov chain has time invariant transition probabilities denoted by $\pi(s_{t+1} = s'|s_t = s) = \pi(s'|s)$, and the probability distribution over the initial state is $\pi_0(s)$. The *aggregate endowment* at t is $Y(s_t) = y_t^1 + y_t^2$.

Let c^i denote the stochastic process of consumption for agent i. Household i orders consumption streams according to

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)] \pi_t(s^t),$$

where $\pi_t(s^t)$ is the probability of the history $s^t = (s_0, s_1, \ldots, s_t)$.

a. Give a formula for $\pi_t(s^t)$.

b. Let $\theta \in (0, 1)$ be a Pareto weight on household 1. Consider the planning problem

$$\max_{c^1, c^2} \left\{ \theta \ln(c^1) + (1 - \theta) \ln(c^2) \right\}$$

where the maximization is subject to

$$c_t^1(s^t) + c_t^2(s^t) \leq Y(s_t).$$

Solve the Pareto problem, taking θ as a parameter.

c. Define a *competitive equilibrium* with history-dependent Arrow-Debreu securities traded once and for all at time 0. Be careful to define all of the objects that compose a competitive equilibrium.

d. Compute the competitive equilibrium price system (i.e., find the prices of all of the Arrow-Debreu securities).

e. Tell the relationship between the solutions (indexed by θ) of the Pareto problem and the competitive equilibrium allocation. If you wish, refer to the two welfare theorems.

f. Briefly tell how you can compute the competitive equilibrium price system *before* you have figured out the competitive equilibrium allocation.

g. Now define a recursive competitive equilibrium with trading every period in one-period Arrow securities only. Describe all of the objects of which such an equilibrium is composed. (Please denominate the prices of one-period time $t+1$ state-contingent Arrow securities in units of time t consumption.) Define the "natural borrowing limits" for each consumer in each state. Tell how to compute these natural borrowing limits.

h. Tell how to compute the prices of one-period Arrow securities. How many prices are there (i.e., how many numbers do you have to compute)? Compute all of these prices in the special case that $\beta = .95$ and $\pi(s_j|s_i) = P_{ij}$ where $P = \begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}$.

i. Within the one-period Arrow securities economy, a new asset is introduced. One of the households decides to market a one-period-ahead riskless claim to one unit of consumption (a one-period real bill). Compute the equilibrium prices of this security when $s_t = 0$ and when $s_t = 1$. Justify your formula for these prices in terms of first principles.

j. Within the one-period Arrow securities equilibrium, a new asset is introduced. One of the households decides to market a two-period-ahead riskless claim to one unit of consumption (a two-period real bill). Compute the equilibrium prices of this security when $s_t = 0$ and when $s_t = 1$.

k. Within the one-period Arrow securities equilibrium, a new asset is introduced. One of the households decides at time t to market five-period-ahead claims to consumption at $t+5$ contingent on the value of s_{t+5}. Compute the equilibrium prices of these securities when $s_t = 0$ and $s_t = 1$ and $s_{t+5} = 0$ and $s_{t+5} = 1$.

Exercise 8.15 **Diverse beliefs, III**

An economy consists of two consumers named $i = 1, 2$. Each consumer evaluates streams of a single nonstorable consumption good according to

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)] \pi_t^i(s^t).$$

Here $\pi_t^i(s^t)$ is consumer i's subjective probability over history s^t. A feasible allocation satisfies $\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t)$ for all $t \geq 0$ and for all s^t. The consumers' endowments of the one good are functions of a state variable $s_t \in \mathbf{S} = \{0, 1, 2\}$. In truth, s_t is described by a time invariant Markov chain with initial distribution $\pi_0 = [0 \quad 1 \quad 0]'$ and transition density defined by the stochastic matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .5 & 0 & .5 \\ 0 & 0 & 1 \end{bmatrix}$$

where $P_{ij} = \text{Prob}[s_{t+1} = j - 1 | s_t = i - 1]$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. The endowments of the two consumers are

$$\begin{aligned} y_t^1 &= s_t/2 \\ y_t^2 &= 1 - s_t/2. \end{aligned}$$

In part I, both consumers know the true probabilities over histories s^t (i.e., they know both π_0 and P). In part II, the two consumers have different subjective probabilities.

Part I:

Assume that both consumers know (π_0, P), so that $\pi_t^1(s^t) = \pi_t^2(s^t)$ for all $t \geq 0$ for all s^t.

a. Show how to deduce $\pi_t^i(s^t)$ from (π_0, P).

b. Define a competitive equilibrium with sequential trading of Arrow securities.

c. Compute a competitive equilibrium with sequential trading of Arrow securities.

d. By hand, simulate the economy. In particular, for every possible realization of the histories s^t, describe time series of c_t^1, c_t^2 and the wealth levels for the two consumers.

Part II:

Now assume that while consumer 1 knows (π_0, P), consumer 2 knows π_0 but thinks that P is

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 \\ .4 & 0 & .6 \\ 0 & 0 & 1 \end{bmatrix}.$$

e. Deduce $\pi_t^2(s^t)$ from $(\pi_0, \hat{P})$ for all $t \geq 0$ for all s^t.

f. Formulate and solve a Pareto problem for this economy.

g. Define an equilibrium with time 0 trading of a complete set of Arrow-Debreu history-contingent securities.

h. Compute an equilibrium with time 0 trading of a complete set of Arrow-Debreu history-contingent securities.

i. Compute an equilibrium with sequential trading of Arrow securities. For every possible realization of s^t for all $t \geq 0$, please describe time series of c_t^1, c_t^2 and the wealth levels for the two consumers.

① a) Def: A competitive equilibrium is an initial endowment d_0, an initial distribution λ_0, a consumption allocation $\{c_t(\lambda^t)\}_{\forall t,\lambda^t}$ where $\lambda^t=\{\lambda_t\}_{\forall t}$ is the entire history until time t, and a price system $\{q_t^0(\lambda^t)\}_{\forall t,\lambda^t}$ such that

- Given y_0 and λ_0, the consumer solves

$$\max_{c_t(\lambda^t)} \sum_{t=0}^{\infty}\sum_{\lambda^t} \beta^t \pi_t(\lambda^t)\, u[c_t(\lambda^t)]$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty}\sum_{\lambda^t} q_t^0(\lambda^t)\, c_t(\lambda^t) \leq \sum_{t=0}^{\infty}\sum_{\lambda^t} q_t^0(\lambda^t)\, d_t(\lambda^t)$$

- Markets clear (feasibility): $c_t(\lambda^t) = d_t(\lambda^t) \quad \forall t, \lambda^t$

b) The consumer problem is the previous one. Forming the Lagrangian,

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{\lambda^t} \beta^t \pi_t(\lambda^t) \frac{c_t(\lambda^t)^{1-\gamma}}{1-\gamma} + \mu\left\{\sum_{t=0}^{\infty}\sum_{\lambda^t} q_t^0(\lambda^t)\left[d_t(\lambda^t) - c_t(\lambda^t)\right]\right\}$$

Taking the FOC

$$c_t(\lambda^t): \ \beta^t \pi_t(\lambda^t)\, c_t(\lambda^t)^{-\gamma} - \mu q_t^0(\lambda^t) = 0$$

Notice that (i) there are no assets and (ii) goods are non-storable. Hence, at each time t, $c_t(\lambda^t) = d_t(\lambda^t)$ (given locally non-satiation). Therefore, we can write prices as

$$q_t^0(\lambda^t) = \beta^t \pi_t(\lambda^t) \frac{d_t(\lambda^t)^{-\gamma}}{\mu}$$

Setting $q_0^0(\lambda^0)=1$ as numeraire, rewritting the FOC

$$\beta^0 \pi_0(\lambda^0)\, c_0(\lambda^0)^{-\gamma} - \mu = 0 \implies \underbrace{c_0(\lambda^0)}_{d_0(\lambda^0)}{}^{-\gamma} = \mu$$

Hence, prices read

$$q_t^0(\lambda^t) = \beta^t \pi_t(\lambda^t)\left[\frac{d_t(\lambda^t)}{d_0(\lambda^0)}\right]^{-\gamma} \qquad (1)$$

Therefore, the price at $t=0$ of a claim to 1 unit of consumption at $t=5$, contingent on a given history λ^t is

$$q_5^0(\lambda^5) = q_5^0\left(\{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\} = \{0.97, 0.97, 1.03, 0.97, 1.03\}\right) =$$

$$= \beta^5 \pi(\lambda_1|\lambda_0)\pi(\lambda_2|\lambda_1)\pi(\lambda_3|\lambda_2)\pi(\lambda_4|\lambda_3)\pi(\lambda_5|\lambda_4)\left[\frac{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 d_0}{d_0}\right]^{-\gamma}$$

$$= \beta^5\left[P_{11}P_{11}P_{12}P_{21}P_{12}\right](\bar\lambda_1\bar\lambda_1\bar\lambda_2\bar\lambda_1\bar\lambda_2)^{-\gamma} = 0.0021$$

c) $$q_5^0(\lambda^5) = q_5^0\left(\{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\} = \{1.03, 1.03, 1.03, 1.03, 0.97\}\right) =$$

$$= \beta^5 \pi(\lambda_1|\lambda_0)\pi(\lambda_2|\lambda_1)\pi(\lambda_3|\lambda_2)\pi(\lambda_4|\lambda_3)\pi(\lambda_5|\lambda_4)\left[\frac{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 d_0}{d_0}\right]^{-\gamma}$$

$$= \beta^5\left[P_{12}P_{22}P_{22}P_{22}P_{21}\right](\bar\lambda_2\bar\lambda_2\bar\lambda_2\bar\lambda_2\bar\lambda_1)^{-\gamma} = 0.0095$$

Julian has it much more simple. They only ask us to give the formulas

d) An asset that provides a claim on the entire endowment sequence is calculated by adding the price $q_t^0(\lambda^t)$ of Arrow-Debreu securities for all time t and history λ^t.

$$P_t^0(\lambda^t) = \sum_{t=0}^{\infty} \sum_{\lambda^t} q_t^0(\lambda^t)\, d_t(\lambda^t)$$

