Mathematics III
Problem Set 3: Dynamic Optimization I

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Deadline is *Mon 11 December at 2:00pm.* Please submit your homework to *José Elías Gallegos Dago's* mailbox at Stockholm University in paper form. You may work in groups of up-to four, but each student must submit an individual solution. Please indicate the names of the other group members on your solution.

Exercise 1: Bang-Bang Control (20 points)

Consider the following dynamic optimization problem,

$$\max_{u\in[-1,1]} \int_0^1 (2x - x^2)dt \quad s.t. \quad \dot{x} = u, \quad x(0) = 0, \quad x(1) = 0.$$

(a) Write down the conditions of the *maximum principle.* Are these conditions sufficient?

(b) Show that $\lambda(t)$ is decreasing.

(c) Suppose that $\lambda(t)$ has a unique crossing with the zero-line at $\hat{t} = 1/2$. Use this to derive $u^\star(t)$, $\lambda^\star(t)$ and $x^\star(t)$. Is the assumption correct?

Exercise 2: A Distance Problem (20 points)

Suppose $x_0 < x_1 < x_0 + T$. Solve the problem

$$\max_{u\in[0,1]} \int_0^T x dt \quad s.t. \quad \dot{x} = u, \quad x(0) = x_0, \quad x(T) = x_1.$$

What "classes of problems" does this problem attempt to solve? Discuss.

Exercise 3: Growth in Consumption (30 points)

Suppose a person seeks to maximize,

$$\max_{x(t)} \int_0^\infty u(c)e^{-\delta t}dt \quad \text{s.t.} \quad \dot{k} = b(1-x)k, \; x \in [0,\, 1]\,,$$

where $c = f(xk)$, $f(0) = 0$, $f'(\cdot) > 0$ and $f''(\cdot) < 0$.

(a) Derive the *sufficient* first-order conditions associated with the maximization problem.

(b) Assume that $f(xk) = A(xk)^{\alpha}$ where $\alpha \in (0,1)$ and that $u(c) = log(c)$. Show that on the optimal path the growth of c is given by $\alpha(b - \delta)$. When does c grow forever?

Exercise 4: Autonomous Systems (30 points)

Consider the optimal control problem,

$$\max_{u(t)} \int_0^1 \left[-\frac{1}{2}u^2 - x \right] dt \quad \text{s.t.} \quad \dot{x} = 2(1-u),\ x(0) = 1,$$

where $x(1)$ is free.

(a) State the necessary and sufficient conditions for a candidate solution.

(b) Find $u^{\star}(t)$, $x^{\star}(t)$ and $\lambda^{\star}(t)$ using these conditions.

(c) Show that the Hamiltonian is constant along the optimal trajectory. Explain why?