The price of such asset will crucially depend on the initial state $\lambda_0 \in \{\bar{\lambda}_1, \bar{\lambda}_2\}$. Let us start at $\bar{\lambda}_1$. We know that $d_0 = 1$, $q_0^0(\lambda^0) = 1$ and (1). Hence,

$$P_t^0(\bar{\lambda}_1) = 1 + \sum_{t=1}^{\infty} \sum_{\lambda^t} \beta^t \pi_t(\lambda^t)\, d_t(\lambda^t)^{1-\gamma} =$$

$$= 1 + \sum_{t=1}^{\infty} \sum_{\lambda^t} \beta^t \left[\pi(\lambda_t|\lambda_{t-1}) \cdots \pi(\lambda_1|\lambda_0)\right]\left[\lambda_t \cdots \lambda_0\right]^{1-\gamma} =$$

$$= 1 + \sum_{t=1}^{\infty} \sum_{\lambda^t} \beta^t \prod_{s=1}^{t} \left[\pi(\lambda_s|\lambda_{s-1})\, \lambda_s^{1-\gamma}\right] =$$

$$= 1 + \beta\left[P_{11}(\bar{\lambda}_1)^{1-\gamma} + P_{12}(\bar{\lambda}_2)^{1-\gamma}\right] + \beta^2\left[P_{11}^2(\bar{\lambda}_1^2)^{1-\gamma} + P_{11}P_{12}(\bar{\lambda}_1\bar{\lambda}_2)^{1-\gamma} + P_{12}P_{22}(\bar{\lambda}_2^2)^{1-\gamma} + P_{12}P_{21}(\bar{\lambda}_2\bar{\lambda}_1)^{1-\gamma}\right] + \cdots =$$

$$= 1 + \beta P_{11}(\bar{\lambda}_1)^{1-\gamma}\left\{1 + \beta\left[P_{11}(\bar{\lambda}_1)^{1-\gamma} + P_{12}(\bar{\lambda}_2)^{1-\gamma}\right] + \cdots\right\} + \beta P_{12}(\bar{\lambda}_2)^{1-\gamma}\left\{1 + \beta\left[P_{21}(\bar{\lambda}_1)^{1-\gamma} + P_{22}(\bar{\lambda}_2)^{1-\gamma}\right] + \cdots\right\} =$$

$$= 1 + \beta P_{11}(\bar{\lambda}_1)^{1-\gamma}\underbrace{\left\{1 + \sum_{t=2}^{\infty}\sum_{\lambda^t}\beta^t \prod_{s=2}^{t}\left[\pi(\lambda_s|\lambda_{s-1})\,\lambda_s^{1-\gamma}\right]\right\}}_{P_t^0(\bar{\lambda}_1)} + \beta P_{12}(\bar{\lambda}_2)^{1-\gamma}\underbrace{\left\{1 + \sum_{t=2}^{\infty}\sum_{\lambda^t}\beta^t \prod_{s=2}^{t}\left[\pi(\lambda_s|\lambda_{s-1})\,\lambda_s^{1-\gamma}\right]\right\}}_{P_t^0(\bar{\lambda}_2)} =$$

$$= 1 + \beta\left\{P_{11}(\bar{\lambda}_1)^{1-\gamma} P_t^0(\bar{\lambda}_1) + P_{12}(\bar{\lambda}_2)^{1-\gamma} P_t^0(\bar{\lambda}_2)\right\} \tag{2}$$

If we had started from $\bar{\lambda}_2$,

$$P_t^0(\bar{\lambda}_2) = 1 + \beta\left\{P_{21}(\bar{\lambda}_1)^{1-\gamma} P_t^0(\bar{\lambda}_1) + P_{22}(\bar{\lambda}_2)^{1-\gamma} P_t^0(\bar{\lambda}_2)\right\} \tag{3}$$

We can write (2)-(3) in matrix form

$$P_t^0(\lambda^t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\, \Pi\, P_t^0(\lambda^t) \tag{4}$$

where $P_t^0(\lambda^t) = \begin{bmatrix} P_t^0(\bar{\lambda}_1) \\ P_t^0(\bar{\lambda}_2) \end{bmatrix}$, $\Pi = \begin{bmatrix} P_{11}(\bar{\lambda}_1)^{1-\gamma} & P_{12}(\bar{\lambda}_2)^{1-\gamma} \\ P_{21}(\bar{\lambda}_1)^{1-\gamma} & P_{22}(\bar{\lambda}_2)^{1-\gamma} \end{bmatrix} = P\begin{bmatrix} (\bar{\lambda}_1)^{1-\gamma} & 0 \\ 0 & (\bar{\lambda}_2)^{1-\gamma} \end{bmatrix}$. Solving for $P_t^0(\lambda^t)$ in (4),

$$P^0 = \mathbb{1} + \beta P \lambda P^0 \implies I = \mathbb{1}[P^0]^{-1} + \beta P\lambda \implies P^0 = [I - \beta P\lambda]^{-1} \cdot \mathbb{1}$$

in particular,

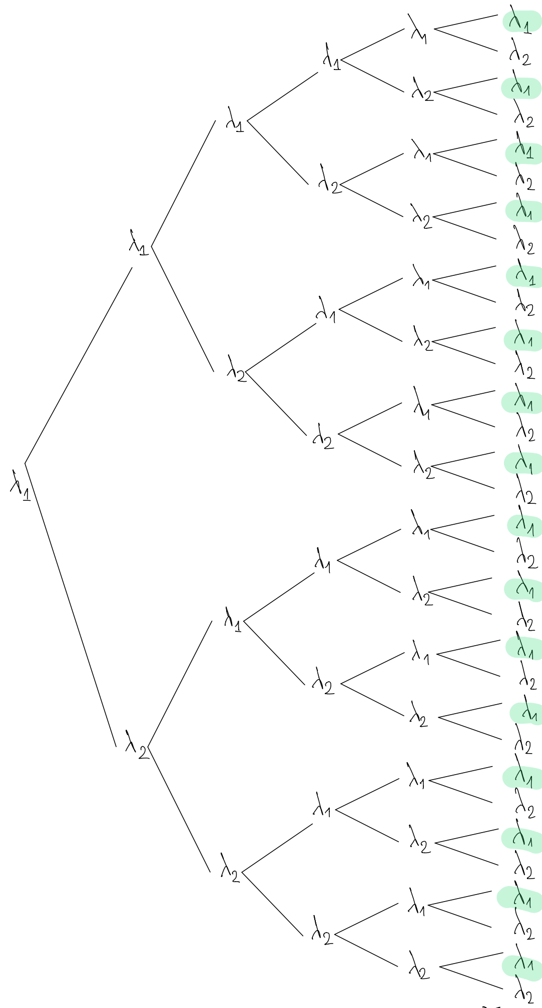
$$P^0 = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \beta\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} (\bar{\lambda}_1)^{1-\gamma} & 0 \\ 0 & (\bar{\lambda}_2)^{1-\gamma} \end{pmatrix}\right]^{-1} \cdot \mathbb{1} =$$

$$= \begin{bmatrix} 1 - \beta P_{11}(\bar{\lambda}_1)^{1-\gamma} & -\beta P_{12}(\bar{\lambda}_2)^{1-\gamma} \\ -\beta P_{21}(\bar{\lambda}_1)^{1-\gamma} & 1 - \beta P_{22}(\bar{\lambda}_2)^{1-\gamma} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{\left[1-\beta P_{11}(\bar{\lambda}_1)^{1-\gamma}\right]\left[1-\beta P_{22}(\bar{\lambda}_2)^{1-\gamma}\right] - \beta^2 P_{12} P_{21} (\bar{\lambda}_1 \bar{\lambda}_2)^{1-\gamma}} \begin{bmatrix} 1+\beta(\bar{\lambda}_2)^{1-\gamma}(P_{12}-P_{22}) \\ 1+\beta(\bar{\lambda}_1)^{1-\gamma}(P_{21}-P_{11}) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{7081}{374} \\ \frac{6283}{374} \end{bmatrix}$$

Therefore, $P_t^0(\bar{\lambda}_1) = 18.93$ and $P_t^0(\bar{\lambda}_2) = 16.80$



e) It would be dramatically tedious to calculate the price by doing these $\longrightarrow$ iterations. Hence, lets make use of some of the Markov processes' properties.

Define ${}_5Q_5^0(\lambda^5)$ the price of an asset that pays $d_5(\lambda^5) = 1$ unit of consumption at time $t=5$, contingent on $\lambda_5 = \bar{\lambda}_1$. We can write it as

$${}_5Q_5^0(\{\lambda^t\} = \{\lambda^{t-1}, \bar{\lambda}_1\}) = \sum_{\lambda^t \in \bar{\Lambda}_1} q_5^0(\{\lambda^t\} = \{\lambda^{t-1}, \bar{\lambda}_1\})$$

where $\bar{\Lambda}_1$ is the set of λ^t histories up to $t=5$ such that $\lambda_5 = \bar{\lambda}_1$. Introducing (1),

$${}_5Q_5^0(\{\lambda^t\} = \{\lambda^{t-1}, \bar{\lambda}_1\}) = \sum_{\lambda^t \in \bar{\Lambda}_1} \beta^5 \pi_5(\lambda^5)\left[d_5(\lambda^5)\right]^{-\gamma} =$$

$$= \sum_{\lambda^t \in \bar{\Lambda}_1} \beta^5 \pi(\lambda_1 | \lambda_0)\, \pi(\lambda_2|\lambda_1)\, \pi(\lambda_3|\lambda_2)\, \pi(\lambda_4|\lambda_3)\, \pi(\bar{\lambda}_1|\lambda_4)\left[\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5\right]^{-\gamma}$$

Notice that λ^t follows a Markov process. From its properties we know that

$$\pi_t(\lambda^t) = \pi_{t-1}(\lambda^{t-1}) P = \pi_{t-2}(\lambda^{t-2}) P^2 = \cdots = \pi_0(\lambda^0) P^t$$

which in our case of interest collapses to $\pi_5(\{\lambda^4, \bar{\lambda}_1\}) = \pi_0(\bar{\lambda}_1) P^5$.
Similarly for the endowment process

$$d_t = \lambda_t d_{t-1} = \lambda_t \lambda_{t-1} d_{t-2} = \cdots = \lambda_t \cdots \lambda_1 d_0$$

where $\lambda_t = \begin{bmatrix} \bar{\lambda}_1 & 0 \\ 0 & \bar{\lambda}_2 \end{bmatrix}$, which in our case of interest collapses to

$$d_5 = \lambda_5 \lambda_4 \lambda_3 \lambda_2 \lambda_1 = \begin{bmatrix} \bar{\lambda}_1 & 0 \\ 0 & \bar{\lambda}_2 \end{bmatrix}^5$$

Therefore,

$${}_5Q_5^0(\lambda^5) = \left[{}_5Q_5^0(\{\lambda^4, \bar{\lambda}_1\}) \quad {}_5Q_5^0(\{\lambda^4, \bar{\lambda}_2\})\right] = \beta^5 \underbrace{\pi_0(\lambda^0)}_{[1 \quad 0]} \left[P \begin{pmatrix} \bar{\lambda}_1 & 0 \\ 0 & \bar{\lambda}_2 \end{pmatrix}^{-\gamma}\right]^5 = \left[0.44 \quad 0\right]$$

f) Def: A (recursive) competitive equilibrium is an initial distribution of wealth, a pricing function $Q(\lambda'|\lambda)$, a set of value functions $\{v^i(a,\lambda)\}_{\forall i}$ and policy functions $\{h^i(a,\lambda), g^i(a,\lambda;\lambda')\}_{\forall i}$ such that

- For all i, given a_0^i and the pricing function, the policy functions solve the agent's problem

$$v_t^i(a,\lambda^t) = \max_{\{c, \{\hat{a}(\lambda_{t+1})\}_{\lambda_{t+1}}} \left\{ u_i(c) + \beta E_t\, v_{t+1}^i(\hat{a}(\lambda_{t+1}), \lambda^{t+1}) \right\}$$

$$\text{s.t. } d_t^i(\lambda^t) + a \geq c + \sum_{\lambda_{t+1}} \hat{a}(\lambda_{t+1})\, Q_{t+1}(\lambda_{t+1}|\lambda^t)$$

$$-\hat{a}(\lambda_{t+1}) \leq A_{t+1}^i(\lambda^{t+1}) \quad \forall \lambda_{t+1}$$

$$c \geq 0$$

- For all realization of $\{\lambda^t\}$, the consumption and asset portfolio implied by the policy functions satisfy

$$\sum_i c_t^i = \sum_i d_t^i$$

$$\sum_i \hat{a}_{t+1}^i(\lambda') = 0$$

Taking the Markov specification, the consumer problem becomes

$$v^i(a,\lambda) = \max_{\{c, \{\hat{a}(\lambda')\}_{\lambda'}\}} \left\{ u_i(c) + \beta \sum_{\lambda'} \pi(\lambda'|\lambda)\, v^i(\hat{a}(\lambda'), \lambda') \right\}$$

$$\text{s.t. } d^i(\lambda) + a \geq c + \sum_{\lambda'} \hat{a}(\lambda')\, Q(\lambda'|\lambda)$$

$$-\hat{a}(\lambda') \leq \bar{A}^i(\lambda') \quad \forall \lambda'$$

$$c \geq 0$$

g) The value of agent i's original endowment process is

$$\bar{A}_t^i(\lambda^t) = \sum_{\tau=t}^{\infty} \sum_{\lambda^\tau|\lambda^t} q_\tau^t(\lambda^\tau)\, d_t^i(\lambda^\tau)$$

We define the natural debt limit at time t as the maximum value that the agent could repay (in the limit case in which $c_\tau^i = 0 \;\; \forall \tau \geq t$). Such natural limit is assumed to be constant across time and individuals,

$$\bar{A}(\lambda^t) = q_t^t(\lambda_t)\, d_t(\lambda_t) + \underbrace{\sum_{\tau=t+1}^{\infty} \sum_{\lambda^\tau|\lambda^{t+1}} q_\tau^t(\lambda^\tau)\, d_\tau(\lambda^\tau)}_{\bar{A}(\lambda^{t+1})}$$

$$\Longrightarrow \quad \bar{A}(\lambda) = q(\lambda)\, d(\lambda) + \bar{A}(\lambda')$$

h) Forming the lagrangian

$$\mathcal{L} = u_i(c) + \beta \sum_{\lambda'} \pi(\lambda'|\lambda) V^i(\hat{a}(\lambda'), \lambda') + \mu^i \left[d^i(\lambda) + a - \sum_{\lambda'} \hat{a}(\lambda') Q(\lambda'|\lambda) - c \right]$$

Taking the FOC's

$$c: \; u_i'(c) - \mu^i = 0$$

$$\hat{a}(\lambda'): \; \beta \pi(\lambda'|\lambda) V_1^i(\hat{a}(\lambda'), \lambda') - \mu^i Q(\lambda'|\lambda) = 0$$

$$c_t: \; u_i'(c_t) - \mu_t^i = 0$$

$$\hat{a}_{t+1}(\lambda_{t+1}): \; \beta \pi(\lambda_{t+1}|\lambda_t) \underbrace{V_1^i(\hat{a}_{t+1}(\lambda_{t+1}), \lambda_{t+1})}_{u_i'[c_{t+1}^i(\lambda_{t+1})]} - \mu^i Q(\lambda_{t+1}|\lambda_t) = 0$$

(Benveniste – Scheinkman condition)

Solving for Q,

$$Q(\lambda'|\lambda) = \beta \pi(\lambda'|\lambda) \left(\frac{c'}{c} \right)^{-\gamma}$$

$$Q(\lambda_{t+1}|\lambda_t) = \beta \pi(\lambda_{t+1}|\lambda_t) \left[\frac{c_{t+1}(\lambda_{t+1})}{c_t(\lambda_t)} \right]^{-\gamma}$$

where $c = h^i(a, \lambda)$ and $c' = h^i(g^i(a, \lambda; \lambda'), \lambda')$. Knowing that $c = d$ (market clearing) and $d' = \lambda' d$,

$$Q(\lambda'|\lambda) = \beta \pi(\lambda'|\lambda) (\lambda')^{-\gamma}$$

Using the just-derived expression,

$$Q(\lambda'|\lambda) = \begin{cases} \beta \left[P_{11} \bar{\lambda}_1^{-\gamma} \right] = 0.81 & \text{for } \lambda = \bar{\lambda}_1 = \lambda' \\ \beta \left[P_{12} \bar{\lambda}_2^{-\gamma} \right] = 0.18 & \text{for } \lambda = \bar{\lambda}_1, \; \lambda' = \bar{\lambda}_2 \\ \beta \left[P_{21} \bar{\lambda}_1^{-\gamma} \right] = 0.10 & \text{for } \lambda = \bar{\lambda}_2, \; \lambda' = \bar{\lambda}_1 \\ \beta \left[P_{22} \bar{\lambda}_2^{-\gamma} \right] = 0.81 & \text{for } \lambda = \bar{\lambda}_2 = \lambda' \end{cases}$$

i) We denote the two-period asset price as

$$Q(\lambda''|\lambda) = \sum_{\lambda'} Q(\lambda'|\lambda) Q(\lambda''|\lambda') = Q(\bar{\lambda}_1|\lambda) Q(\lambda''|\bar{\lambda}_1) + Q(\bar{\lambda}_2|\lambda) Q(\lambda''|\bar{\lambda}_2)$$

we have to consider all possible intermediate λ' cases $\rightarrow \lambda'$

Hence, if the asset pays in both λ'' cases (assuming we started at $\lambda = \bar{\lambda}_1$),

$$p(\lambda''|\bar{\lambda}_1) = \sum_{\lambda''} Q(\lambda''|\bar{\lambda}_1) =$$

$$= \overbrace{\underbrace{Q(\bar{\lambda}_1|\bar{\lambda}_1) Q(\bar{\lambda}_1|\bar{\lambda}_1)}_{0.65} + \underbrace{Q(\bar{\lambda}_2|\bar{\lambda}_1) Q(\bar{\lambda}_1|\bar{\lambda}_2)}_{0.018}}^{\lambda'' = \bar{\lambda}_1} + \overbrace{\underbrace{Q(\bar{\lambda}_1|\bar{\lambda}_1) Q(\bar{\lambda}_2|\bar{\lambda}_1)}_{0.14} + \underbrace{Q(\bar{\lambda}_2|\bar{\lambda}_1) Q(\bar{\lambda}_2|\bar{\lambda}_2)}_{0.14}}^{\lambda'' = \bar{\lambda}_2} = 0.96$$

② a) $\pi_t(s^t) = \pi(s_t | s_{t-1})\, \pi(s_{t-1} | s_{t-2}) \cdots \pi(s_1 | s_0)\, \pi_0(s_0)$

b) Forming the lagrangian,

$$\mathcal{L} = \theta \ln\left[c_t^1(s^t)\right] + (1-\theta)\ln\left[c_t^2(s^t)\right] + \mu\left[Y(s_t) - c_t^1(s^t) - c_t^2(s^t)\right]$$

Taking the FOC's,

$$c_t^1(s^t): \quad \frac{\theta}{c_t^1(s^t)} - \mu = 0$$

$$c_t^2(s^t): \quad \frac{1-\theta}{c_t^2(s^t)} - \mu = 0$$

Combining the FOC's,

$$\frac{c_t^1(s^t)}{c_t^2(s^t)} = \frac{\theta}{1-\theta}$$

(5)

Using feasibility

$$c_t^1(s^t) + \frac{1-\theta}{\theta} c_t^1(s^t) = Y_t(s_t) \implies \begin{array}{l} c_t^1(s^t) = \theta\, Y(s_t) \\ c_t^2(s^t) = (1-\theta)\, Y(s_t) \end{array}$$

no initial $Y(s_0)$ isn't required

c) Def: A competitive equilibrium is a initial distribution s_0, a consumption allocation $\{c_t^i(s^t)\}_{\forall t, i, s^t}$ where $s^t = \{s_t\}_{\forall t}$ is the entire history until time t, and a price system $\{q_t^0(s^t)\}_{\forall t, s^t}$ such that

- Given s_0, each consumer i solves

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)\, u\left[c_t^i(s^t)\right]$$
$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, y_t^i(s^t)$$

- Markets clear (feasibility): $\displaystyle\sum_{i=1}^{I=2} c_t^i(s^t) = \sum_{i=1}^{I=2} y_t^i(s^t) = Y(s_t)$

d) Forming the lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \ln\left[c_t^i(s^t)\right] + \mu^i \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\left[y_t^i(s^t) - c_t^i(s^t)\right]$$

Taking the FOC

$$c_t^i(s^t): \quad \beta^t \pi_t(s^t) \frac{1}{c_t^i(s^t)} - \mu^i q_t^0(s^t) = 0$$

Rewritting it for $t=0$ and normalizing $q_0^0(s_0) = 1$, $\mu^i = \left[c_0^i(s_0)\right]^{-1}$. Hence, we can rewrite the FOC as

$$q_t^0(s^t) = \beta^t \pi_t(s^t) \frac{c_0^i(s_0)}{c_t^i(s^t)} \qquad (\star)$$

$\star$ another (better) way by the end

Since individuals face common price, beliefs and discount factor,

$$\frac{c_0^1(s_0)}{c_t^1(s^t)} = \frac{c_0^2(s_0)}{c_t^2(s^t)} \implies \frac{c_t^1(s^t)}{c_t^2(s^t)} = \frac{c_0^1(s_0)}{c_0^2(s_0)} = \frac{c_0^1(s_0)}{Y(s_0) - c_0^1(s_0)}$$

Since $Y(s_0) = c_0^1(s_0) + c_0^2(s_0)$, we can rewrite $c_0^1(s_0)$ as a fraction $\xi \in [0,1]$ of $Y(s_0)$,

$$\frac{c_t^1(s^t)}{c_t^2(s^t)} = \frac{\xi Y(s_0)}{(1-\xi)Y(s_0)} = \frac{\xi}{1-\xi} \qquad (6)$$

Let me now show that optimal consumption will not be constant across time. From the FOC

$$\frac{1}{c_t^1(s^t)}\frac{1}{\mu^1} = \frac{1}{c_t^2(s^t)}\frac{1}{\mu^2} \implies c_t^1(s^t) = \frac{\mu^2}{\mu^1} c_t^2(s^t)$$

from feasibility,

$$c_t^1(s^t) = \frac{\mu^2}{\mu^1}\left[Y(s_t) - c_t^1(s^t)\right] \implies c_t^1(s^t) = \frac{\mu^2}{1+\mu^1} Y(s_t)$$

Since $Y(s_t)$ is not constant, we cannot prove that optimal consumption will be constant! Hence, we cannot write $c_t^1(s^t) = \xi Y(s_t)\ \forall t$. However, we can write price as

$$q_t^0(s^t) = \beta^t \pi_t(s^t) \frac{Y(s_0)}{Y(s_t)}$$

That is, the ratio of consumption (what % will individual 2 consume more than individual 1) will be constant! But not individual consumption across time.

$\star$ We can rewrite $(\star)$ as

$$c_t^i(s^t) = \beta^t \pi_t(s^t) c_0^i(s_0) \frac{1}{q_t^0(s^t)}$$

since feasibility is binding (Inada),

$$c_t^1(s^t) + c_t^2(s^t) = Y_t(s^t) \implies \frac{\beta^t \pi_t(s^t)}{q_t^0(s^t)} \Big[\underbrace{c_0^1(s_0) + c_0^2(s_0)}_{Y_0(s_0)} \Big] = Y_t(s^t) \implies$$

$$\implies q_t^0(s^t) = \beta^t \pi_t(s^t) \frac{Y_0(s_0)}{Y_t(s^t)} \qquad (7)$$

e) Notice that the optimal allocation of the social planner problem (5) equalizes competitive allocation (6) if $\xi = \theta$ (satisfying welfare theorems)
By IWTh, any competitive allocation is Pareto optimal.

f) Recall the pricing formula ($\star$). We can plug-in the Pareto optimal allocation that we found in (b),

$$q_t^0(s^t) = \beta^t \pi_t(s^t) \frac{\theta Y_0(s_0)}{\theta Y_t(s^t)} = \beta^t \pi_t(s^t) \frac{Y_0(s_0)}{Y_t(s^t)}$$

which coincides with (7)!

g) Def: A (recursive) competitive equilibrium is an initial distribution of wealth, a pricing function $Q(s'|s)$, a set of value functions $\{v^i(a,s)\}_{\forall i}$ and policy functions $\{h^i(a,s), g^i(a,s;s')\}_{\forall i}$ such that

- For all i, given a_0^i and the pricing function, the policy functions solve the agent's problem

$$v_t^i(a, s^t) = \max_{\{c, \{\hat{a}(s_{t+1})\}_{s_{t+1}}} \left\{ u_i(c) + \beta E_t\, v_{t+1}^i\left(\hat{a}(s_{t+1}), s^{t+1}\right)\right\}$$

$$\text{s.t. } y_t^i(s^t) + a \geq c + \sum_{t+1} \hat{a}(s_{t+1})\, Q_{t+1}(s_{t+1}|s^t)$$

$$-\hat{a}(s_{t+1}) \leq A_{t+1}^i(s^{t+1}) \quad \forall s_{t+1}$$

$$c \geq 0$$

- For all realizations of $\{s^t\}$, the consumption and asset portfolio implied by the policy functions satisfy

$$\sum_i c_t^i = \sum_i y_t^i$$

$$\sum_i \hat{a}_{t+1}^i(s') = 0$$

Taking the Markov specification, the consumer problem becomes

$$v^i(a, s) = \max_{\{c, \{\hat{a}(s')\}_{s'}\}} \left\{ u_i(c) + \beta \sum_{s'} \pi(s'|s)\, v^i(\hat{a}(s'), s')\right\}$$

$$\text{s.t. } y^i(s) + a \geq c + \sum_{s'} \hat{a}(s')\, Q(s'|s)$$

$$-\hat{a}(s') \leq \bar{A}^i(s') \quad \forall s'$$

$$c \geq 0$$

We define the "natural borrowing limits" (one per state) as

$$\bar{A}^i(s) = y^i(s) + \sum_{s'} Q(s'|s) \underbrace{\bar{A}^i(s'|s)}_{\bar{A}^i(s')}$$

the natural debt limit at t+1 does not depend on how we got there! It depends on the realization at t+1 (s_{t+1}) and all future realizations

Hence,

$$\bar{A}^1(0) = \overbrace{y^1(0)}^{0} + Q(0|0)A^1(0) + Q(1|0)A^1(1)$$

$$\bar{A}^1(1) = \overbrace{y^1(1)}^{1} + Q(0|1)A^1(0) + Q(1|1)A^1(1)$$

$$\bar{A}^2(0) = \overbrace{y^2(0)}^{1} + Q(0|0)A^2(0) + Q(1|0)A^2(1)$$

$$\bar{A}^2(0) = \overbrace{y^2(1)}^{1} + Q(0|1)A^2(0) + Q(1|1)A^2(1)$$

Hence, to find out the limits $\bar{A}^i(s)$ $\forall i,s$ we would need to calculate the pricing functions / kernels $Q(s'|s)$ $\forall s, s'$

h) Forming the lagrangian

$$\mathcal{L} = u_i(c) + \beta \sum_{s'} \pi(s'|s) V^i(\hat{a}(s'), s') + \mu^i \left[y^i(s) + a - \sum_{s'} \hat{a}(s')Q(s'|s) - c \right]$$

Taking the FOC's

$$c: \; u_i'(c) - \mu^i = 0$$

$$\hat{a}(s'): \; \beta\pi(s'|s) V_1^i(\hat{a}(s'), s') - \mu^i Q(s'|s) = 0$$

Solving for Q and using $u_i(c) = \ln(c)$,

$$Q(s'|s) = \beta\pi(s'|s)\frac{c}{c'}$$

where $c = h^i(a,s)$ and $c' = h^i(g^i(a,s;s'), s')$. We proved before that $c^1 = \xi Y(s)$ $\forall t$. Hence,

$$Q(s'|s) = \beta\pi(s'|s)\frac{Y(s)}{Y(s')} \qquad (8)$$

☆ Using (again) feasibility

$$c_1' + c_2' = Y(s') \implies \beta\frac{\pi(s'|s)}{Q(s'|s)}\underbrace{[c_1 + c_2]}_{Y(s)} = Y(s') \implies Q(s'|s) = \beta\pi(s'|s)\frac{Y(s)}{Y(s')}$$

Notice there will be 4 prices: two per state

$$Q(0|0) = \beta P_{00}\frac{y^1(\bar{s}_1) + y^2}{y^1(\bar{s}_1) + y^2} = 0.76 \qquad Q(1|0) = \beta P_{01}\frac{y^1(\bar{s}_1) + y^2}{y^1(\bar{s}_2) + y^2} = 0.095$$

$$Q(0|1) = \beta P_{10}\frac{y^1(\bar{s}_2) + y^2}{y^1(\bar{s}_1) + y^2} = 0.57 \qquad Q(1|1) = \beta P_{11}\frac{y^1(\bar{s}_2) + y^2}{y^1(\bar{s}_2) + y^2} = 0.67$$

i) Define ${}_1P(s'|s) = \sum_{s'} Q(s'|s)$ as the price of an asset that pays 1 unit of consumption the next period,

$$ {}_1P(s'|s) = \begin{cases} \beta\left[P_{11}\dfrac{y^1(\bar{s}_1)+y^2}{y^1(\bar{s}_1)+y^2} + P_{12}\dfrac{y^1(\bar{s}_1)+y^2}{y^1(\bar{s}_2)+y^2}\right] = 0.86 & \text{if } s=\bar{s}_1 \\ \beta\left[P_{21}\dfrac{y^1(\bar{s}_2)+y^2}{y^1(\bar{s}_1)+y^2} + P_{22}\dfrac{y^1(\bar{s}_2)+y^2}{y^1(\bar{s}_2)+y^2}\right] = 1.24 & \text{if } s=\bar{s}_2 \end{cases} $$

j) Define ${}_2P(s''|s)$ as the price of an asset that pays 1 unit of consumption in 2 periods. We can write it as

$$ {}_2P(s''|s) = \sum_{s''} Q(s''|s) = \sum_{s''} \beta^2 \pi(s''|s)\frac{Y(s)}{Y(s'')} $$

where $\pi(s''|s) = \pi(s''|s')\pi(s'|s)$

$$ {}_2P(s''|s) = \begin{cases} \beta^2\left[P_{11}^2\dfrac{y^1(\bar{s}_1)+y^2}{y^1(\bar{s}_1)+y^2} + P_{11}P_{12}\dfrac{y^1(\bar{s}_1)+y^2}{y^1(\bar{s}_2)+y^2} + P_{12}P_{21}\dfrac{y^1(\bar{s}_1)+y^2}{y^1(\bar{s}_1)+y^2} + P_{12}P_{22}\dfrac{y^1(\bar{s}_1)+y^2}{y^1(\bar{s}_2)+y^2}\right] = 0.77 & \text{if } s=\bar{s}_1 \\ \beta^2\left[P_{21}P_{11}\dfrac{y^1(\bar{s}_2)+y^2}{y^1(\bar{s}_1)+y^2} + P_{21}P_{12}\dfrac{y^1(\bar{s}_2)+y^2}{y^1(\bar{s}_2)+y^2} + P_{22}P_{21}\dfrac{y^1(\bar{s}_2)+y^2}{y^1(\bar{s}_1)+y^2} + P_{22}^2\dfrac{y^1(\bar{s}_2)+y^2}{y^1(\bar{s}_2)+y^2}\right] = 1.31 & \text{if } s=\bar{s}_2 \end{cases} $$

k) The j-step pricing is given by

$$ Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} Q(s_{t+1}|s_t)\, Q_{j-1}(s_{t+j}|s_{t+1}) $$

In this case $j=5$,

$$ Q_5(s_{t+5}|s_t) = \sum_{s_{t+1}} Q(s_{t+1}|s_t)\, Q_4(s_{t+5}|s_{t+1}) \tag{9} $$

Following a recursive argument

not $Q_4(s_{t+4}|s_t)$ →

$$ Q_4(s_{t+5}|s_{t+1}) = \sum_{s_{t+2}} Q(s_{t+2}|s_{t+1})\, Q_3(s_{t+5}|s_{t+2}) \tag{10} $$

$$ Q_3(s_{t+5}|s_{t+2}) = \sum_{s_{t+3}} Q(s_{t+3}|s_{t+2})\, Q_2(s_{t+5}|s_{t+3}) \tag{11} $$

$$ Q_2(s_{t+5}|s_{t+3}) = \sum_{s_{t+4}} Q(s_{t+4}|s_{t+3})\, Q(s_{t+5}|s_{t+4}) \tag{12} $$

Plugging (7) into (12),

$$ Q_2(s_{t+5}|s_{t+3}) = \sum_{s_{t+4}} \beta\,\pi(s_{t+4}|s_{t+3})\frac{Y(s_{t+3})}{Y(s_{t+4})}\,\beta\,\pi(s_{t+5}|s_{t+4})\frac{Y(s_{t+4})}{Y(s_{t+5})} = $$

$$ = \sum_{s_{t+4}} \beta^2\,\pi(s_{t+4}|s_{t+3})\,\pi(s_{t+5}|s_{t+4})\frac{Y(s_{t+3})}{Y(s_{t+5})} \tag{13} $$

Plugging (13) into (11)

$$Q_3(s_{t+5}|s_{t+2}) = \sum_{s_{t+3}} \beta\, \pi(s_{t+3}|s_{t+2}) \frac{\gamma(s_{t+2})}{\gamma(s_{t+3})} \sum_{s_{t+4}} \beta^2\, \pi(s_{t+4}|s_{t+3})\, \pi(s_{t+5}|s_{t+4}) \frac{\gamma(s_{t+3})}{\gamma(s_{t+5})} =$$

$$= \beta^3 \sum_{s_{t+3}} \pi(s_{t+3}|s_{t+2}) \frac{\gamma(s_{t+2})}{\gamma(s_{t+3})} \frac{\gamma(s_{t+3})}{\gamma(s_{t+5})} \sum_{s_{t+4}} \pi(s_{t+4}|s_{t+3})\, \pi(s_{t+5}|s_{t+4})$$

$$= \beta^3 \sum_{s_{t+3}} \pi(s_{t+3}|s_{t+2}) \frac{\gamma(s_{t+2})}{\gamma(s_{t+5})} \sum_{s_{t+4}} \pi(s_{t+4}|s_{t+3})\, \pi(s_{t+5}|s_{t+4}) \qquad (14)$$

Plugging (14) into (10)

$$Q_4(s_{t+5}|s_{t+1}) = \sum_{s_{t+2}} \beta\, \pi(s_{t+2}|s_{t+1}) \frac{\gamma(s_{t+1})}{\gamma(s_{t+2})}\, \beta^3 \sum_{s_{t+3}} \pi(s_{t+3}|s_{t+2}) \frac{\gamma(s_{t+2})}{\gamma(s_{t+5})} \sum_{s_{t+4}} \pi(s_{t+4}|s_{t+3})\, \pi(s_{t+5}|s_{t+4})$$

$$= \beta^4 \sum_{s_{t+2}} \pi(s_{t+2}|s_{t+1}) \frac{\gamma(s_{t+1})}{\gamma(s_{t+5})} \sum_{s_{t+3}} \pi(s_{t+3}|s_{t+2}) \sum_{s_{t+4}} \pi(s_{t+4}|s_{t+3})\, \pi(s_{t+5}|s_{t+4}) \qquad (15)$$

Finally (thank God) plugging (15) into (9)

$$Q_5(s_{t+5}|s_t) = \sum_{s_{t+1}} \beta\, \pi(s_{t+1}|s_t) \frac{\gamma(s_t)}{\gamma(s_{t+1})}\, \beta^4 \sum_{s_{t+2}} \pi(s_{t+2}|s_{t+1}) \frac{\gamma(s_{t+1})}{\gamma(s_{t+5})} \sum_{s_{t+3}} \pi(s_{t+3}|s_{t+2}) \sum_{s_{t+4}} \pi(s_{t+4}|s_{t+3})\, \pi(s_{t+5}|s_{t+4}) =$$

$$= \beta^5 \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \frac{\gamma(s_t)}{\gamma(s_{t+5})} \sum_{s_{t+2}} \pi(s_{t+2}|s_{t+1}) \sum_{s_{t+3}} \pi(s_{t+3}|s_{t+2}) \sum_{s_{t+4}} \pi(s_{t+4}|s_{t+3})\, \pi(s_{t+5}|s_{t+4})$$

$$= \beta^5 \underbrace{\mathrm{Prob}(s_{t+5}|s_t)}_{\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \sum_{s_{t+2}} \pi(s_{t+2}|s_{t+1}) \sum_{s_{t+3}} \pi(s_{t+3}|s_{t+2}) \sum_{s_{t+4}} \pi(s_{t+4}|s_{t+3})\, \pi(s_{t+5}|s_{t+4})} \frac{\gamma(s_t)}{\gamma(s_{t+5})}$$

where $\mathrm{Prob}(s_{t+5}|s_t)$ is the transition probability from $s_t = j$ at t to $s_{t+5} = k$ at $t+5$. Such transition is given by P^t, which we can write as

$$P^t = D \lambda^t D^{-1}$$

where λ is a diagonal matrix containing eigenvalues and D contains eigenvectors. To obtain eigenvalues,

$$|P - \lambda I| = 0 \implies \begin{vmatrix} 0.8-\lambda & 0.2 \\ 0.3 & 0.7-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 1.5\lambda + 0.5 = 0 \implies \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 0.5 \end{matrix}$$

To obtain eigenvectors,

$$Pv = \lambda_1 v \implies \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \implies v_1 = v_2 \implies v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Pw = \lambda_2 w \implies \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \implies w_1 = -\frac{2w_2}{3} \implies w = \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$$

Therefore, $D = \begin{pmatrix} 1 & 1 \\ 1 & -\frac{3}{2} \end{pmatrix}$ and

$$D^{-1} = \frac{1}{-5/2} \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{pmatrix}$$

And hence

$$P^5 = D \lambda^5 D^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.5^5 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{49}{80} & \frac{31}{80} \\ \frac{93}{160} & \frac{67}{160} \end{pmatrix}$$

Therefore

$$Q_5(0|0) = \beta^5 P^5_{11} \frac{Y(0)}{Y(0)} = 0.47$$

$$Q_5(1|0) = \beta^5 P^5_{12} \frac{Y(0)}{Y(1)} = 0.15$$

$$Q_5(0|1) = \beta^5 P^5_{21} \frac{Y(1)}{Y(0)} = 0.90$$

$$Q_5(1|1) = \beta^5 P^5_{22} \frac{Y(1)}{Y(1)} = 0.32$$

3 a) $\pi_t(s^t) = \pi(s_t|s_{t-1}) \, \pi(s_{t-1}|s_{t-2}) \cdots \pi(s_1|s_0) \pi_0(s_0)$

where $\pi(s_j|s_{j-1})$ are derived from P. We consider $\pi_0(s_0)$ certain (it was already observed). The distribution at $t=1$ is

$$\pi_1' = \pi_0' P$$

at $t=2$,

$$\pi_2' = \pi_1' P = \pi_0' P^2$$

hence, at t

$$\pi_t' = \pi_0' P^t$$

Notice that in this particular case P is idempotent ($P^t = P$). Hence,

$$\pi_t' = \pi_0' P = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix}$$

b) Def: A sequential-trading competitive equilibrium is an initial distribution of wealth $\{a_0^i(s_0)\}_{\forall i}$, a consumption allocation $\{\tilde{c}_t^i(s^t)\}_{\forall i,t,s^t}$ and a price system $\{\tilde{Q}_t(s_{t+1}|s^t)\}_{\forall t, s^t, s_{t+1}}$ such that

- Given initial wealth distribution and price system, the allocation solves each agent's problem

$$\max_{\{\tilde{c}_t^i(s^t), \{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}\}_{\forall t, s^t}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \, u_i\left[\tilde{c}_t^i(s^t)\right]$$

$$\text{s.t.} \quad \tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s_t, s^{t-1})$$

given $\tilde{a}_0^i(s_0)$

$\forall t, s^t$

$\tilde{c}_t^i(s^t) \geq 0$

non-Ponzi constraint : $-\tilde{a}_{t+1}^i(s_{t+1}, s^t) \leq A_{t+1}^i(s^{t+1})$

- For all realization of $\{s^t\}$ the consumption allocation and implied asset portfolio satisfy

$$\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t) \qquad \forall t, s^t$$

$$\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$$

c) The most straight-forward algorithm is the following

1) Compute 0-trading

Forming the lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^i(s^t) \ln c_t^i(s^t) + \mu_i \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \left[y_t^i(s^t) - c_t^i(s^t) \right]$$

Taking the FOC

$$c_t^i(s^t) : \beta^t \pi_t^i(s^t) \frac{1}{c_t^i(s^t)} - \mu_i q_t^0(s^t) = 0$$

Setting the FOC at $t=0$ and normalizing $q_0^0(s_0) = 1$, we obtain that

$$\mu_i = \frac{1}{c_0^i(s_0)}$$

Hence, we can rewrite the FOC as

$$q_t^0(s^t) = \beta^t \pi_t^i(s^t) \frac{c_0^i(s_0)}{c_t^i(s^t)} \tag{16}$$

Using feasibility

$$c_t^1(s^t) + c_t^2(s^t) = y_t^1(s^t) + y_t^2(s^t) \equiv 1 \implies \frac{\beta^t \pi_t(s^t)}{q_t^0(s^t)} \underbrace{\left[c_0^1(s_0) + c_0^2(s_0) \right]}_{y_0^1(s_0) + y_0^2(s_0) = 1} = 1 \implies$$

$$\implies q_t^0(s^t) = \beta^t \pi_t(s^t) \tag{17}$$

Notice that (16)-(17) implies $c_t^i(s^t) = c_0^i(s_0) = \bar{c}^i \ \forall i, t, s^t$. Using the budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \bar{c}^i = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t) \implies \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \bar{c}^1 = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \frac{s_t}{2} \implies$$

$$\implies \bar{c}^1 \sum_{t=0}^{\infty} \beta^t \underbrace{\sum_{s^t} \pi_t(s^t)}_{1} = \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \underbrace{\sum_{s^t} \pi_t(s^t) s_t}_{\substack{\text{at } t=0,\ 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 2 = 1 \\ t \geq 1,\ 0.5 \cdot 0 + 0 \cdot 1 + 0.5 \cdot 2 = 1}} \implies$$

$$\Longrightarrow \frac{\bar{c}^1}{1-\beta} = \frac{1}{2(1-\beta)} \Longrightarrow \begin{array}{l} \bar{c}^1 = \frac{1}{2} \\ \bar{c}^2 = \frac{1}{2} \quad (\bar{c}^1 + \bar{c}^2 = 1) \end{array}$$

2) Set $\tilde{c}^i_t(s^t) = c^i_t(s^t)$ $\left[\text{Sequential trading} \equiv \text{Arrow-Debreu time-0}\right]$

3) Compute equilibrium price

$$\tilde{Q}_t(s_{t+1}|s_t) = \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = \frac{\beta^{t+1}\pi_{t+1}(s^{t+1})}{\beta^t\,\pi_t(s^t)} = \beta\,\frac{\pi(s_{t+1}|s_t)\,\pi(s_t|s_{t-1})\cdots\pi_0(s_0)}{\pi(s_t|s_{t-1})\cdots\pi_0(s_0)} = \beta\pi(s_{t+1}|s_t)$$

4) Compute "natural debt limits"

$$\bar{A}^i(s) = y^i(s) + \sum_{s'} Q(s'|s)\,\underbrace{\bar{A}^i(s'|s)}_{\bar{A}^i(s')}$$

Using the above equation,

$$\bar{A}^i(s) = y^i(s) + \sum_{s'} \beta\pi(s'|s)\,\bar{A}^i(s')$$

For $i=1$, $\forall s$,

$$\bar{A}^1(0) = 0 + \beta[1\cdot\bar{A}^1(0) + 0\cdot\bar{A}^1(1) + 0\cdot\bar{A}^1(2)] \Longrightarrow \bar{A}^1(0) = \beta\,\bar{A}^1(0) \Longrightarrow \bar{A}^1(0) = 0$$

$$\bar{A}^1(1) = \tfrac{1}{2} + \beta[\tfrac{1}{2}\cdot\bar{A}^1(0) + 0\cdot\bar{A}^1(1) + \tfrac{1}{2}\cdot\bar{A}^1(2)] \Longrightarrow \bar{A}^1(1) = \tfrac{1}{2} + \beta\tfrac{1}{2}[\bar{A}^1(0) + \bar{A}^1(2)] \Longrightarrow \bar{A}^1(1) = \tfrac{1}{2}\left[1 + \beta\bar{A}^1(2)\right]$$

$$\bar{A}^1(2) = 1 + \beta[0\cdot\bar{A}^1(0) + 0\cdot\bar{A}^1(1) + 1\cdot\bar{A}^1(2)] \Longrightarrow \bar{A}^1(2) = 1 + \beta\bar{A}^1(2) \Longrightarrow \bar{A}^1(2) = \frac{1}{1-\beta}\ ;\quad \bar{A}^1(1) = \frac{1}{2(1-\beta)}$$

For $i=2$, $\forall s$,

$$\bar{A}^2(0) = 1 + \beta[1\cdot\bar{A}^2(0) + 0\cdot\bar{A}^2(1) + 0\cdot\bar{A}^2(2)] \Longrightarrow \bar{A}^2(0) = 1 + \beta\,\bar{A}^2(0) \Longrightarrow \bar{A}^2(0) = \frac{1}{1-\beta}$$

$$\bar{A}^2(1) = \tfrac{1}{2} + \beta[\tfrac{1}{2}\cdot\bar{A}^2(0) + 0\cdot\bar{A}^2(1) + \tfrac{1}{2}\cdot\bar{A}^2(2)] \Longrightarrow \bar{A}^2(1) = \tfrac{1}{2} + \beta\tfrac{1}{2}[\bar{A}^2(0) + \bar{A}^2(2)] \Longrightarrow \bar{A}^2(1) = \tfrac{1}{2}\left\{1 + \beta\left[\tfrac{1}{1-\beta} + \bar{A}^2(2)\right]\right\}$$

$$\bar{A}^2(2) = 0 + \beta[0\cdot\bar{A}^2(0) + 0\cdot\bar{A}^2(1) + 1\cdot\bar{A}^2(2)] \Longrightarrow \bar{A}^2(2) = \beta\bar{A}^2(2) \Longrightarrow \bar{A}^2(2) = 0\ ;\quad \bar{A}^2(1) = \frac{1}{2(1-\beta)}$$

5) Compute portfolios of Arrow securities

Compute first individual's wealth,

$$W^i_t(s^t) = \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t} \underbrace{q^t_\tau(s^\tau)}_{\frac{q^0_\tau(s^\tau)}{q^0_t(s^t)}}\,c^i_\tau(s^\tau) = \frac{1}{2}\sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\frac{\beta^\tau\pi_\tau(s^\tau)}{\beta^t\pi_t(s^t)} = \frac{1}{2}\sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\beta^{\tau-t}\,\pi(s^\tau|s^t) =$$

$$= \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\underbrace{\sum_{s^\tau|s^t}\pi(s^\tau|s^t)}_{1} = \frac{1}{2(1-\beta)}$$

We know that $\Upsilon^i_t(s^t) = W^i_t(s^t) - \bar{A}^i_t(s^t)$. Setting $\Upsilon^i_t(s^t) = \tilde{a}^i_t(s^t)$, and using $\bar{A}^i(s^t)$,

$$\tilde{a}^i(s) = \Upsilon^i(s) = \frac{1}{2(1-\beta)} - \bar{A}^i(s)$$

For $i=1$, $\forall s$,

$$\tilde{a}^1(0) = \Upsilon^1(0) = \frac{1}{2(1-\beta)} - \bar{A}^1(0) = \frac{1}{2(1-\beta)}$$

$$\tilde{a}^1(1) = \Upsilon^1(1) = \frac{1}{2(1-\beta)} - \bar{A}^1(1) = \frac{1}{2(1-\beta)} - \frac{1}{2(1-\beta)} = 0$$

$$\tilde{a}^1(2) = \Upsilon^1(2) = \frac{1}{2(1-\beta)} - \bar{A}^1(2) = \frac{1}{2(1-\beta)} - \frac{1}{1-\beta} = -\frac{1}{2(1-\beta)}$$

For $i=2$, $\forall s$,

$$\tilde{a}^2(0) = \Upsilon^2(0) = \frac{1}{2(1-\beta)} - \bar{A}^2(0) = \frac{1}{2(1-\beta)} - \frac{1}{1-\beta} = -\frac{1}{2(1-\beta)}$$

$$\tilde{a}^2(1) = \Upsilon^2(1) = \frac{1}{2(1-\beta)} - \bar{A}^2(1) = \frac{1}{2(1-\beta)} - \frac{1}{2(1-\beta)} = 0$$

$$\tilde{a}^2(2) = \Upsilon^2(2) = \frac{1}{2(1-\beta)} - \bar{A}^2(2) = \frac{1}{2(1-\beta)}$$

Where we have satisfied market clearing, $a^1(s) + a^2(s) = 0 \quad \forall t, s$

d) Notice that there are only 2 histories possible, namely $\bar{s}^t = \{1,0,0,\dots\}$ and $\underline{s}^t = \{1,2,2,\dots\}$.

1) $\bar{s}^t = \{1,0,0,\dots\}$

$$\bar{c}_t^1(\bar{s}^t) = \left\{\frac{1}{2}, \frac{1}{2}, \dots\right\} \quad ; \quad \tilde{a}_t^1(\bar{s}^t) = \left\{0, \frac{1}{2(1-\beta)}, \frac{1}{2(1-\beta)}, \dots\right\}$$

$$\bar{c}_t^2(\bar{s}^t) = \left\{\frac{1}{2}, \frac{1}{2}, \dots\right\} \quad ; \quad \tilde{a}_t^2(\bar{s}^t) = \left\{0, -\frac{1}{2(1-\beta)}, -\frac{1}{2(1-\beta)}, \dots\right\}$$

2) $\underline{s}^t = \{1,2,2,\dots\}$

$$\bar{c}_t^1(\underline{s}^t) = \left\{\frac{1}{2}, \frac{1}{2}, \dots\right\} \quad ; \quad \tilde{a}_t^1(\underline{s}^t) = \left\{0, -\frac{1}{2(1-\beta)}, -\frac{1}{2(1-\beta)}, \dots\right\}$$

$$\bar{c}_t^2(\underline{s}^t) = \left\{\frac{1}{2}, \frac{1}{2}, \dots\right\} \quad ; \quad \tilde{a}_t^2(\underline{s}^t) = \left\{0, \frac{1}{2(1-\beta)}, \frac{1}{2(1-\beta)}, \dots\right\}$$

e) $\pi_t^i(s^t) = \pi^i(s_t|s_{t-1})\, \pi^i(s_{t-1}|s_{t-2}) \cdots \pi^i(s_1|s_0)\, \pi_0(s_0)$

where $\pi^i(s_j|s_{j-1})$ are derived from P. We consider $\pi_0(s_0)$ certain (it was already observed). The distribution at $t=1$ is

$$\pi_1^{i\prime} = \pi_0^{i\prime} P$$

at $t=2$,

$$\pi_2^{i\prime} = \pi_1^{i\prime} P = \pi_0^{i\prime} P^2$$

hence, at t

$$\pi_t^{i\prime} = \pi_0^{i\prime} P^t$$

Notice that in this particular case P is idempotent $(P^t = P)$.
Hence,

$$\pi_t^{1'} = \pi_0^{1'}P = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix}$$

$$\pi_t^{2'} = \pi_0^{2'}\hat{P} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & 0 & 0.6 \end{bmatrix}$$

Consumer 2 is more pessimistic: believes state 2 is more likely, in which his endowment is 0.

f)
$$\max_{c_t^1(s^t),\, c_t^2(s^t)} \left\{ \theta \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^1(s^t) \ln\left[c_t^1(s^t)\right] + (1-\theta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^2(s^t) \ln\left[c_t^2(s^t)\right] \right\}$$

$$\text{s.t.} \quad c_t^1(s^t) + c_t^2(s^t) \leq Y(s_t) = 1$$

Forming the lagrangian,

$$\mathcal{L} = \theta \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^1(s^t) \ln\left[c_t^1(s^t)\right] + (1-\theta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^2(s^t) \ln\left[c_t^2(s^t)\right] + \mu_t(s^t)\left[1 - c_t^1(s^t) - c_t^2(s^t)\right]$$

Taking the FOC's

$$c_t^1(s^t): \quad \theta \beta^t \pi_t^1(s^t) \frac{1}{c_t^1(s^t)} - \mu_t(s^t) = 0$$

$$c_t^2(s^t): \quad (1-\theta) \beta^t \pi_t^2(s^t) \frac{1}{c_t^2(s^t)} - \mu_t(s^t) = 0$$

Combining the FOC's

$$\frac{c_t^1(s^t)\, \pi_t^2(s^t)}{c_t^2(s^t)\, \pi_t^1(s^t)} = \frac{\theta}{1-\theta}$$

Using feasibility

$$c_t^1(s^t) + \frac{1-\theta}{\theta} \frac{c_t^1(s^t)\, \pi_t^2(s^t)}{\pi_t^1(s^t)} = 1 \implies c_t^1(s^t) = \frac{\theta \pi_t^1(s^t)}{\theta \pi_t^1(s^t) + (1-\theta)\pi_t^2(s^t)}$$

$$c_t^2(s^t) = \frac{(1-\theta)\pi_t^2(s^t)}{\theta \pi_t^1(s^t) + (1-\theta)\pi_t^2(s^t)}$$

g) Def: A competitive equilibrium is an initial distribution s_0, a consumption allocation $\{c_t^i(s^t)\}_{\forall i,t,s^t}$ where $s^t = \{s_t\}_{\forall t}$ is the entire history until time t, and a price system $\{q_t^0(s^t)\}_{\forall t, s^t}$ such that

- Given s_0, each consumer i solves

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^i(s^t) \ln\left[c_t^i(s^t)\right]$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\, y_t^i(s^t)$$

- Markets clear (feasibility): $\sum_{i=1}^{I=2} c_t^i(s^t) = \sum_{i=1}^{I=2} y_t^i(s^t) \equiv 1$

h) Forming the lagrangian,

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^i(s^t) \ln\left[c_t^i(s^t)\right] + \mu^i \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)\left[y_t^i(s^t) - c_t^i(s^t)\right]$$

Taking the FOC,

$$c_t^i(s^t): \quad \beta^t \pi_t^i(s^t) \frac{1}{c_t^i(s^t)} - \mu^i q_t^0(s^t) = 0$$

Rewritting it for $t=0$, normalizing $q_0^0(s_0) = 1$; and since we know π_0, $\pi_0^i(s_0) = 1$ $\forall i$. As a result $\mu^i = \left[c_0^i(s_0)\right]^{-1}$. Hence, we can write the FOC as

$$q_t^0(s^t) = \beta^t \pi_t^i(s^t) \frac{c_0^i(s_0)}{c_t^i(s^t)} \tag{18}$$

Using feasibility

$$c_t^1(s^t) + c_t^2(s^t) = y_t^1(s^t) + y_t^2(s^t) \equiv 1 \implies \frac{\beta^t}{q_t^0(s^t)}\left[\pi_t^1(s^t)c_0^1(s_0) + \pi_t^2(s^t)c_0^2(s_0)\right] = 1 \implies$$

$$\xrightarrow{c_0^2(s_0) = 1 - c_0^1(s_0)} \frac{\beta^t}{q_t^0(s^t)}\left[\pi_t^1(s^t)c_0^1(s_0) + \pi_t^2(s^t)c_0^2(s_0)\right] = 1 \implies$$

$$\implies q_t^0(s^t) = \beta^t \pi_t^2(s^t) + \beta^t\left[\pi_t^1(s^t) - \pi_t^2(s^t)\right] c_0^1(s_0) \tag{19}$$

We need to determine $c_0^i(s_0)$ $\forall i$. Using the budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\, c_t^i(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\, y_t^i(s^t) \xrightarrow[(18)]{i=1} \sum_{t=0}^{\infty}\sum_{s^t} \cancel{q_t^0(s^t)}\,\beta^t \pi_t^1(s^t)\frac{c_0^1(s_0)}{\cancel{q_t^0(s^t)}} = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\frac{s_t}{2} \implies$$

$$\xrightarrow{(19)} c_0^1(s_0)\sum_{t=0}^{\infty}\beta^t \underbrace{\sum_{s^t}\pi_t^1(s^t)}_{1} = \sum_{t=0}^{\infty}\sum_{s^t}\left\{\beta^t\left[\pi_t^1(s^t)c_0^1(s_0) + \pi_t^2(s^t)c_0^2(s_0)\right]\right\}\frac{s_t}{2}$$

$$\implies \frac{c_0^1(s_0)}{1-\beta} = \frac{c_0^1(s_0)}{2}\underbrace{\sum_{t=0}^{\infty}\beta^t \underbrace{\sum_{s^t}\pi_t^1(s^t)\, s_t}_{\substack{t=0,\ 0\cdot 0 + 1\cdot 1 + 0\cdot 2 = 1 \\ t\geq 1,\ 0.5\cdot 0 + 0\cdot 1 + 0.5\cdot 2 = 1}}}_{\frac{1}{1-\beta}} + \frac{c_0^2(s_0)}{2}\underbrace{\sum_{t=0}^{\infty}\beta^t \underbrace{\sum_{s^t}\pi_t^2(s^t)\, s_t}_{\substack{t=0,\ 0\cdot 0 + 1\cdot 1 + 0\cdot 2 = 1 \\ t\geq 1,\ 0.4\cdot 0 + 0\cdot 1 + 0.6\cdot 2 = 1.2}}}_{1 + 1.2\beta + 1.2\beta^2 + \cdots}$$

$$1 + \underbrace{1.2\beta + 1.2\beta^2}_{V = 1.2\beta + \beta V \implies V = \frac{1.2\beta}{1-\beta}} + \cdots$$

$$1 + \frac{1.2\beta}{1-\beta} = \frac{1+0.2\beta}{1-\beta}$$

$$\implies \frac{c_0^1(s_0)}{1-\beta} = \frac{c_0^1(s_0)}{2(1-\beta)} + \frac{c_0^2(s_0)(1+0.2\beta)}{2(1-\beta)} \implies$$

$$\xrightarrow{c_0^2(s_0) = 1 - c_0^1(s_0)} \quad c_0^1(s_0) = \frac{1}{2}\,\frac{1+0.2\beta}{1+0.1\beta} > \frac{1}{2}, \qquad c_0^2(s_0) = \frac{1}{2}\,\frac{1}{1+0.1\beta} < \frac{1}{2} \qquad (\pi_t^1 = \pi_t^2)$$

consumer 2's pessimism pushes him to order more consumption for history $s^t = \{1, 2, 2, \ldots\}$. Hence he decreases initial consumption (c_0^2) and thus, by feasibility, individual 1 consumes more in the first period we can now obtain price through (19),

$$q_t^0(s^t) = \beta^t \left[\pi_t^1(s^t) c_0^1(s_0) + \pi_t^2(s^t) c_0^2(s_0)\right] = \beta^t \left[\pi_t^1(s^t) \frac{1}{2} \frac{1+0.2\beta}{1+0.1\beta} + \pi_t^2(s^t) \frac{1}{2} \frac{1}{1+0.1\beta}\right]$$

and through (18),

$$c_t^i(s^t) = \beta^t \pi_t^i(s^t) \frac{c_0^i(s_0)}{q_t^0(s^t)} = \cancel{\beta^t} \pi_t^i(s^t) \frac{c_0^i(s_0)}{\cancel{\beta^t} \left[\pi_t^1(s^t) \frac{1}{2} \frac{1+0.2\beta}{1+0.1\beta} + \pi_t^2(s^t) \frac{1}{2} \frac{1}{1+0.1\beta}\right]} \qquad (20)$$

$$= \begin{cases} \pi_t^1(s^t) \dfrac{\frac{1}{2} \frac{1+0.2\beta}{1+0.1\beta}}{\left[\pi_t^1(s^t) \frac{1}{2} \frac{1+0.2\beta}{1+0.1\beta} + \pi_t^2(s^t) \frac{1}{2} \frac{1}{1+0.1\beta}\right]} \\ \pi_t^2(s^t) \dfrac{\frac{1}{2} \frac{1}{1+0.1\beta}}{\left[\pi_t^1(s^t) \frac{1}{2} \frac{1+0.2\beta}{1+0.1\beta} + \pi_t^2(s^t) \frac{1}{2} \frac{1}{1+0.1\beta}\right]} \end{cases}$$

i) Def : A sequential-trading competitive equilibrium is an initial distribution of wealth $\{a_0^i(s_0)\}_{\forall i}$, a consumption allocation $\{\tilde{c}_t^i(s^t)\}_{\forall i,t,s^t}$ and a price system $\{\tilde{Q}_t(s_{t+1} | s^t)\}_{\forall t, s^t, s_{t+1}}$ such that

- Given initial wealth distribution and price system, the allocation solves each agent's problem

$$\max_{\{\tilde{c}_t^i(s^t), \{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}\}_{\forall t, s^t}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t^i(s^t) u_i\left[\tilde{c}_t^i(s^t)\right]$$

$$\text{s.t.} \quad \tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1} | s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s_t, s^{t-1})$$

$$\text{given } \tilde{a}_0^i(s_0) \qquad \forall t, s^t$$

$$\tilde{c}_t^i(s^t) \geq 0$$

$$\text{non-Ponzi constraint} : -\tilde{a}_{t+1}^i(s_{t+1}, s^t) \leq A_{t+1}^i(s^{t+1})$$

- For all realization of $\{s^t\}$ the consumption allocation and implied asset portfolio satisfy

$$\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t) \qquad \forall t, s^t$$

$$\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$$

To compute the equilibrium we will follow the same algorithm as before,

2) Set $\tilde{c}_t^i(s^t) = c_t^i(s^t)$ [Sequential trading $\equiv$ Arrow-Debreu time-0]

3) Compute equilibrium price

$$\tilde{Q}_t(s_{t+1} | s_t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \frac{\beta^{t+1} \pi_{t+1}^i(s^{t+1}) \frac{c_0^i(s_0)}{c_{t+1}^i(s^{t+1})}}{\beta^t \pi_t^i(s^t) \frac{c_0^i(s_0)}{c_t^i(s^t)}} = \beta \frac{\pi^i(s_{t+1}|s_t) \pi^i(s_t|s_{t-1}) \cdots \pi_0^i(s_0) \; c_t^i(s^t)}{\pi^i(s_t|s_{t-1}) \cdots \pi_0^i(s_0) \; c_{t+1}^i(s^{t+1})} =$$

$$= \beta\, \pi^i(s_{t+1}|s_t)\, \frac{c_t^i(s^t)}{c_{t+1}^i(s^{t+1})}$$

Using feasibility,

$$\tilde{Q}_t(s_{t+1}|s_t)\Big[\underbrace{c_{t+1}^1(s^{t+1}) + c_{t+1}^2(s^{t+1})}_{=1}\Big] = \beta\Big[\pi^1(s_{t+1}|s_t)\, c_t^1(s^t) + \pi^2(s_{t+1}|s_t)\, c_t^2(s^t)\Big]$$

$$\Longrightarrow \tilde{Q}_t(s_{t+1}|s_t) = \beta\Big[\pi^1(s_{t+1}|s_t)\, c_t^1(s^t) + \pi^2(s_{t+1}|s_t)\, c_t^2(s^t)\Big]$$

Recall that there are only 4 possible transitions: $1\to 0$, $1 \to 2$, $0 \to 0$ and $2 \to 2$. Hence,

such transition only occurs at $t=0$

$$\tilde{Q}_0(0|1) = \beta\Big[\underbrace{\pi^1(0|1)}_{0.5}\, \underbrace{c_0^1(1)}_{\frac{1}{2}\frac{1+0.2\beta}{1+0.1\beta}} + \underbrace{\pi^2(0|1)}_{0.4}\, \underbrace{c_0^2(1)}_{\frac{1}{2}\frac{1}{1+0.1\beta}}\Big] = \frac{\beta}{2}\left(\frac{9+\beta}{10+\beta}\right)$$

$$\tilde{Q}_0(2|1) = \beta\Big[\underbrace{\pi^1(2|1)}_{0.5}\, \underbrace{c_0^1(1)}_{\frac{1}{2}\frac{1+0.2\beta}{1+0.1\beta}} + \underbrace{\pi^2(2|1)}_{0.6}\, \underbrace{c_0^2(1)}_{\frac{1}{2}\frac{1}{1+0.1\beta}}\Big] = \frac{\beta}{2}\left(\frac{11+\beta}{10+\beta}\right)$$

$\tilde{Q}_0(2|1) > \tilde{Q}_0(0|1)$ consequence of $i=2$ prefering to shift consumption to s^t, since he thinks such state is more likely

$$\tilde{Q}_t(0|0) = \beta\Big[\underbrace{\pi^1(0|0)}_{1}\, c_t^1(0) + \underbrace{\pi^2(0|0)}_{1}\, c_t^2(0)\Big] = \beta$$

no need to calculate them since $c_t^1(0) + c_t^2(0) = 1$

$$\tilde{Q}_t(2|2) = \beta\Big[\underbrace{\pi^1(2|2)}_{1}\, c_t^1(2) + \underbrace{\pi^2(2|2)}_{1}\, c_t^2(2)\Big] = \beta$$

no need to calculate them since $c_t^1(2) + c_t^2(2) = 1$

4) Compute "natural debt limits"

$$\bar{A}^i(s) = y^i(s) + \sum_{s'} Q(s'|s)\, \underbrace{\bar{A}^i(s'|s)}_{\bar{A}^i(s')}$$

For $i=1$, $\forall s$,

not possible!

$$\bar{A}^1(0) = 0 + \underbrace{Q(0|0)}_{\beta}\bar{A}^1(0) + Q(1|0)\bar{A}^1(1) + Q(2|0)\bar{A}^1(2) \Longrightarrow \bar{A}^1(0) = \beta\bar{A}^1(0) \Longrightarrow \bar{A}^1(0) = 0$$

not possible!

$$\bar{A}^1(1) = \frac{1}{2} + \underbrace{Q(0|1)}_{\frac{\beta}{2}\left(\frac{9+\beta}{10+\beta}\right)}\underbrace{\bar{A}^1(0)}_{0} + Q(1|1)\bar{A}^1(1) + \underbrace{Q(2|1)}_{\frac{\beta}{2}\left(\frac{11+\beta}{10+\beta}\right)}\bar{A}^1(2) \Longrightarrow \frac{1}{2}\left[1 + \beta\left(\frac{11+\beta}{10+\beta}\right)\bar{A}^1(2)\right]$$

not possible!

$$\bar{A}^1(2) = 1 + Q(0|2)\underbrace{\bar{A}^1(0)}_{0} + Q(1|2)\bar{A}^1(1) + \underbrace{Q(2|2)}_{\beta}\bar{A}^1(2) \Longrightarrow \bar{A}^1(2) = 1 + \beta\bar{A}^1(2) \Longrightarrow \bar{A}^1(2) = \frac{1}{1-\beta}$$

Hence, $\bar{A}^1(1) = \frac{1}{2}\left[\frac{10+2\beta}{(1-\beta)(10+\beta)}\right]$

For $i=2$, $\forall s$,

not possible!

$$\bar{A}^2(0) = 1 + \underbrace{Q(0|0)}_{\beta}\bar{A}^1(0) + Q(1|0)\bar{A}^1(1) + Q(2|0)\bar{A}^1(2) \Rightarrow \bar{A}^2(0) = 1 + \beta\bar{A}^2(0) \Rightarrow \bar{A}^2(0) = \frac{1}{1-\beta}$$

not possible!

$$\bar{A}^2(1) = \frac{1}{2} + \underbrace{Q(0|1)}_{\frac{\beta}{2}\left(\frac{9+\beta}{10+\beta}\right)}\underbrace{\bar{A}^2(0)}_{\frac{1}{1-\beta}} + Q(1|1)\bar{A}^2(1) + \underbrace{Q(2|1)}_{\frac{\beta}{2}\left(\frac{11+\beta}{10+\beta}\right)}\bar{A}^2(2) \Rightarrow \bar{A}^2(1) = \frac{1}{2}\left\{1 + \beta\left[\frac{9+\beta}{(10+\beta)(1-\beta)} + (11+\beta)\bar{A}^1(2)\right]\right\}$$

not possible!

$$\bar{A}^2(2) = 0 + Q(0|2)\bar{A}^2(0) + Q(1|2)\bar{A}(1) + \underbrace{Q(2|2)}_{\beta}\bar{A}^2(2) \Rightarrow \bar{A}^2(2) = \beta\bar{A}^2(2) \Rightarrow \bar{A}^1(2) = 0$$

Hence, $\bar{A}^2(1) = \frac{1}{2}\left[\frac{10}{(1-\beta)(10+\beta)}\right]$

Notice that all debt limits are the same as in the case with $\pi^1 = \pi^2$, except for $s_t = 1$. In that case $\bar{A}^1(1) > \bar{A}^2(1)$. Notice that $i=1$ gets all future endowments if $\bar{s}^t$ realizes, while $i=2$ gets all future endowments if $\underline{s}^t$. And since prices are greater if $\underline{s}^t$ realizes ($Q(2|1) > Q(0|1)$), the present value of $i=1$ future endowments is greater than the present value of $i=2$ future endowments, at initial state $s_0 = 1$.

5) Compute portfolios of Arrow securities

Compute first individual's wealth,

$$W_t^i(s^t) = \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t} \underbrace{q_\tau^t(s^\tau)}_{\frac{q_\tau^0(s^\tau)}{q_t^0(s^t)}} c_\tau^i(s^\tau) = \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t} \frac{\beta^\tau \pi_\tau^i(s^\tau)\frac{c_0^i(s_0)}{c_\tau^i(s^\tau)}}{\beta^t\pi_t^i(s^t)\frac{c_0^i(s_0)}{c_t^i(s^t)}} c_\tau^i(s^\tau) = \sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}\beta^{\tau-t}\pi(s^\tau|s^t)\, c_t^i(s^t) =$$

$$= c_t^i(s^t)\sum_{t=0}^{\infty}\beta^{\tau-t}\underbrace{\sum_{s^\tau|s^t}\pi(s^\tau|s^t)}_{1} = \frac{c_t^i(s^t)}{1-\beta}$$

We know that $\Upsilon_t^i(s^t) = W_t^i(s^t) - \bar{A}_t^i(s^t)$. Setting $\Upsilon_t^i(s^t) = \tilde{a}_t^i(s^t)$, and using $\bar{A}^i(s^t)$,

$$\tilde{a}^i(s) = \Upsilon^i(s) = \frac{c_t^i(s^t)}{1-\beta} - \bar{A}^i(s)$$

we now need to obtain $c_t^i(s^t)$. From (20),

$$c_t^i(s^t) = \frac{c_0^i(s_0)}{\pi_t^1(s^t)\frac{1}{2}c_0^1(s_0) + \pi_t^2(s^t)\, c_0^2(s_0)}$$

where $\underset{1}{c_0^1(s_0)} = \frac{1}{2}\frac{1+0.2\beta}{1+0.1\beta}$ and $\underset{1}{c_0^2(s_0)} = \frac{1}{2}\frac{1}{1+0.1\beta}$. Hence, for $i=1$, $\forall s \wedge 1$, $\forall t \geq 1$,

$$c_t^1(\bar{s}^t = \{1,0,0,\dots\}) = \frac{\pi_t^1(\bar{s}^t)\, c_0^1(1)}{\pi_t^1(\bar{s}^t)\, c_0^1(1) + \pi_t^2(\bar{s}^t)\, c_0^2(1)} = \frac{\frac{1}{2}\frac{1}{2}\frac{1+0.2\beta}{1+0.1\beta}}{\frac{1}{2}\frac{1}{2}\frac{1+0.2\beta}{1+0.1\beta} + 0.4\frac{1}{2}\frac{1}{1+0.1\beta}} = \frac{5+\beta}{9+\beta}$$

$$c_t^1(\underline{s}^t = \{1,2,2,\dots\}) = \frac{\pi_t^1(\underline{s}^t)\, c_0^1(1)}{\pi_t^1(\underline{s}^t)\, c_0^1(1) + \pi_t^2(\underline{s}^t)\, c_0^2(1)} = \frac{\frac{1}{2}\frac{1}{2}\frac{1+0.2\beta}{1+0.1\beta}}{\frac{1}{2}\frac{1}{2}\frac{1+0.2\beta}{1+0.1\beta} + 0.6\frac{1}{2}\frac{1}{1+0.1\beta}} = \frac{5+\beta}{11+\beta}$$

we could do exactly the same for $i=2$. Instead, I'll use market clearing

$$c_t^2(\bar{s}^t = \{1,0,0,\dots\}) = 1 - c_t^1(\bar{s}^t = \{1,0,0,\dots\}) = \frac{4}{9+\beta}$$

$$c_t^2(\underline{s}^t=\{1,2,2,\dots\}) = 1 - c_t^1(\underline{s}^t=\{1,2,2,\dots\}) = \frac{6}{11+\beta}$$

Finally we can obtain the financial wealth. For $i=1$, $\forall s$,

$$\tilde{a}^1(0) = \Upsilon^1(0) = \frac{c_t^1(0)}{1-\beta} - \underbrace{\bar{A}^1(0)}^{0} = \frac{5+\beta}{(9+\beta)(1-\beta)}$$

$$\tilde{a}^1(1) = \Upsilon^1(1) = \frac{\overbrace{c_t^1(1)}^{c_0^1(s_0)}}{1-\beta} - \bar{A}^1(1) = \frac{1}{2}\,\frac{1+0.2\beta}{1+0.1\beta}\,\frac{1}{1-\beta} - \frac{1}{2}\,\frac{10+2\beta}{(10+\beta)(1-\beta)} = 0$$

$$\tilde{a}^1(2) = \Upsilon^1(2) = \frac{c_t^1(1)}{1-\beta} - \bar{A}^1(2) = \frac{5+\beta}{11+\beta}\,\frac{1}{1-\beta} - \frac{1}{1-\beta} = -\frac{6}{(11+\beta)(1-\beta)}$$

we could do exactly the same for $i=2$. Instead, I'll use market clearing: $\tilde{a}^2(s) = -\tilde{a}^1(s)$ $\forall s$.

$i=1$ borrows from $i=2$ at $s=0$ (when $i=1$'s endowment is 0) and lends to $i=2$ at $s=2$ (when $i=2$'s endowment is 0). Hence, they are perfectly insured!

Notice that there are only 2 histories possible, namely $\bar{s}^t=\{1,0,0,\dots\}$ and $\underline{s}^t=\{1,2,2,\dots\}$.

1) $\bar{s}^t = \{1,0,0,\dots\}$

$$\bar{c}_t^1(\bar{s}^t) = \left\{\frac{5+\beta}{10+\beta}, \frac{5+\beta}{9+\beta}, \frac{5+\beta}{9+\beta}, \dots\right\} \quad ; \quad \tilde{a}_t^1(\bar{s}^t) = \left\{0, \frac{5+\beta}{(9+\beta)(1-\beta)}, \frac{5+\beta}{(9+\beta)(1-\beta)}, \dots\right\}$$

$$\bar{c}_t^2(\bar{s}^t) = \left\{\frac{5}{10+\beta}, \frac{4}{9+\beta}, \frac{4}{9+\beta}, \dots\right\} \quad ; \quad \tilde{a}_t^2(\bar{s}^t) = \left\{0, -\frac{5+\beta}{(9+\beta)(1-\beta)}, -\frac{5+\beta}{(9+\beta)(1-\beta)}, \dots\right\}$$

2) $\underline{s}^t = \{1,2,2,\dots\}$

$$\bar{c}_t^1(\underline{s}^t) = \left\{\frac{5+\beta}{10+\beta}, \frac{5+\beta}{11+\beta}, \frac{5+\beta}{11+\beta}, \dots\right\} \quad ; \quad \tilde{a}_t^1(\underline{s}^t) = \left\{0, -\frac{6}{(11+\beta)(1-\beta)}, -\frac{6}{(11+\beta)(1-\beta)}, \dots\right\}$$

$$\bar{c}_t^2(\underline{s}^t) = \left\{\frac{5}{10+\beta}, \frac{6}{11+\beta}, \frac{6}{11+\beta}, \dots\right\} \quad ; \quad \tilde{a}_t^2(\underline{s}^t) = \left\{0, \frac{6}{(11+\beta)(1-\beta)}, \frac{6}{(11+\beta)(1-\beta)}, \dots\right\}$$